CIRCULAR DISTANCE IN DIRECTED GRAPHS

Bohdan Zelinka, Liberec

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Abstract. Circular distance $d^\circ(x,y)$ between two vertices x, y of a strongly connected directed graph G is the sum $d(x,y)+d(y,x)$, where d is the usual distance in digraphs. Its basic properties are studied.

Keywords: strongly connected digraph, circular distance, directed cactus

MSC 1991: 05C38, 05C20

In an undirected graph the distance between two vertices is usually defined as the length of the shortest path connecting these vertices. This distance is a metric on the vertex set of the graph. Analogously in a directed graph (usually the strong connectedness is supposed) the distance $d(x,y)$ from a vertex x to a vertex y is defined as the length of the shortest directed path from x to y. In general, $d(x,y)$ thus defined is not a metric, because it is not symmetric. In this paper we define a certain distance in a digraph which is a metric.

Let G be a strongly connected directed graph, let x, y be two vertices of G. The *circular distance* $d^\circ(x,y)$ between the vertices x, y in the graph G is defined as

$$d^\circ(x,y) = d(x,y) + d(y,x),$$

where d denotes the usual distance in digraphs (see above). In other words, $d^\circ(x,y)$ is the length of the shortest directed walk going from x to y and then back to x.

Note that in the walk mentioned, vertices and edges may repeat. In the graph in Fig. 1 such shortest walk for x and y contains all edges of the graph and the edge e occurs twice in it.

The following proposition is evident.

Proposition 1. *The circular distance $d^\circ(x,y)$ is a metric on the vertex set $V(G)$ of the graph G.*

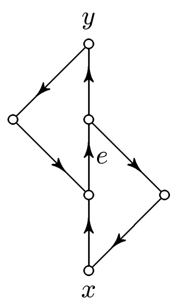

Fig. 1

The properties of the circular distance are considerably different from the properties of the usual distance in graphs.

The length of the shortest cycle (directed circuit) in the graph G will be called the *directed girth* of G and denoted by $g(G)$.

Proposition 2. *Let x, y be two distinct vertices of a strongly connected graph G, let $g(G)$ be the directed girth of G. Then*

$$d^{\circ}(x, y) \geqslant g(G).$$

Proof. Let P_1 (or P_2) be the shortest path from x to y (or from y to x, respectively). The circular distance $d^{\circ}(x, y)$ is equal to the sum of lengths of P_1 and P_2. The union of P_1 and P_2 must contain a cycle; the length of this cycle is greater than or equal to $g(G)$ and less than or equal to the sum of lengths of P_1 and P_2; this implies the assertion. □

Analogously as for the usual distance, we may introduce the circular radius $\varrho^{\circ}(G)$ and the circular diameter $\delta^{\circ}(G)$. For each vertex x of G we define the *circular elongation* $e^{\circ}(x)$ as the maximum of $d^{\circ}(x, y)$ for all $y \in V(G)$. Then the minimum of $e^{\circ}(x)$ for all $x \in V(G)$ is the *circular radius* $\varrho^{\circ}(G)$ of G. The set of vertices x for which $e^{\circ}(x) = \varrho^{\circ}(G)$ is called the *circular center* $C^{\circ}(G)$ of G. The maximum of $d^{\circ}(x, y)$ over all pairs x, y of vertices of G is the *circular diameter* $\delta^{\circ}(G)$ of G.

In the case of infinite graphs it may happen that the maximum of $d^{\circ}(x, y)$ does not exist. Then we put $\delta^{\circ}(G) = \infty$ and also $\varrho^{\circ}(G) = \infty$. In the sequel we shall consider only finite radii and diameters.

The following proposition can be proved in the same way as the analogous statement for the usual distance in graphs; it follows from the triangle inequality.

Proposition 3. *For the circular radius $\varrho^\circ(G)$ and the circular diameter $\delta^\circ(G)$ of a strongly connected directed graph G the following inequality holds:*

$$\varrho^\circ(G) \leqslant \delta^\circ(G) \leqslant 2\varrho^\circ(G).$$

Now we have a theorem.

Theorem 1. *Let r, d be positive integers, $2 \leqslant r \leqslant d \leqslant 2r$. Then there exists a strongly connected directed graph G such that $\varrho^\circ(G) = r$, $\delta^\circ(G) = d$.*

Proof. If $r = d$, then G is the cycle of length r. In it $d^\circ(x, y) = r$ for any two distinct vertices x, y.

If $d = r + 1$, distinguish the cases $r = 2$ and $r \geqslant 3$. If $r = 2$, then let $V(G) = \{u, v_1, v_2\}$ and let the edges of G be uv_1, v_1u, uv_2, v_2u, v_1v_2 (Fig. 2). We have $d^\circ(u, v_1) = d^\circ(u, v_2) = 2$, $d^\circ(v_1, v_2) = 3$, $e^\circ(u) = 2$, $e^\circ(v_1) = e^\circ(v_2) = 3$ and thus $\varrho^\circ(G) = 2$, $\delta^\circ(G) = 3$. If $r \geqslant 3$, then let $V(G) = \{v_0, v_1, \ldots, v_{r-1}, w\}$. Let the edges be v_iv_{i+1} for $i = 0, \ldots, r-2$, $v_{r-1}v_0$, v_0w and wv_i for $i = 1, \ldots, r-1$. (Fig. 3 for $r = 8$.) We have $d^\circ(v_1, w) = r + 1 = d$, $d^\circ(v_1, v_0) = r$, $d^\circ(v_1, v_i) = r$ for $i = 2, \ldots, r-1$. Further we have $d^\circ(v_0, w) = 3 \leqslant r$, $d^\circ(v_i, w) = r - i + 2 \leqslant r$ for $i = 2, \ldots, r-1$. Finally, $d^\circ(v_i, v_j) \leqslant r$ for any i and j, because $v_0, \ldots, v_{r-1}$ form a cycle of length r. We have $e^\circ(v_1) = e^\circ(w) = d$, $e^\circ(v_0) = e^\circ(v_i) = r$ for $i = 2, \ldots, r-1$. Hence $\delta^\circ(G) = d$, $\varrho^\circ(G) = r$.

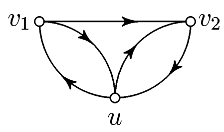

Fig. 2

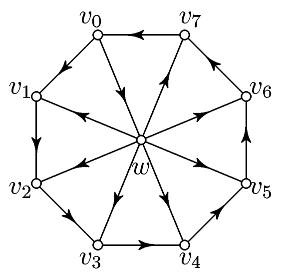

Fig. 3

If $d \geqslant r+2$, let the graph G consist of two cycles C_1, C_2 with exactly one common vertex a; let the length of C_1 be r and let the length of C_2 be $d-r$. Let u_1 (or u_2) be an arbitrary vertex of C_1 (or C_2, respectively) different from a. Then $d^\circ(a, u_1) = r$, $d^\circ(a, u_2) = d - r \leqslant r$, $d^\circ(u_1, u_2) = d$. This implies $e^\circ(a) = r$, $e^\circ(u_1) = e^\circ(u_2) = d$ and again $\delta^\circ(G) = d$, $\varrho^\circ(G) = r$. □

If to the graph G for the case $d = r + 1$, $r \geqslant 3$ we add the edge wv_0 (Fig. 4), we obtain a graph G' such that the circular center $C^\circ(G') = \{v_0, v_1, \ldots, v_{r-1}\}$, while the center $C(G')$ for the usual distance $d(x, y)$ is $\{w\}$ and thus $C^\circ(G') \cap C(G') = \emptyset$. We have a proposition.

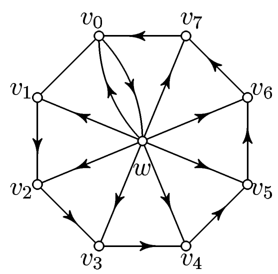

Fig. 4

Proposition 4. *The circular center $C^\circ(G)$ and the usual center $C(G)$ of a digraph G may be disjoint.*

Note that always $d^\circ(x, y) \neq 1$; this follows from the definition. Evidently also $\varrho^\circ(G) \neq 1$ and $\delta^\circ(G) \neq 1$.

Theorem 2. *Let (M, m) be a metric space such that the set M is finite and the metric m attains only integral values. Then there exists a strongly connected directed graph G such that $M \subseteq V(G)$ and $d^\circ(x, y) = m(x, y) + 1$ for any two distinct vertices x, y of M. Moreover, all vertices of $V(G) - M$ have indegree 1 and outdegree 1.*

P r o o f. Choose an arbitrary total ordering $<$ on M. For any two vertices x, y of M such that $x < y$ we form the edge xy; in this way we obtain a tournament with the vertex set M. Further, for any x and y of M such that $x < y$ we add a directed path $P(x, y)$ of length $m(x, y)$ from y to x. The inner vertices of any path $P(x, y)$ are not in M and any two such paths have no inner vertex in common. The graph thus obtained is G. We see that all vertices of $V(G) - M$ have indegree 1 and outdegree 1. Consider two vertices x, y of M such that $x < y$ and let d denote the usual distance in a digraph. Then evidently $d(x, y) = 1$. The path $P(x, y)$ is the shortest path from y to x, because any other path from y to x must contain at least one vertex $z \in M$; then its length is at least $m(y, z) + m(z, x)$ and by the triangle inequality this is greater than or equal to $m(y, x)$. Therefore $d(y, x) = m(x, y)$ and $d^\circ(x, y) = m(x, y) + 1$. □

A certain analogue of trees are directed cacti. A directed cactus is a graph in which each block is a cycle [1].

The following proposition is easy to prove.

Proposition 5. *Let x, y be two distinct vertices of a directed cactus G. Then there exists exactly one directed path $P(x,y)$ from x to y in G.*

Now we prove a theorem.

Theorem 3. *If x, y are two distinct vertices of a directed cactus G, then $d^{\circ}(x,y)$ is equal to the sum of lengths of all cycles in G which have common edges with the path $P(x,y)$.*

Proof. We will proceed by induction according to the number k of blocks which contain edges of $P(x,y)$. If $k=1$, then x and y are in the same block (cycle) B and this block is the (edge-disjoint) union of $P(x,y)$ and $P(y,x)$, therefore $d^{\circ}(x,y)$ is equal to the length of the cycle B. Now let $k \geqslant 2$ and suppose that for $k-1$ the assertion is true. Let the first edge of $P(x,y)$ be in the block B_1 and let a be the terminal vertex of the last edge of $P(x,y)$ being in B_1. Then a is an articulation between B_1 and another block B_2 which contains the edge of $P(x,y)$ outgoing from a. The path $P(a,y)$ is part of $P(x,y)$ and there are $k-1$ blocks containing edges of $P(a,y)$, namely all those containing edges of $P(x,y)$ except B_1. By the induction hypothesis $d^{\circ}(a,y)$ is the sum of lengths of these blocks. Not only $P(x,y)$, but also $P(y,x)$ goes through a and therefore $d^{\circ}(x,y) = d^{\circ}(x,a) + d^{\circ}(a,y)$, which is the sum of lengths of all cycles which contain edges of $P(x,y)$. □

Now we prove a theorem which concerns circular centers of directed cacti.

Theorem 4. *The circular center of a finite directed cactus G either consists of one vertex, or is equal to the vertex set of one block of G.*

Proof. Let $\varrho^{\circ}(G) = r$. First suppose that the circular center $C^{\circ}(G)$ contains two vertices u_1, u_2 which are not contained in the same block. Then there exists an articulation a of G which separates (in the same sense as in an undirected graph) the vertices u_1, u_2. By V_1 (or V_2) we denote the set of vertices of G which are separated by a from u_2 and not from u_1 (or from u_1 and not from u_2, respectively). By V_3 we denote the set of vertices of G which are separated by a from both u_1, u_2. Suppose that there exists a vertex v such that $d^{\circ}(a,v) \geqslant r$. If $v \in V_1 \cup V_3$, then

$$d^{\circ}(u_2,v) = d^{\circ}(u_2,a) + d^{\circ}(a,v) \geqslant d^{\circ}(u_2,a) + r > r;$$

we have a contradiction with the assumption that r is the circular radius and $u_2 \in C^{\circ}(G)$. If $v \in V_2 \cup V_3$, then

$$d^{\circ}(u_1,v) = d^{\circ}(u_1,a) + d^{\circ}(a,v) \geqslant d^{\circ}(u_1,a) + r > r;$$

again we have a contradiction. Evidently $V(G) = V_1 \cup V_2 \cup V_3 \cup \{a\}$ and therefore $d^\circ(a,x) < r$ for all $x \in V(G)$. Then $\varrho^\circ(G) < r$, which is again a contradiction. We have proved that $C^\circ(G)$ must be a subset of the vertex set of a block of G. Let B be such a block; it is a cycle. Let its length be b. If $B = G$, then evidently each vertex of B belongs to the circular center and $C^\circ(G) = G = B$. If not, then $r > b$. For each $x \in V(B)$ let $W(x)$ be the set of all vertices of G which are separated by x from all other vertices of B. The sets $W(x)$ for all $x \in V(B)$ and the set $V(B)$ are pairwise disjoint and their union is $V(G)$. Let p be the number of vertices $x \in V(B)$ with the property that there exists a vertex $y \in W(x)$ such that $d^\circ(x,y) \geqslant r - b$. Suppose $p = 0$. Let $v \in C^\circ(G) \subseteq V(B)$, let $x \in V(G)$. If $x = v$, then $d^\circ(v,x) = 0 < r$. If $x \in V(B) - \{v\}$, then $d^\circ(v,x) = b < r$. If $x \in W(v)$, then $d^\circ(v,x) < r$ according to the assumption. If $x \in V(G) - \big(V(B) \cup W(v)\big)$, then there exists $y \in V(B) - \{v\}$ such that $x \in W(y)$. Then

$$d^\circ(v,x) = d^\circ(v,y) + d^\circ(y,x) = b + d^\circ(y,x) < b + r - b = r.$$

This is a contradiction with the assumption that $C^\circ(G) \subseteq V(B)$. Therefore $p \neq 0$. Suppose $p = 1$ and let w be a vertex of $V(B)$ such that there exists $y \in W(w)$ for which $d^\circ(w,y) \geqslant r - b$. We may assume that y is the vertex of $W(w)$ with the maximum circular distance from w. If $d^\circ(w,y) > r - b$, then each vertex of $V(B) - \{w\}$ has the circular distance from y equal to $b + d^\circ(w,y) > r$. As we have supposed $C^\circ(G) \subseteq V(B)$, we have $C^\circ(G) = \{w\}$. If $d^\circ(w,y) = r - b$, then the circular distance of each vertex of $W(w)$ from w is at most $r - b$ and the circular distance of any other vertex from w is less than r; we have a contradiction with the assumption that $\varrho^\circ(G) = r$. Finally, suppose $p \geqslant 2$. Let w_1, w_2 be two distinct vertices of $V(B)$ such that there exist vertices y_1, y_2 with $d^\circ(w_1,y_1) \geqslant r - b$, $d^\circ(w_2,y_2) \geqslant r - b$. If $d^\circ(u_1,y_1) > r - b$, then only w_1 can be in $C^\circ(G)$. The case $d^\circ(w_2,y_2) > r - b$ is analogous. Therefore $d^\circ(w_1,y_1) = d^\circ(w_2,y_2) = r - b$ and there exists no vertex in $W(w_1)$ with the circular distance from w_1 greater than $r - b$ and no vertex in $W(w_2)$ with the circular distance from w_2 greater than $r - b$. For each vertex $u \in V(B) - \{w_1\}$ we have $d^\circ(w_1,u) = r$ and for each vertex $u \in V(B) - \{w_2\}$ we have $d^\circ(w_2,u) = r$. In no set $W(x)$ for $x \in V(B)$ there is a vertex whose circular distance from x would be greater than $r - b$; this can be proved in the same way as for $x = w_1$. Therefore for each $v \in V(G)$ and $u \in V(B)$ we have $d^\circ(u,v) \leqslant r$ and $C^\circ(G) = V(B)$. □

In Fig. 5 we see a directed cactus in which the circular center is a one-element set; in Fig. 6 we see a directed cactus in which the circular center is the vertex set of a block. In both the figures the vertices of the circular center are black.

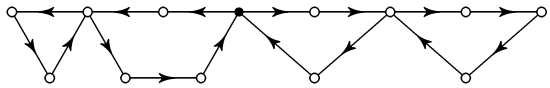

Fig. 5

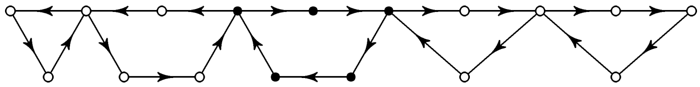

Fig. 6

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Author's address: *Bohdan Zelinka,* Technická univerzita Liberec, katedra diskrétní matematiky a statistiky, Voroněžská 13, 461 17 Liberec 1, Czech Republic.