

CHARACTERIZING THE INTERVAL FUNCTION
OF A CONNECTED GRAPH

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(Received December 3, 1996)

Abstract. As was shown in the book of Mulder [4], the interval function is an important tool for studying metric properties of connected graphs. An axiomatic characterization of the interval function of a connected graph was given by the present author in [5]. (Using the terminology of Bandelt, van de Vel and Verheul [1] and Bandelt and Chepoi [2], we may say that [5] gave a necessary and sufficient condition for a finite geometric interval space to be graphic).

In the present paper, the result given in [5] is extended. The proof is based on new ideas.

Keywords: graphs, distance, interval function

MSC 1991: 05C12

The letters h – n will be reserved for denoting non-negative integers. By a graph we will mean a finite undirected graph without multiple edges and loops (i.e. a graph in the sense of Chartrand and Lesniak [3], for example). If U is a nonempty set, then we denote by $\Omega(U)$ the set of all mappings of U into the set of all subsets of U .

Let G be a connected graph, and let $V(G)$, $E(G)$ and d_G denote its vertex set, its edge set, and its distance function, respectively. Following Mulder [4], we define the interval function I_G of G as follows:

$$I_G(x, z) = \{y \in V(G); y \text{ belongs to an } x\text{-}z \text{ path of length } d_G(x, z) \text{ in } G\}$$

for all $x, z \in V(G)$. Obviously, $I_G \in \Omega(V(G))$.

Proposition 1. *Let G be a connected graph, and let J denote the interval function of G . Put $U = V(G)$. Then J fulfils the following Axioms A–G (for arbitrary $u, v, x, y \in U$):*

A if $v \in J(u, x)$, then $J(v, x) \subseteq J(u, x)$;

- B if $v \in J(u, x)$ and $y \in J(v, x)$, then $v \in J(u, y)$;
- C $u \in J(u, x)$;
- D $|J(u, u)| = 1$;
- E $J(u, x) = J(x, u)$;
- F if $|J(u, v)| = 2 = |J(x, y)|$, $v \in J(u, x)$ and $u \in J(v, y)$, then $x \in J(v, y)$;
- G if $|J(u, v)| = 2 = |J(x, y)|$ and $v \in J(u, x)$, then either $x \in J(v, y)$ or $y \in J(u, x)$ or $v \in J(u, y)$.

The validity of Axioms A–E follows from 1.1.2 in [4]. The verification of Axiom G was given in [5].

Verification of Axiom F: Let the assumption in F hold. Then $d_G(v, y) \leq d_G(v, x) + 1 = d_G(u, x) \leq d_G(u, y) + 1 = d_G(v, y)$. Hence $x \in J(v, y)$.

As will be shown in our theorem, Axioms A–G can be used for characterizing the interval function of a connected graph. A similar result was originally proved in [5].

In the theorem of [5], instead of Axiom F the following Axiom F_0 was used (u, v, x, y are arbitrary elements in U):

- F_0 if $|J(u, v)| = 2 = |J(x, y)|$, $v \in J(u, x)$, $u \in J(v, y)$ and $y \in J(u, x)$, then $x \in J(v, y)$.

Because of the proof of our theorem we prefer Axiom F to Axiom F_0 .

Proposition 2. *Let U be a nonempty set, let $J \in \Omega(U)$, and let J fulfil Axioms A–E and G. Then it fulfils Axiom F if and only if it fulfils Axiom F_0 .*

Proof. Obviously, F implies F_0 . Conversely, let J fulfil Axiom F_0 . Consider arbitrary $u, v, x, y \in U$ such that $|J(u, v)| = 2 = |J(x, y)|$, $v \in J(u, x)$ and $u \in J(v, y)$. We will show that $x \in J(v, y)$. Suppose, to the contrary, that $x \notin J(v, y)$. Axiom F_0 implies that $y \notin J(u, x)$. By Axiom G, $v \in J(u, y)$. Since $u \in J(v, y)$, we conclude that $u = v$, which is a contradiction. Thus J fulfils Axiom F. \square

Proofs of the following two lemmas are not difficult and will be omitted. Note that the proof of Lemma 2 depends on the fact that U is finite.

Lemma 1. *Let U be a nonempty set, let $J \in \Omega(U)$, and let J fulfil Axioms A, B and E. Let $x_0, \dots, x_{n+m} \in U$, let*

$$(1) \quad x_{n+1} \in J(x_n, x_0), \dots, x_{n+m} \in J(x_{n+m-1}, x_0)$$

and

$$(2) \quad x_0 \in J(x_{n+1}, x_1), \dots, x_{k-2} \in J(x_{n+k-1}, x_{k-1}),$$

where $2 \leq k \leq m$ and $n \geq 1$. Then

$$(3) \quad \begin{aligned} & x_{n+i+1} \in J(x_{n+i}, x_i), \dots, x_{n+k} \in J(x_{n+k-1}, x_i) \text{ and} \\ & x_{i-1} \in J(x_{n+i+1}, x_i), \dots, x_{i-1} \in J(x_{n+k}, x_i) \text{ for each } i, 1 \leq i \leq k-1. \end{aligned}$$

Lemma 2. Let U be a nonempty finite set, let $J \in \Omega(U)$, and let J fulfil Axioms A–E. If $u, x \in U$ and $u \neq x$, then

$$J(u, x) - \{u\} = \bigcup_{\substack{v \in J(u, x) \\ |J(u, v)|=2}} J(v, x).$$

Remark 1. Let U be a finite nonempty set, let $J \in \Omega(U)$, let J fulfil Axioms A–E, let $u, x \in U$ and $u \neq x$. By Axiom C, $u \in J(u, x)$. Lemma 2 implies that there exists $w \in U$ such that $w \in J(u, x)$ and $|J(u, w)| = 2$. Consider an arbitrary $v \in J(u, x)$ such that $|J(u, v)| = 2$. Recall that U is finite. Lemma 2 implies that there exist $w_0, w_1, \dots, w_j \in U$, where $j \geq 1$, such that $w_0 = u, w_1 = v, w_j = x$,

$$|J(w_0, w_1)| = \dots = |J(w_{j-1}, w_j)| = 2$$

and

$$w_1 \in J(w_0, x), \dots, w_j \in J(w_{j-1}, x).$$

Let U be a finite nonempty set, and let $J \in \Omega(U)$. We will say that a graph G is the graph of J if $V(G) = U$ and t and z are adjacent in G if and only if $|J(t, z)| = 2$ for all distinct $t, z \in U$. Obviously, there exists exactly one graph of J . As follows from Remark 1, if J fulfils Axioms A–E, then the graph of J is connected.

It is clear that if G is a connected graph, then G is the graph of I_G .

The following theorem extends (and partially modifies) the result of [5]:

Theorem. Let U be a finite nonempty set, let $J \in \Omega(U)$, and let G denote the graph of J . Then the following three statements are equivalent:

- (I) G is connected and $J = I_G$;
- (II) J fulfils Axioms A–G (for arbitrary $u, v, x, y \in U$);
- (III) J fulfils Axioms A–F (for arbitrary $u, v, x, y \in U$) and $I_G(t, z) \subseteq J(t, z)$ for all $t, z \in U$.

Remark 2. Let U be a nonempty set, and let $J \in \Omega(U)$. It is not difficult to show that J is a geometric interval space in the sense of Bandelt, van de Vel and Verheul [1], Verheul [7] and Bandelt and Chepoi [2] if and only if J fulfils Axioms A–E. By our theorem, a finite geometric interval space is graphic in the sense of [1] and [2] if and only if it fulfils Axioms F and G.

In proving our theorem, we will need one more lemma. Let U be a finite nonempty set, and let $J_1, J_2 \in \Omega(U)$. Assume that J_1 and J_2 have the same graph. Let G denote the graph of J_1 and J_2 , and let $n \geq 0$. We will write

$$J_1 \subseteq_{(n)} J_2 \text{ (or } J_1 =_{(n)} J_2)$$

if and only if $J_1(r, s) \subseteq J_2(r, s)$ for all $r, s \in U$ such that $d_G(r, s) \leq n$ (or $J_1(r, s) = J_2(r, s)$ for all $r, s \in U$ such that $d_G(r, s) \leq n$, respectively).

Lemma 3. *Let U be a finite nonempty set, let $n \geq 0$, $J \in \Omega(U)$, and let G denote the graph of J . Assume that J fulfils Axioms A–F (for arbitrary $u, v, x, y \in U$) and that $I_G \subseteq_{(n)} J$. Then $I_G =_{(n)} J$.*

Proof of Lemma 3. Let D denote the diameter of G . Instead of d_G and I_G we write d and I , respectively. We proceed by induction on n . The case when $n \leq 1$ is obvious. Assume that $n \geq 2$. Since $I \subseteq_{(n)} J$, we have $I \subseteq_{(n-1)} J$. By the induction hypothesis, $I =_{(n-1)} J$. If $D \leq n - 1$, then $I =_{(n)} J$. Let $D \geq n$.

Consider arbitrary $r, s \in U$ such that $d(r, s) = n$. We want to prove that $J(r, s) \subseteq I(r, s)$. First, assume that $z \in I(r, s)$ for each $z \in J(r, s)$ such that $|J(r, z)| = 2$. By virtue of Lemma 2, $J(r, s) \subseteq I(r, s)$. Now, assume that there exists $t \in J(r, s)$ such that $|J(r, t)| = 2$ and $t \notin I(r, s)$.

There exist $x_0, \dots, x_n \in I(r, s)$ such that $x_0 = s, x_n = r$ and the sequence

$$(4) \quad x_n, x_{n-1}, \dots, x_0$$

is a path in G . By virtue of Remark 1, there exist $x_{n+1}, \dots, x_{n+m} \in U$, where $m \geq 1$, such that $x_{n+1} = t, x_{n+m} = x_0$,

$$(5) \quad |J(x_n, x_{n+1})| = \dots = |J(x_{n+m-1}, x_{n+m})| = 2$$

and (1) holds. Since the sequence

$$(6) \quad x_n, x_{n+1}, \dots, x_{n+m}$$

is a path in G , $m \geq n$. Since $x_{n+1} \notin I(x_n, x_0)$, we have $d(x_{n+1}, x_0) \geq n$. Hence $m \geq n + 1$.

We will show that

$$(7) \quad x_{m-1} \notin J(x_{n+m}, x_m).$$

Since $m > n$, (1) implies that $x_m \in J(x_{m-1}, x_{n+m})$. If $x_{m-1} \in J(x_m, x_{n+m})$, then Axioms B–D imply that $x_{m-1} = x_m$, which contradicts (5). Thus (7) holds.

By virtue of (7), there exists k , $1 \leq k \leq m$, such that

$$(8) \quad x_{k-1} \notin J(x_{n+k}, x_k)$$

and if $k \geq 2$, then (2) holds. Recall that $d(x_{n+1}, x_0) \geq n$. There exists h , $0 \leq h \leq k-1$, such that

$$(9) \quad d(x_{n+h+1}, x_h) \geq n$$

and

$$(10) \quad \text{if } h \leq k-2, \text{ then } d(x_{n+h+2}, x_{h+1}) \leq n-1.$$

By Lemma 1, if $k \geq 2$, then (3) holds. Combining this fact with (1), we get

$$(11) \quad x_{n+h+1} \in J(x_{n+h}, x_h).$$

Moreover, (3) implies that

$$(12) \quad \text{if } h \leq k-2, \text{ then } x_h \in J(x_{n+h+2}, x_{h+1}).$$

Clearly,

$$(13) \quad |J(x_h, x_{h+1})| = 2 = |J(x_{n+h}, x_{n+h+1})|.$$

Obviously, $d(x_{n+h+1}, x_{h+1}) \leq n$. It follows from (9) that $d(x_{n+h+1}, x_{h+1}) \geq n-1$. We distinguish two cases.

Case 1. Let $d(x_{n+h+1}, x_{h+1}) = n$. This implies that $x_{n+h} \in I(x_{n+h+1}, x_{h+1})$. Since $I \subseteq_{(n)} J$, we have

$$(14) \quad x_{n+h} \in J(x_{n+h+1}, x_{h+1}).$$

Combining (11), (13) and (14) with Axiom F, we get

$$(15) \quad x_h \in J(x_{n+h+1}, x_{h+1}).$$

It follows from (8) that $h \leq k-2$. According to (12), $x_h \in J(x_{n+h+2}, x_{h+1})$. By (10), $d(x_{n+h+2}, x_{h+1}) \leq n-1$. Since $I =_{(n-1)} J$, $x_h \in I(x_{n+h+2}, x_{h+1})$. Therefore, $d(x_{n+h+2}, x_h) \leq n-2$. This implies that $d(x_{n+h+1}, x_h) < n$, which contradicts (9).

Case 2. Let $d(x_{n+h+1}, x_{h+1}) = n - 1$. It follows from (9) that $d(x_{n+h+1}, x_h) = n$. Hence $x_{h+1} \in I(x_{n+h+1}, x_h)$. Since $I \subseteq_{(n)} J$, we get

$$(16) \quad x_{h+1} \in J(x_{n+h+1}, x_h).$$

Combining (11) and (16) with Axiom B, we get

$$(17) \quad x_{n+h+1} \in J(x_{n+h}, x_{h+1}).$$

Since $d(x_{n+h}, x_{h+1}) \leq n - 1$ and $I =_{(n-1)} J$, we see that $x_{n+h+1} \in I(x_{n+h}, x_{h+1})$. We have $d(x_{n+h+1}, x_{h+1}) \leq n - 2$. This means that $d(x_{n+h+1}, x_h) < n$, which contradicts (9) again.

Thus $J(r, s) \subseteq I(r, s)$. We conclude that $I =_{(n)} J$, which completes the proof of the lemma. \square

Proof of the theorem. Instead of d_G and I_G we write d and I , respectively. By Proposition 1, (I) implies (II).

Now, we will prove that (II) implies (III). Suppose, to the contrary, that (II) holds but (III) does not. Then there exists $n \geq 2$ such that $I \subseteq_{(n-1)} J$ but it is not true that $I \subseteq_{(n)} J$. Since J fulfils Axioms A–F, Lemma 3 implies that $I =_{(n-1)} J$. Clearly, there exist $r, s \in U$ such that $d(r, s) = n$ and $I(r, s) - J(r, s) \neq \emptyset$.

First, assume that $z \in J(r, s)$ for each $z \in I(r, s)$ such that $|I(r, z)| = 2$. Then we get $I(r, s) \subseteq J(r, s)$, which is a contradiction. Now, assume that there exists $t \in I(r, s)$ such that $|I(r, t)| = 2$ and $t \notin J(r, s)$. Obviously, there exist $x_0, \dots, x_{n-1}, x_n \in U$ such that $x_0 = s, x_{n-1} = t, x_n = r$ and (4) is a path in G . Thus $x_{n-1} \notin J(x_n, x_0)$.

By virtue of Remark 1, there exist $x_{n+1}, \dots, x_{n+m} \in U$, where $m \geq 1$, such that $x_{n+m} = x_0$, and (1) and (5) hold. Since (6) is a path in G , we have $m \geq n$. If $m = n$, then (7) holds. If $m > n$, then similarly as in the proof of Lemma 3 we get (7) again.

There exists $k, 1 \leq k \leq m$, such that (8) holds and if $k \geq 2$, then (2) holds. Recall that $x_{n-1} \notin J(x_n, x_0)$ and $d(x_n, x_0) = n$. This implies that there exists $h, 0 \leq h \leq k - 1$, such that

$$(18) \quad x_{n+h-1} \notin J(x_{n+h}, x_h) \text{ and } d(x_{n+h}, x_h) = n$$

and

$$(19) \quad \text{if } h \leq k - 2, \text{ then either } x_{n+h} \in J(x_{n+h+1}, x_{h+1}) \\ \text{or } d(x_{n+h+1}, x_{h+1}) \leq n - 1.$$

By Lemma 1, if $k \geq 2$, then (3) holds. Combining this fact with (1), we get (11). Moreover, it is easy to see that (13) holds.

By (18), $d(x_{n+h}, x_h) = n$. Thus $x_{n+h-1} \in I(x_{n+h}, x_{h+1})$. Since $I \subseteq_{(n-1)} J$, we get $x_{n+h-1} \in J(x_{n+h}, x_{h+1})$. If $x_{h+1} \in J(x_{n+h}, x_h)$, then, combining Axioms A and E, we get $x_{n+h-1} \in J(x_{n+h}, x_h)$, which contradicts (18). Thus

$$(20) \quad x_{h+1} \notin J(x_{n+h}, x_h).$$

Obviously, $d(x_{n+h+1}, x_{h+1}) \leq n$. We distinguish two cases.

Case 1. Let (15) hold. As follows from (8), $h \leq k - 2$.

First, assume that $d(x_{n+h+1}, x_{h+1}) = n$. By virtue of (19), (14) holds. Combining (11), (13) and (14) with Axioms E and F, we get $x_{h+1} \in J(x_{n+h}, x_h)$, which contradicts (20).

Now, assume that $d(x_{n+h+1}, x_{h+1}) \leq n - 1$. Combining (15) with the fact that $J \subseteq_{(n-1)} I$, we get $x_h \in J(x_{n+h+1}, x_{h+1})$. Therefore, $d(x_{n+h+1}, x_h) \leq n - 2$. This implies that $d(x_{n+h}, x_h) < n$, which contradicts (18).

Case 2. Let $x_h \notin J(x_{n+h+1}, x_{h+1})$. Combining this fact with (11), (13), (20) and Axiom G, we see that (17) holds. Since $d(x_{n+h}, x_{h+1}) = n - 1$, the fact that $J \subseteq_{(n-1)} I$ implies that $x_{n+h+1} \in I(x_{n+h}, x_{h+1})$. Thus $d(x_{n+h+1}, x_{h+1}) = n - 2$. This means that $d(x_{n+h+1}, x_h) \leq n - 1$. It follows from (18) that $d(x_{n+h+1}, x_h) = n - 1$. This implies that $x_{h+1} \in I(x_{n+h+1}, x_h)$. Since $I \subseteq_{(n-1)} J$, (16) holds. Combining (11) and (16) with Axiom A, we see that $x_{h+1} \in J(x_{n+h}, x_h)$, which contradicts (20).

Thus $I(r, s) \subseteq J(r, s)$, which is a contradiction. We conclude that (II) implies (III).

By virtue of Lemma 3, (III) implies (I), which completes the proof of the theorem. \square

Remark 3. Let G be a connected graph. An axiomatic characterization of the set of all ordered triples (u, v, w) of vertices in G with the properties that $d_G(u, v) = 1$ and $d_G(u, w) = d_G(v, w) + 1$ was given by the present author in [6].

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