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NOTE ON TWO COMPATIBILITY CRITERIA: JACOBI-MAYER BRACKET VS. DIFFERENTIAL GRÖBNER BASIS

(submitted by V.V. Lychagin)

Abstract. We compare two compatibility criteria for overdetermined PDEs: one via geometric theory of differential equations and another via differential algebra approach. Whenever both are applicable, we show that the former is more effective, though in some very special cases they are equivalent.

Introduction

In this paper we investigate and compare two recent results on compatibility of overdetermined systems of partial differential equations, which we formulate below. For simplicity of exposition we restrict to the case of scalar PDEs, though the same comparison results hold true in the general context (see §3.1).

Let $\mathcal{E} \subset J^k(M)$ be a system of scalar differential equations on a manifold M, represented as a finite set of equations (relations) $\{F_i = 0\}$ on the jet-space. For $G_j \in C^\infty(J^l M)$ denote by $\langle \mathbf{G} \rangle = \langle \{G_j\} \rangle$ (here and in what follows we denote collections of functions in bold) the algebraic ideal in $C^\infty(J^l M)$ generated by G_j, i.e. $\{\sum h_i G_i \,|\, h_i \in C^\infty(J^l M)\}$. For

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polynomial systems $\mathcal{E}$ the functions h_i may be assumed polynomial and for linear systems $h_i \in C^\infty(M)$ (depending on the context).

Denote also $\langle \mathbf{F}\rangle_l = \langle \hat{\Delta}_i F_i \mid \mathrm{ord}(F_i)+\mathrm{ord}(\Delta_i) \le l\rangle$, where $\hat{\Delta}$ are the lifts of scalar differential operators $\Delta \in \mathrm{diff}(M)$ on M (see §1.1 and [KLV]). The differential ideal generated by $\mathbf{F}$ is $\langle\!\langle \mathbf{F}\rangle\!\rangle = \langle \mathbf{F}\rangle_\infty$.

Let us write $A \overset{\mathbf{F}}{\rightsquigarrow} B$ if B is obtained from A as quotient by the $C^\infty(J^l M)$-algebraic ideal $\langle \mathbf{F}\rangle_l$ with $l = \mathrm{ord}(A)$.

Suppose the system $\mathcal{E}$ is regular in the usual sense ([S]). Then we have (more details will be provided in the subsequent sections):

Theorem 1 ([KL$_3$])**.** *Let $\mathcal{E}$ be a complete intersection, i.e. the characteristic varieties* $\mathrm{Char}^{\mathbb{C}}(F_i)$ *are jointly transversal. Then the system is compatible (formally integrable) iff the Jacobi brackets $\{F_i, F_j\} \overset{\mathbf{F}}{\rightsquigarrow} 0$.*

Theorem 2. *Let $\mathcal{E}$ be a polynomial type system and let $\{G_j\}$ be its differential Gröbner basis (dGB). Then $\mathcal{E}$ is compatible iff each element $G_j \overset{\mathbf{F}}{\rightsquigarrow} 0$.*

While the first theorem is recent and substantial, the second is folklore and easy (we will prove it in §2.2). It can be deduced somehow from the pioneer works by Ritt [R] and Kolchin [K], though the dGB notion appeared later (see the paper [M$_2$], where the question is discussed).

Simple compatibility criteria are very important for solving PDEs via auxiliary integrals ([KL$_1$, KL$_2$]), in particular cases also known as Lagrange-Charpit method ([Gu]), non-classical symmetries ([BC]), direct reduction ([CK]) etc.

The purpose of this paper is to discuss effectiveness of these compatibility criteria. Let us call two criteria equivalent if they calculate the same number of obstructions. In the case of brackets approach, this is the number of all pair-wise brackets and for dGB basis this is the number of elements in the basis. While different elements-obstructions involve different calculations, it is known that complications with Gröbner basis are mostly due to its length. So principally the comparison by the number of elements is reasonable.

Of course, the most economical criterion is one that uses the minimal number of obstructions, i.e. which deals with the basis of syzygies. As follows from calculations of Spencer cohomologies in [KL$_2$, KL$_3$], the criterion of Theorem 1 is the most economical. But it is clear that criterion of Theorem 2 is rarely such (even simple re-numeration of the coordinates can increase the number of elements in the dGB, while it is irrelevant for the calculation with the brackets). More exactly, we show (more details and terminology see in the main text):

Theorem 3. *For a differential-polynomial scalar system of complete intersection type the second criterion is never more optimal than the first. Moreover, the two criteria compare as follows. Consider a total degree order. Then:*

(1) *If the system is triangular-linear with the leading terms forming a complete intersection, then the above criteria are equivalent.*
(2) *If the system is generic linear (even differentially triangular), then the first criteria is optimal, while the second is not.*
(3) *If the system is generic non-linear, then optimality of the second criterion becomes less with the growth of non-linearity (degree of the leading terms).*

Thus for the most part of PDE systems, for which both criteria apply, we prove advantage of the method of Theorem 1: The complexity of the dGB algorithm is poor, while the bracket approach turns out to be quite effective.

The above result is only one visible comparison between the following two approaches: Formal Theory (jet-spaces or basically equivalent exterior differential systems) and Differential Algebra. We discuss other relations below in the text.

1. Preliminaries

Here we recollect basic facts important for understanding Theorems 1 and 2.

1.1. **Jacobi-Mayer bracket.** For (non-linear) scalar differential operators $F, G \in \operatorname{diff}(M) = C^\infty(J^\infty(M))$ the Jacobi bracket is defined by the formula:

$$\{F, G\} = \ell_F(G) - \ell_G(F),$$

where ℓ_F is the operator of universal linearization along F and similarly for G. If $F \in \operatorname{diff}_k(M) = C^\infty(J^k(M))$ is the operator of k-th order and $G \in \operatorname{diff}_l(M)$, then the bracket satisfies: $\{F, G\} \in \operatorname{diff}_{k+l-1}(M)$. For linear operators $F, G \in \operatorname{Diff}(M)$ the bracket $\{F, G\}$ is the usual commutator.

Let's write the Jacobi bracket in the canonical coordinates (x^i, p_σ) on the jet-space $J^\infty(M)$ (see [KLV]). Recall that these are the base M coordinates $(x^1, \dots, x^n)$ coupled with the vertical coordinates p_σ (with $\sigma = (i_1, \dots, i_n)$, $i_j \geq 0$, being a multi-index) fixed by the condition

$p_\sigma([u]_x^k)=\partial^{|\sigma|}u/\partial x^\sigma$ for $|\sigma|=\sum i_j\le k$ (here $[u]_x^k\in J^k(M)$ is the k-jet of the function u at x).

The total derivative operator $\mathcal{D}^\sigma:\operatorname{diff}_k(M)\to\operatorname{diff}_{k+|\sigma|}(M)$ is defined by: $\mathcal{D}^\sigma=\mathcal{D}_1^{i_1}\dots\mathcal{D}_n^{i_n}$, $\mathcal{D}_j=\sum p_{\sigma+1_j}\partial_{p_\sigma}$. Thus we get

$$\langle\mathbf{F}\rangle_l=\langle\mathcal{D}^{\sigma_i}F_i\mid\operatorname{ord}(F_i)+|\sigma_i|\le l\rangle.$$

Expressed in the canonical coordinates the linearization operator has the form: $\ell_F=\sum F_{p_\sigma}\mathcal{D}^\sigma$. Thus for $F\in\operatorname{diff}_k(M)$, $G\in\operatorname{diff}_l(M)$ we have:

$$\{F,G\}=\sum_{|\sigma|\le l}\mathcal{D}^\sigma(F)G_{p_\sigma}-\sum_{|\tau|\le k}\mathcal{D}^\tau(G)F_{p_\tau}.$$

If in the above summation we restrict to equalities for $|\sigma|,|\tau|$, we get the Mayer bracket $[F,G]$. They equal modulo the ideal

$$\langle F,G\rangle_{k+l-1}=\langle\mathcal{D}^\sigma F,\mathcal{D}^\tau G\,:\,|\sigma|\le l-1,|\tau|\le k-1\rangle,$$

so that $\{F,G\}\overset{F,G}{\rightsquigarrow}[F,G]$. We call their common value in the quotient space the Jacobi-Mayer bracket.

1.2. **Compatibility and Solvability.** System $\mathcal{E}=\{F_i=0\}$ can be defined by PDEs of different orders. In this case it is important to describe prolongations successively (we refer to the definition of prolongations and other notions for the systems of pure degree to [S, KLV], the general theory of various degrees is sketched in [KL$_2$]).

Let $\mathcal{E}_k=\{F_j=0\mid\deg(F_j)\le k\}$ be the locus of equations $\langle\mathbf{F}\rangle_k$ in the jet-space J^kM. We say that $\mathcal{E}$ is *compatible* up to the level k if $\mathcal{E}_{k-1}$ has one prolongation and $\mathcal{E}_k=\mathcal{E}_{k-1}^{(1)}\cap\{F_j=0\mid\deg F_j=k\}$ is foliated over $\mathcal{E}_{k-1}$ via the surjection $\pi_{k,k-1}:J^kM\to J^{k-1}M$. If $\mathcal{E}_k$ is compatible at all levels the system $\mathcal{E}$ is called *formally integrable*.

A finite type system $\mathcal{E}$ is the system without (complex) characteristics. Equivalently this means that a sufficiently high prolongation of the symbol of $\mathcal{E}$ vanishes [S]. Then for the same jet-level k for any $x_{k-1}\in\mathcal{E}_{k-1}$ the set $\pi_{k,k-1}^{-1}(x_{k-1})\cap\mathcal{E}_k$ is discrete and formal integrability implies the local one (thanks to Frobenius theorem).

For general systems one needs to examine compatibility on many levels to conclude formal integrability. But due to [KL$_3$] the compatibility conditions are known if the system is of complete intersection type. This is a condition of general position if $\mathcal{E}$ is given by $r\le n=\dim M$ PDEs (finite type corresponds to $r=n$). It can be formulated by the requirement of joint transversality for characteristic varieties ([KL$_2$]): $\operatorname{codim}\operatorname{Char}^{\mathbb{C}}(\mathcal{E})=r$.

In this case the obstructions to compatibility are the Jacobi-Mayer brackets $\{F_i, F_j\}$. If they vanish due to the system (in the sense of Theorem 1), $\mathcal{E}$ is formally integrable.

Otherwise we need to add the brackets to the system and continue investigation of integrability by computing new compatibility conditions. On the level of geometry if some projection $\pi_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$ is not epimorphic (surjective, but we always consider the regular case, so the distinction becomes inessential), we redefine $\mathcal{E}_{k-1}$ to be the image.

This is the essence of prolongation-projection method and Cartan-Kuranishi theorem guarantees that it terminates in the regular case. If we stop at some non-empty equation (for empty ones Cartan used the term "contradiction") we get formal solutions, so the system becomes (formally) solvable (or consistent as is said in differential algebra context).

In this note we study only the first step to solvability investigation: calculation of the compatibility conditions. And for this test we compare two criteria from the introduction.

1.3. **Differential algebra.** Let us fix an order on the set of derivatives p_σ, which is compatible with the operator of differentiation $\mathcal{D}^\tau$ (it increases the order and preserves the inequalities). A good candidate, which we use below, is the total order, for which $p_\sigma < p_\tau$ when $|\sigma| < |\tau|$, and one can use the lexicographic order (involving ordering of coordinate variables $x^1, \dots, x^n$), when $|\sigma| = |\tau|$.

Then every differential polynomial F (a usual polynomial in x^i, p_σ) possesses the leading term $\mathrm{HT}(F) = \max\{p_\sigma \,|\, \partial_{p_\sigma} F \neq 0\}$. This term occurs in F in maximal degree $\mathrm{HD}(F) = \max\{k \,|\, \partial^k_{\mathrm{HT}(F)} F \neq 0\}$ and the coefficient before it equals $\mathrm{HC}(F) = \frac{1}{\mathrm{HD}(F)!} \partial^{\mathrm{HD}(F)}_{\mathrm{HT}(F)} F$. Thus the leading monomial $\mathrm{HM}(F)$ equals the result of iterations of the operator $F \mapsto \mathrm{HC}(F) \cdot \mathrm{HT}(F)^{\mathrm{HD}(F)}$, applied to the leading coefficient etc.

In reducing a differential polynomial F by a set $\mathbf{G} = \{G_j\}$ we use simplifications by elements of the algebraic ideal generated by $\mathcal{D}^{\sigma_j} G_j$, $|\sigma_j| \leq \mathrm{ord}(F) - \mathrm{ord}(G_j)$ to get a minimal possible element w.r.t. the order. During this process only differential polynomials with leading monomials smaller than or equal to $\mathrm{HM}(F)$ can be used. We thus obtain the normal form $\mathrm{NF}(F; \mathbf{G})$.

However, since in differential polynomial algebra we cannot divide, one is allowed to multiply F by some (not identically zero) differential polynomials in order to obtain pseudonormal form $\mathrm{PN}(F; \mathbf{G})$, so that we get for some differential polynomial V with $\mathrm{HT}(V) < \mathrm{HT}(F)$ (usually such

V are included into the set of inequalities/singularities of the system):

$$F \dashrightarrow V \cdot F \stackrel{\text{def}}{=} \mathrm{PN}(F; \mathbf{G}).$$

Note that the pseudonormal form, contrary to the normal form, is not unique due to non-uniqueness of the factor V choice [R, K].

This process is called a pseudo-reduction and it is denoted by

$$F \underset{p}{\overset{\mathbf{G}}{\rightsquigarrow}} \mathrm{PN}(F; \mathbf{G}).$$

Note that $A \underset{p}{\overset{\mathbf{G}}{\rightsquigarrow}} B$ implies $A \overset{\mathbf{G}}{\rightsquigarrow} B$, but not otherwise.

Similar to the Buchberger's S-polynomial one introduces the differential S-polynomial $\mathrm{dS}(F_1, F_2)$ as follows. Let α_i be the minimal among multi-indices that satisfy the equality: $\mathcal{D}^{\alpha_1}\,\mathrm{HT}(F_1) = \mathcal{D}^{\alpha_2}\,\mathrm{HT}(F_2)$. Then $k_i = \mathrm{HD}(\mathcal{D}^{\alpha_i} F_i)$ equals $\mathrm{HD}(F_i)$ if $\alpha_i = 0$ and 1 when $|\alpha_i| > 0$ (prolongation of any equation is quasi-linear). Denote $m_i = \mathrm{LCM}\{k_1, k_2\}/k_i$.

Let $W_i = \mathrm{HC}((\mathcal{D}^{\alpha_i} F_i)^{m_i})$ and $Z_i = W_i / \mathrm{GCD}\{W_1, W_2\}$. Then we define:

$$\mathrm{dS}(F_1, F_2) = Z_2 \cdot (\mathcal{D}^{\alpha_1} F_1)^{m_1} - Z_1 \cdot (\mathcal{D}^{\alpha_2} F_2)^{m_2}.$$

If I is a differential ideal, then subset of its elements $\{G_j\}$ is called a *differential Gröbner basis*, if for every element $F \in I$ the pseudonormal form is uniquely defined and equals zero: $F \underset{p}{\overset{\mathbf{G}}{\rightsquigarrow}} 0$.

The Buchberger algorithm can be combined with Kolchin-Ritt algorithm to produce effectively a dGB ([M_1, H]). In brief it is done as follows. One chooses a set of generators $\mathbf{G} = \{G_j\}$ of the differential ideal I (in the case of PDE systems it is usually presented as such). Then if for a pair of functions G_i, G_j their dS-polynomial does not $\mathbf{G}$ pseudo-reduce to zero, this $\mathrm{PN}(dS(G_i, G_j), \mathbf{G})$ is added to the basis. The resulted set in many cases is a dGB.

But we would like to minimize it by removing elements, that pseudo-reduce to zero via the rest and then removing differential monomials, which can be dS-reduced by the leading terms. So we will understand by a dGB such a basis, which is also minimal and reduced.

Given a dGB one can answer many questions about the system. For instance, the system is solvable iff the dGB contains no non-zero polynomial in the variables x^i only. It is also possible to determine formal integrability (see [M_2] for a sufficient condition) and we are going to investigate compatibility.

2. Comparison results

2.1. **Compatibility via dGB.** We shall start with Theorem 2. Since no exact reference is known to the author, a simple proof will be provided, which almost directly follows from the definitions. We actually prove more: Namely the system is compatible to the level k iff for a Gröbner basis $\mathbf{G}$ we have: $G_j \overset{\mathbf{F}}{\rightsquigarrow} 0$ whenever $\mathrm{ord}(G_j) \le k$. The formal integrability is obtained from this for $k = \infty$.

Actually, let $I = I(\mathbf{F})$ be the differential ideal of the system $\mathcal{E} = \{F_i = 0\}$. It is filtered by the order of operators: $I_k = I \cap \mathrm{diff}_k(M)$.

Compatibility to the level k can be reformulated as

$$\langle \mathbf{F} \rangle_l = \langle\!\langle \mathbf{F} \rangle\!\rangle \cap \mathrm{diff}_l(M) \quad \text{for} \quad l \le k$$

or via the ideal I as the claim: $H \overset{\mathbf{F}}{\rightsquigarrow} 0 \; \forall H \in I_k$. Since elements of the Gröbner basis G_i are in the ideal, the necessity follows. But any other element of I pseudo-reduces to 0 by $\mathbf{G}$ and so it reduces to zero via $\mathbf{F}$:

$$H \underset{p}{\overset{\mathbf{G}}{\rightsquigarrow}} 0 \; \forall H \in I_k \quad \& \quad G_i \overset{\mathbf{F}}{\rightsquigarrow} 0 \; \forall i \quad \Longrightarrow \quad H \overset{\mathbf{F}}{\rightsquigarrow} 0 \; \forall H \in I_k,$$

which constitutes the sufficiency.

Remark 1. *If the system is linear, then the reduction $G_j \overset{\mathbf{F}}{\rightsquigarrow} 0$ of Theorem 2 can be relaxed to the requirement $G_j \underset{p}{\overset{\mathbf{F}}{\rightsquigarrow}} 0$. Equivalently one can describe generators in the module of compatibility relations via dS-polynomials, see Theorem 2 of* [M$_2$] *(proved in* [M$_1$]*). This set determines the Janet resolution for the ideal I of $\mathcal{E}$ (compatibility of compatibility etc), which always terminates* [J].

We would like now to describe why Theorem 2 is similar to Theorem 1 and then track the differences. When we calculate a dGB $\{G_j\}$ from the set of generators $\{F_i\}$ of I we calculate the pair-wise differential S-polynomials $dS(F_a, F_b)$. So the first condition is that $dS(F_a, F_b) \overset{\mathbf{F}}{\rightsquigarrow} 0$ and this is similar to the Jacobi-Mayer bracket vanishing condition.

In fact, these two conditions coincide iff the system F is differentially triangular (for definition see [H] or §3.1) or is equivalent to it via a (linear) transformation of dependent and independent variables. If addition of (pseudo-reduced) dS-polynomials to the generators does not yield a dGB, we need to proceed with dS-polynomials and pseudo-reduction and this usually goes for many times, which shows that the dGB-compatibility algorithm is less effective than the Jacobi-Mayer bracket approach.

2.2. **Proof of Theorem 3.** We consider at first systems of linear equations, in which case each equation $\nabla(u) = 0$ will be identified with the differential operator $\nabla : C^\infty(M) \to \mathbb{R}$ (there is non-uniqueness, but this will have no effect on our purpose). Let I be the differential ideal of the system $\mathcal{E}$ in the algebra $\mathrm{Diff}(M)$ of scalar linear differential operators on M equipped with the operation of composition.

Let $\mathbb{M} = \{p_\sigma\}$ be the commutative monoid of all differential monomials (with operation $p_\sigma * p_\tau = p_{\sigma+\tau}$) and $\mathrm{In} : I \to \mathbb{M}$ be the (initial) homomorphism $f \mapsto \mathrm{HT}(f)$. There is a similar homomorphism $\ln : I \to \mathbb{Z}^n_{\geq 0}$ obtained by post-composition $\mathbb{M} \to \mathbb{Z}^n_{\geq 0}$, $p_\sigma \mapsto \sigma$, but we'll be concerned with the first one.

Define $\mathrm{In}(I) = \{\mathrm{In}(f) \,|\, f \in I\} \subset \mathbb{M}$. By assumption in (1) of Theorem 3 this ideal is also a complete intersection. In addition, a collection $\mathbf{G}$ is a dGB for I iff $\mathrm{In}(\mathbf{G})$ is a basis of $\mathrm{In}(I)$.

It is known [V] for polynomial ideals I that if $\mathrm{In}(\mathbf{G})$ is a regular sequence, then $\mathbf{G}$ is a Gröbner basis for the algebraic ideal generated by $\mathbf{G}$. If $\mathcal{E}$ is a linear system with constant coefficients, then the same holds for dGB. But if the coefficients are functions, the compatibility conditions exist.

In case (1) they are brackets as well as reduced dS-polynomials. To see this denote the leaders of $\mathrm{In}(I)$ by p_{σ_k}. By assumption they have no common derivative, so that the minimal α_s satisfying $\mathcal{D}_i^{\alpha_i} p_{\sigma_i} = \mathcal{D}_j^{\alpha_j} p_{\sigma_j}$, $i \neq j$, are $\alpha_i = \sigma_j, \alpha_j = \sigma_i$ (in the case of finite type we must have $\sigma_1 = (k_1, 0, \dots, 0), \sigma_2 = (0, k_2, 0, \dots, 0), \dots, \sigma_n = (0, \dots, 0, k_n)$).

We have: $F_k = \sum_{\sigma \leq \sigma_k} a_{k\sigma}(x) p_\sigma$ and so

$$dS(F_i, F_j) = a_{j\sigma_j} \mathcal{D}^{\sigma_j} F_i - a_{i\sigma_i} \mathcal{D}^{\sigma_i} F_j = \sum_{\sigma<\sigma_i} a_{j\sigma_j} a_{i\sigma} p_{\sigma+\sigma_j} - \sum_{\sigma<\sigma_j} a_{i\sigma_i} a_{j\sigma} p_{\sigma+\sigma_i} + \dots$$

Here we omit the terms, involving derivatives of α_σ, of total degree less than those that are shown. Thus further reductions by F_i with $\mathrm{HT}(F_i) = p_{\sigma_i}$ and by F_j with $\mathrm{HT}(F_j) = p_{\sigma_j}$ are possible, we can reduce all terms of order $|\sigma_i| + |\sigma_j|$ and the result of (pseudo-)reduction is

$$dS(F_i, F_j) \overset{F_i, F_j}{\rightsquigarrow_p} [F_i, F_j].$$

So these brackets are to be added to the dGB, but maybe this does not suffice, we need to calculate more dS-polynomials etc. However, for compatibility purposes this is enough. Actually, the compatible case is characterized by vanishing of the brackets due to the system $\mathbf{F}$ and so by vanishing of all further dS-polynomials.

If the system is linear, but its In-image is not a complete intersection, then the calculations with dS-polynomials involve differentiations of smaller degrees. So before arriving to the Jacobi-Mayer brackets we calculate some more intermediate dS-polynomials, whence claim (2).

In general non-linear situation (3) we use the space of non-linear differential operators $\mathrm{diff}(M)$ instead of the algebra $\mathrm{Diff}(M)$ of linear operators and it is obvious that the cardinality of dGB grows much higher.

2.3. **Examples.** In all examples below we impose the total degree order with $x > y$.

1) Let $F_1 = u_{xx} + u_{xy} - \lambda(x, y, u, u_x, u_y)$ and $F_2 = u_{yy} - \mu(x, y, u, u_x, u_y)$. This system is in the triangular form. The higher terms are u_{xx} and u_{yy} respectively. Let's construct the differential S polynomial:

$$\begin{aligned} dS(F_1, F_2) &= \mathcal{D}_x^2 F_2 - \mathcal{D}_y^2 F_1 \\ &= \mathcal{D}_y^2 \lambda - \mathcal{D}_x^2 \mu - u_{xyyy} \overset{F_2}{\underset{p}{\rightsquigarrow}} \mathcal{D}_y^2 \lambda - \mathcal{D}_x^2 \mu - \mathcal{D}_x \mathcal{D}_y \mu = [F_1, F_2]. \end{aligned}$$

So we see the equivalence.

2) Let us consider the system with constant coefficients (which is always compatible): $F_1 = a_{11} u_{xx} + a_{12} u_{yy}$ and $F_2 = a_{21} u_{xxx} + a_{22} u_{yyy}$. This linear system cannot be brought to a triangular form unless some of the coefficients a_{ij} vanish. The differential Buchberger algorithm works as follows:

$$F_3 = dS(F_1, F_2) = a_{21} \mathcal{D}_x F_1 - a_{11} F_2 = a_{12} a_{21} u_{xyy} - a_{11} a_{22} u_{yyy},$$

$$F_4 = dS(F_1, F_3) = a_{12} a_{21} \mathcal{D}_y^2 F_1 - a_{11} \mathcal{D}_x F_3 \overset{F_3}{\underset{p}{\rightsquigarrow}} \Delta \cdot u_{yyyy} \overset{\Delta^{-1}}{\longmapsto} u_{yyyy},$$

if $\Delta = a_{11}^3 a_{22}^2 + a_{12}^3 a_{21}^2 \neq 0$. All the other differential S-polynomials reduce to zero. So $\{F_i\}_{i=1}^4$ is a dGB, which can be reduced to the dGB (F_1, F_3, F_4), when $\Delta \neq 0$, and to the dGB (F_1, F_3), when $\Delta = 0$.

In any case the number of calculated dS-polynomials is bigger than one Mayer bracket $[F_1, F_2] = 0$ for complete intersections, i.e. $\Delta \neq 0$, or another simple obstruction (see [KL_2]) if the system has characteristic covectors, i.e. $\Delta = 0$.

3) If in the above example $a_{ij} = a_{ij}(x, y)$ are polynomials, then the number of calculated dS-polynomials (as well as the elements in a dGB) grows. This is just because in addition to the above differential polynomials we should add at least the compatibility condition. But we still have only one compatibility condition in terms of brackets: $[F_1, F_2] = 0$ (when $\Delta \neq 0$).

In fact, the beginning of the differential Buchberger method is the same in the variable coefficients situation. We have non-zero $F_3 = dS(F_1, F_2)$

and $F_4 = dS(F_1, F_3)$ given by the same formulae. The next differential polynomials $dS(F_2, F_3)$, $dS(F_1, F_4)$, $dS(F_2, F_4)$ pseudo-reduce to zero modulo $\{F_i\}_{i=1}^4$. But $dS(F_3, F_4)$ is non-zero and it pseudo-reduces to the compatibility condition.

For compatibility we can actually stop here, but the dGB is not yet constructed and we continue. In the most generic case (when compatibility does not hold) we pseudo-reduce to the dGB $\{u_{xx}, u_{yy}\}$, but this requires more intermediate dS-polynomials.

4) Consider now a non-linear situation, where the difference between the two methods becomes more perceptible.

Let us study the question, when the associativity equation (WDVV) $F_1 = u_{yyy} + u_{xxx}u_{xyy} - u_{xxy}^2$ has an auxiliary integral of the form $F_2 = u_{xx}u_{yy} - cu_{xy}^2$ ([KL$_2$]). In other words, when the PDEs $F_1 = 0$ and $F_2 = 0$ are compatible. This is important for establishing exact solutions of WDVV.

The Jacobi-Mayer bracket approach works as follows ($\mathbf{F} = \{F_1, F_2\}$):

$$\frac{u_{xx}^7}{4}\,[F_1, F_2] \overset{\mathbf{F}}{\rightsquigarrow} (\tfrac{3}{2} - c)\cdot\bigl(2T^2 + 4cu_{xy}\sqrt{RS}\cdot T - u_{xx}^4u_{yy}^2u_{xxy}^2\bigr),$$

where

$$R = u_{yy} + u_{xx}u_{xxy}, \quad S = Rc^2u_{xy}^2 - u_{xx}^2u_{yy}u_{xxy}, \quad T = S + Rc^2u_{xy}^2.$$

Thus $c = 3/2$ is the only compatible case in the family. The above calculation is very quick. But with the differential algebra approach computer calculation requires much longer time because the number of calculated dS-polynomials is very big (the same concerns the other symbolic differential algebra programs, not only dGB).

3. Conclusion

3.1. **Coherence and d-triangularity.** Usually a differential system is given by its generating set in the form of PDEs $F_1 = 0, \dots, F_r = 0$, but not as a differential ideal I. Then we investigate compatibility conditions, i.e. check if there are essentially new equations in I.

The Kolchin-Ritt algorithm [K, R] decomposes I into an intersection of special differential triangular systems. A finite subset $\mathbf{F}$ of I is called a differential (d-)triangular set if (in the total degree ordering) for each $F \in \mathbf{F}$ we have $\deg(\mathrm{HT}(F)) > 0$, for a pair $F_i, F_j \in \mathbf{F}$ the leading term $\mathrm{HT}(F_i)$ is not a derivative of $\mathrm{HT}(F_j)$ and moreover no proper derivative of $\mathrm{HT}(F_j)$ appears in F_i.

Remark 2. *Working with triangular sets is important because this allows to reduce the problem to integration of ODEs. In the differential algebra context triangulation is arranged over the independent variables* $x^1, \dots, x^n$*. Another important issue is integration via Riemann invariants, which exist when the system possesses a solvable symmetry group (*[KLV]*). The same idea is basic here too: The system is transformed into a triangular form, but over the dependent variables* $u^1, \dots, u^m$*. It would be interesting to combine the two approaches.*

A d-triangular set $\mathbf{F}$ is called coherent ([K]) if for any pair $F_i, F_j \in \mathbf{F}$ their dS-polynomial satisfies $dS(F_i, F_j) \overset{\mathbf{F}}{\underset{p}{\rightsquigarrow}} 0$ plus a condition that we can cancel the leading coefficients (see [H, M$_1$] for details). We would like to notice that this condition is similar to Jacobi-Mayer bracket vanishing condition.

Also a similar idea occurs in the dGB context, though a coherent basis needs not to be a dGB [M$_1$]. However for orthonomic systems (Janet-Riquier), when in all equations the highest derivatives (possibly with polynomial in x coefficients) are expressed via the rest (linear systems are particular cases), the differential analog of the Buchberger criterion holds: If the system $\mathbf{F}$ generates I and is coherent, then it is a dGB (Theorem 1 [M$_2$]). Note that this is the differential algebra counterpart of our Theorem 1.

We would like also to mention that if a coherent set $\mathbf{F}$ is autoreduced ([K, R]: d-triangular and every element of $\mathbf{F}$ is reduced with respect to all the others), then Rosenfeld lemma relates differential algebra to polynomial algebra [H]. This implies that a dGB of a differential system $\mathcal{E}$ is contained in a (usual algebraic) Gröbner basis of some prolongation $\mathcal{E}^{(s)}$.

This is an analog of the celebrated fact from formal theory that after some s prolongations the system becomes involutive (if not empty), though in the dGB context s can be bigger. Actually, the cousin concept to involutivity is Riquier's passivity on which Ritt based his triangulation-decomposition algorithm.

3.2. **Further discussion of the two approaches to PDE systems.** While the dGB approach seems to be more universal for algebraic differential systems and is designed to deal with more general non-orthonomic systems (with results modulo singular integrals as usual), it has several disadvantages.

The first is the mere fact that for certain differential ideals such a basis is infinite. This causes troubles with computer implementation, cf. [F].

The problem is overcome in newer versions of the dGB algorithms ([M_2]) and modifications of Ritt-Kolchin triangulation-decomposition (Rosenfeld-Gröbner algorithm and others, see [H, RWB]). This however does not solve the complexity problem and our comparison result holds (we did the comparison only for systems of scalar PDEs; in the general case we should use multi-brackets instead of Jacobi-Mayer brackets [KL_4], but this just complicates the exposition, adding nothing essential to the idea).

The second is the aforementioned poor complexity and consequently the benefit of other methods, like bracket approach for Cohen-Macaulay systems. For other types systems the criterion of Theorem 1 does not work (the pair-wise brackets do not form a basis of syzygies), but there may be proposed other effective criteria (e.g. [KL_1]§3.2 for the case of two independent variables).

Finally, there is a problem of an efficient choice of the term order for the Kolchin-Ritt algorithm: There are algorithms, which allow to optimize order during the calculation in algebraic ([BW]), but not in differential-algebraic case. This problem is absent with the approach of Theorem 1, where the brackets are defined invariantly, but the coordinate calculations are straightforward.

3.3. **Formal integrability via dGB.** It was mentioned in [M_2] that Cartan-Kuranishi and Spencer approaches require orthonomic form of equations, i.e. that $\mathcal{E} \subset J^kM$ is foliated over $J^{k-1}M$ (projection is a submersion). While this is basically so – the corresponding requirement is usually a kind of regularity (though in geometric theory [KLV] prolongations can be defined in a more general setting), there is an important modification [KL_2], which adapts the general setup to work with the systems having differential equations of different orders.

In Theorem 3 of [M_2], stating formal integrability of a dGB, it is assumed that a given system $F_1, \dots, F_r$ of one order k is already a dGB for its differential ideal, while this is a very rare situation: starting with a given system and calculating its dGB according to the algorithm and a chosen order one usually gets equations of various orders. The difficulty can be overcome by considering orders subsequently, prolonging and adding new equations ([KL_2]), so that most results of [M_2] remain true in the more general context. For instance, Theorem 3 (loc.cit.) generalizes to the following statement:

Theorem 4. *Let $\mathcal{E} = \{F_1, \dots, F_r\}$ be a system of polynomial differential equations (not necessarily of the same order) and let $G_1, \dots, G_l$ be its dGB. Denote $\mathcal{E}'_k = \{\mathcal{D}_\sigma G_j = 0 : |\sigma| + \operatorname{ord} G_j \le k\}$. Then the differential* *system $\mathcal{E}'$ is formally integrable, meaning ${\mathcal{E}'_{k-1}}^{(1)} \supset \mathcal{E}'_k$ and $\pi_{k,k-1} : \mathcal{E}'_k \to \mathcal{E}'_{k-1}$ is a surjection for each k (away from the singularity set).*

Moreover the system $\mathcal{E}'$ is the result of prolongation-projection scheme applied to the system $\mathcal{E}$.

Still having some regularity restrictions (to set up the prolongation machinery) we note that removing non-regular points becomes equivalent to removing singularity sets, which always appear in the dGB approach. Thus orthonomic requirement in a weak form is not so restrictive. Also note that while in polynomial context a resolution of singularities is usually helpful, the jet approach allows to treat singularities geometrically and study multi-valued solutions.

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