ON MONOTONE-LIKE MAPPINGS IN ORLICZ-SOBOLEV SPACES

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We study the mappings of monotone type in Orlicz-Sobolev spaces. We introduce a new class (S_m) as a generalization of (S_+) and extend the definition of quasimonotone map. We also prove existence results for equations involving monotone-like mappings.

Keywords: pseudomonotone, quasimonotone, Orlicz-Sobolev space, almost solvability

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1. Introduction

Since the pioneering work of Minty in 1962 the theory of monotone mappings from a real reflexive Banach space X into its dual space X^* has been extensively generalized by Brezis, Browder, Hess, Leray and Lions, Visik and many others. In its original form the theory considers mappings T which satisfy the condition

$$\langle u-v, T(u)-T(v)\rangle \geqslant 0 \quad \text{for all } u \text{ and } v \text{ in } X,$$

where $\langle u, w\rangle$ denotes the duality pairing between the element u in X and w in X^*. In order to treat efficiently the solvability problems for nonlinear elliptic and parabolic equations and corresponding variational inequalities within the same framework, various generalizations of the concept of monotone maps have been introduced. Most important of these extensions turned out to be the mappings of class (S_+), pseudomonotone mappings (PM), mappings of the type (M) and quasimonotone mappings (QM). The fact that the classical topological degree can be constructed for the class (S_+) and for the class of pseudomonotone mappings in the weak form indicates that the classes are well-defined (see [2, 3, 19]).

A motivation for the definition of various classes of mappings of monotone type comes from the study of the existence of solutions for the problems associated to elliptic differential operators in divergence form

$$Au(x) = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha a_\alpha\big(x, u(x), \nabla u(x), \dots, \nabla^m u(x)\big), \qquad x \in \Omega \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^N$ and $m \geqslant 1$. If the coefficients $a_\alpha(x, \xi)$ satisfy a polynomial growth condition with respect to $|\xi|$ and suitable analytical conditions, the differential operator (1.1) generates a nonlinear mapping T from the Sobolev space $W^{m,p}(\Omega)$ to its dual space $W^{-m,p'}(\Omega)$ belonging to the class (S_+), (PM) or (QM), respectively.

Differential operator (1.1) is called strongly nonlinear if the coefficients A_α do not satisfy any polynomial growth condition. The study of strongly nonlinear elliptic problems was initiated by Browder in 1973. Since then many contributions have been published into this direction. Browder's original idea was to consider operators of the form

$$Au(x) + Bu(x),$$

where A is a polynomial operator as above and B is a lower order operator having no growth restrictions. This approach led to the concept of generalized pseudomonotone mapping, which is, in general, neither everywhere defined nor bounded in the Sobolev space associated with the operator A. Further contributions in this direction are due to Hess [10] and Landes [15] and many others. Browder was able to show that a degree theory can be extended also for a particular class of mappings where $Bu(x) = g\big(x, u(x)\big)$. A further extension for a more general lower order part was obtained by Kittilä [12].

Another line of development for treating strongly nonlinear elliptic boundary value problems is to employ Orlicz spaces in place of reflexive Lebesgue spaces $L^p(\Omega)$. By this change the polynomial growth condition can be replaced by a more liberal condition associated with an Orlicz function. The theory of mappings of monotone type can be extended also for complementary systems of Orlicz-Sobolev spaces which are not reflexive in general, and existence theorems can be produced accordingly. The study along these lines was initiated by Donaldson [4] and continued by Gossez [6–8]. Further contributions in this direction include [9], [17] and [20], where a degree theory is constructed.

In this paper we continue the study of mappings of monotone type in Orlicz-Sobolev spaces. Our main task is to give a more complete characterization of relevant classes and produce corresponding refined solvability theorems for equations. We introduce a new class (S_m) which can be seen as a generalization of the class (S_+) to the Orlicz-Sobolev space setting. We also extend the definition of quasimonotone mapping and prove that the class (S_m) stands quasimonotone perturbations.

Our paper is organized as follows. In Section 2 we present the basic properties of Orlicz and Orlicz-Sobolev spaces. In the next section we study the classes of monotone-like operators in the complementary system formed by Orlicz-Sobolev spaces. In Section 4 we deal with the conditions for differential operators in divergence form in order to generate mappings of the type described in Section 3. In the last section we generalize the basic existence theorem for equations involving quasimonotone mappings.

2. Notations and definitions

We begin with some preliminaries on Orlicz-Sobolev spaces. Let Ω be a bounded open subset in $\mathbb{R}^N$ and let $M\colon \mathbb{R} \to \mathbb{R}$ be an *N-function*, i.e., even, convex and continuous with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to +\infty$ as $t \to +\infty$. M is an N-function if and only if it can be represented in the form

$$M(t) = \int_0^{|t|} m(s)\,\mathrm{d}s \tag{2.1}$$

where $m\colon [0,\infty[\to [0,\infty[$ is increasing, right continuous, $m(t) = 0$ if and only if $t = 0$ and $m(t) \to +\infty$ as $t \to +\infty$. We extend m to $\mathbb{R}$ by $m(t) = -m(-t)$ for $t < 0$ (odd continuation). The *Orlicz class* $\mathcal{L}_M(\Omega)$ is defined as the set of all real-valued measurable functions u defined in Ω such that

$$\int_\Omega M(u)\,\mathrm{d}x < \infty.$$

The *Orlicz space* $L_M(\Omega)$ is the linear hull of $\mathcal{L}_M(\Omega)$. Then $L_M(\Omega)$ is a Banach space with respect to the Luxemburg norm

$$\|u\|_{(M)} = \inf\Big\{k > 0\colon \int_\Omega M\Big(\frac{u}{k}\Big)\,\mathrm{d}x \leqslant 1\Big\}.$$

One has $L_M(\Omega) = \mathcal{L}_M(\Omega)$ if and only if M satisfies the $\triangle_2$-condition: there exist $\alpha > 0$ and $t_0 > 0$ such that

$$M(2t) \leqslant \alpha\, M(t)$$

for all $t \geqslant t_0$. The closure in $L_M(\Omega)$ of all bounded measurable functions is denoted by $E_M(\Omega)$. Then $E_M(\Omega) \subset \mathcal{L}_M(\Omega)$ and $E_M(\Omega) = \mathcal{L}_M(\Omega)$ if and only if M satisfies the $\triangle_2$-condition. The conjugate N-function $\overline{M}$ is defined by

$$\overline{M}(t) = \sup\bigl\{ts - M(s)\colon\ s \in \mathbb{R}\bigr\}.$$

$\overline{M}$ is also an N-function and $\overline{\overline{M}} = M$. The space $L_{\overline{M}}(\Omega)$ is the dual space of $E_M(\Omega)$. It is well-known that $L_M(\Omega)L_{\overline{M}}(\Omega) \subset L^1(\Omega)$. We recall also Young's inequality:

$$M(x) + \overline{M}(y) \geqslant xy \qquad \text{for all } x, y \in \mathbb{R} \tag{2.2}$$

with equality if and only if $x = \overline{m}(y)$ or $y = m(x)$. A sequence $\{u_n\}$ in $L_M(\Omega)$ converges *modularly* to u if there exists $\lambda > 0$ such that

$$\int_\Omega M\Bigl(\frac{u_n - u}{\lambda}\Bigr)\,\mathrm{d}x \to 0,$$

when $n \to \infty$. Modular convergence coincides with norm convergence if and only if M satisfies the $\triangle_2$-condition. If M_1 and M_2 are N-functions satisfying $\lim\limits_{t\to\infty} M_1(ct)/M_2(t) = 0$ for all $c > 0$, then M_1 grows *essentially more slowly than* M_2 and we denote $M_1 << M_2$.

Remark 2.1. Typical examples of N-functions satisfying the $\triangle_2$-condition are $(1+|t|)\log(1+|t|) - |t|$ and $|t|^p$ for $p > 1$. On the other hand, functions $\mathrm{e}^{|t|} - |t| - 1$ and $\mathrm{e}^{|t|^p} - 1$ for $p > 1$ are N-functions not satisfying the $\triangle_2$-condition.

The Orlicz-Sobolev space of functions in $L_M(\Omega)$ with all distributional derivatives up to the order m in $L_M(\Omega)$ is denoted by $W^m L_M(\Omega)$. The space $W^m E_M(\Omega)$ is defined analogously. These spaces are identified, as usual, to subspaces of the product $\prod L_M(\Omega)$. The spaces $W_0^m L_M(\Omega)$ and $W_0^m E_M(\Omega)$ are defined as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_M(\Omega)$ and as the norm closure of $\mathcal{D}(\Omega)$ in $W^m E_M(\Omega)$, respectively. We recall that there exists an N-function $Q >> M$ such that the embedding $W_0^1 L_M(\Omega) \to E_Q(\Omega)$ is compact (see [5, 6]).

The following spaces of distributions will also be used:

$$W^{-m}L_{\overline{M}}(\Omega) = \Bigl\{f \in \mathcal{D}'(\Omega)\colon\ f = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_{\overline{M}}(\Omega)\Bigr\}$$

$$W^{-m}E_{\overline{M}}(\Omega) = \Bigl\{f \in \mathcal{D}'(\Omega)\colon\ f = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_{\overline{M}}(\Omega)\Bigr\}.$$

They are endowed with their usual quotient norms. It is shown in [6] that if Ω has the segment property, then

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^m L_M(\Omega) & W^{-m}L_{\overline{M}}(\Omega) \\ W_0^m E_M(\Omega) & W^{-m}E_{\overline{M}}(\Omega) \end{pmatrix}$$

constitutes a complementary system, i.e., Y and Z are real Banach spaces in duality with respect to a continuous pairing $\langle\cdot\,,\cdot\rangle$ and Y_0 and Z_0 are closed subspaces of Y and Z respectively such that, by means of $\langle\cdot\,,\cdot\rangle$, the dual of Y_0 can be identified to Z and that of Z_0 to Y. The pairing between $u \in Y$ and $f = \sum\limits_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha f_\alpha \in Z$ is given by

$$\langle u, f\rangle = \sum_{|\alpha|\leqslant m} \int_\Omega (D^\alpha u) f_\alpha \,\mathrm{d}x.$$

A sequence $\{u_n\} \subset Y$ converges *modularly* to u in Y if $D^\alpha u_n \to D^\alpha u$ modularly in $L_M(\Omega)$ for each $|\alpha| \leqslant m$. Standard references on Orlicz and Orlicz-Sobolev spaces include [1, 13, 14].

We end this section by presenting some useful convergence results.

Lemma 2.2.

(i) *If $u_n \to u$ a.e. in Ω, $u_n \to u$ in $L_M(\Omega)$ for $\sigma(L_M, E_{\overline{M}})$ and $v_n \to v$ in $E_{\overline{M}}(\Omega)$ strongly, then $u_n v_n \to u\,v$ in $L^1(\Omega)$*

(ii) *if $u_n \to u$ in $E_M(\Omega)$ strongly and $P << M$, then $\overline{P}^{-1}\big(M(u_n)\big) \to \overline{P}^{-1}\big(M(u_n)\big)$ in $E_{\overline{M}}(\Omega)$ strongly*

(iii) *$u_n \to u$ in $L_M(\Omega)$ modularly if and only if $u_n \to u$ in measure and there exist a convergent sequence $\{f_n\}$ in $L^1(\Omega)$ and $c > 0$ such that $M(c\,u_n) \leqslant f_n$ a.e. in Ω.*

Proof. The proofs of (i) and (ii) can be found in [6] and [20], for example. To prove (iii), assume first that $u_n \to u$ in measure and $M(c\,u_n) \leqslant f_n$ a.e. in Ω, where $c > 0$ and $f_n \to f$ in $L^1(\Omega)$. Then

$$M\left(\tfrac{c}{2}(u_n - u)\right) \leqslant \tfrac12 M(cu_n) + \tfrac12 M(cu) \leqslant \tfrac12 f_n + \tfrac12 f \qquad \text{a.e. in } \Omega.$$

By the dominated convergence theorem

$$M\left(\tfrac{c}{2}(u_n - u)\right) \to 0 \text{ in } L^1(\Omega).$$

Hence $u_n \to u$ in $L_M(\Omega)$ modularly.

Assume next that $u_n \to u$ in $L_M(\Omega)$ modularly. Hence $M\big(2\varepsilon_1(u_n - u)\big) \to 0$ in $L^1(\Omega)$ for some $\varepsilon_1 > 0$ implying $u_n \to u$ in measure. Moreover, choosing $0 < \varepsilon_2 < \min(\varepsilon_1, \frac{1}{2\|u\|_{(M)}})$ we get

$$M(\varepsilon_2 u_n) \leqslant \tfrac12 M\big(2\varepsilon_2(u_n - u)\big) + \tfrac12 M(2\varepsilon_2 u) \qquad \text{a.e. in } \Omega$$

where the right hand side converges in $L^1(\Omega)$. $\square$

The original definitions for various classes of mappings of monotone type were given for mappings acting from a real reflexive Banach space X into its dual space X^*. The norm convergence in X and in X^* is denoted by $\to$ and the weak convergence by $\rightharpoonup$. We recall the following classical notions. A mapping T from X to X^* is said to be

– *monotone*, denote $T \in (MON)$, if $\langle u - v, T(u) - T(v)\rangle \geqslant 0$ for all $u, v \in X$

– *of class* (S_+) if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ and $\limsup\langle u_n - u, T(u_n)\rangle \leqslant 0$ we have $u_n \to u$ in X

– *pseudomonotone*, denote $T \in (PM)$, if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ and $\limsup\langle u_n - u, T(u_n)\rangle \leqslant 0$ we have $T(u_n) \rightharpoonup T(u)$ and $\langle u_n, T(u_n)\rangle \to \langle u, T(u)\rangle$

– *quasimonotone*, $T \in (QM)$, if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ we have $\limsup\langle u_n - u, T(u_n)\rangle \geqslant 0$

– *of class* (M), if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$, $T(u_n) \rightharpoonup \chi$ and $\limsup\langle u_n, T(u_n)\rangle \leqslant \langle \chi, u\rangle$ we have $\chi = T(u)$

– *bounded* if it takes bounded sets of X into bounded sets of X^*

– *demicontinuous* if $u_n \to u$ in X implies $T(u_n) \rightharpoonup T(u)$ in X^*
For bounded demicontinuous mappings we have $(S_+) \subset (PM) \subset (QM)$ and $(QM) \cap (M) = (PM)$. Also the perturbation result $(S_+) + (QM) = (S_+)$ is useful in applications. Note that the above condition of quasimonotony can be written also in the form: for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ and $\limsup\langle u_n - u, T(u_n)\rangle \leqslant 0$ we have $\langle u_n - u, T(u_n)\rangle \to 0$.

Our task now is to study the corresponding classification of monotone-like mappings in the complementary system of Orlicz-Sobolev spaces

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^m L_M(\Omega) & W^{-m} L_{\overline{M}}(\Omega) \\ W_0^m E_M(\Omega) & W^{-m} E_{\overline{M}}(\Omega) \end{pmatrix}$$

where $\Omega \subset \mathbb{R}^N$ is an open and bounded subset with the segment property. Essential modifications are needed in the definitions above since Orlicz-Sobolev spaces are not reflexive, in general, and the differential operators in divergence form with natural growth conditions are neither bounded nor everywhere defined. Moreover, the duality map in Orlicz-Sobolev spaces is not a single-valued (S_+)-mapping, in general.

Definition 3.1. A mapping $T\colon\, Y_0 \subset D(T) \subset Y \to Z$

– is *quasimonotone* (denote $T \in (QM)$), if

$$\begin{cases} \{u_n\} \subset D(T) & \\ u_n \to u \in Y & \text{for } \sigma(Y, Z_0) \\ T(u_n) \to \chi \in Z & \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T(u_n)\rangle & \leqslant \langle u, \chi\rangle \end{cases} \quad \text{imply } \langle u_n, T(u_n)\rangle \to \langle u, \chi\rangle$$

– is *pseudomonotone* ($T \in (PM)$), if

$$\begin{cases} \{u_n\} \subset D(T) & \\ u_n \to u \in Y & \text{for } \sigma(Y, Z_0) \\ T(u_n) \to \chi \in Z & \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T(u_n)\rangle & \leqslant \langle u, \chi\rangle \end{cases} \quad \text{imply} \quad \begin{cases} u \in D(T) \\ \chi = T(u) \\ \langle u_n, T(u_n)\rangle \to \langle u, \chi\rangle \end{cases}$$

– is *of class* (S_m), if

$$\begin{cases} \{u_n\} \subset D(T) & \\ u_n \to u \in Y & \text{for } \sigma(Y, Z_0) \\ T(u_n) \to \chi \in Z & \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T(u_n)\rangle & \leqslant \langle u, \chi\rangle \end{cases} \quad \text{imply} \quad \begin{cases} u \in D(T) \\ \chi = T(u) \\ \langle u_n, T(u_n)\rangle \to \langle u, \chi\rangle \\ u_n \to u \text{ modularly in } Y \end{cases}$$

– is *quasibounded* with respect to $\bar{u} \in Y_0$, if $T(u)$ remains bounded in Z whenever $u \in D(T)$ remains bounded in Y and $\langle u - \bar{u}, T(u)\rangle$ remains bounded from above.

– is *finitely continuous*, if T is continuous from each finite dimensional subspace of Y_0 into Z for $\sigma(Z, Y_0)$

– satisfies *the condition* (M_m), if

$$\begin{cases} \{u_n\} \subset D(T) \\ u_n \to u \in Y \text{ modularly} \\ T(u_n) \to \chi \in Z \text{ for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T(u_n)\rangle \leqslant \langle u, \chi\rangle \end{cases} \quad \text{imply} \quad \begin{cases} u \in D(T) \\ \chi = T(u). \end{cases}$$

Clearly any pseudomonotone mapping and mapping of class (S_m) satisfies the condition (M_m). Quasimonotone mappings satisfying the condition (M_m) are denoted by (QM_m). It is straightforward to check that the sum of two quasibounded mappings with respect to the same $\bar{u} \in Y_0$ is also quasibounded with respect to $\bar{u}$. Zero map belongs to each of the classes defined above except (S_m). In the sequel we shall denote the restriction of any class $(\cdot)$ defined above to the class of quasibounded mappings with respect to $\bar{u}$ by a subscript $(\cdot)_{\bar{u}}$.

For the classes in Definition 3.1, we have the following inclusions and perturbation result.

Theorem 3.2.
(i) $(S_m) \subset (PM) \subset (QM_m)$
(ii) $(S_m)_{\bar{u}} + (QM_m)_{\bar{u}} = (S_m)_{\bar{u}}$

Proof. The first assertion follows immediately from the definitions. To prove (ii), assume $T = T_1 + T_2$, where T_1 and T_2 are quasibounded with respect to $\bar{u} \in Y_0$, $T_1 \in (S_m)$ and $T_2 \in (QM_m)$. Clearly Y_0 is a subset of $D(T) = D(T_1) \cap D(T_2)$ and T is quasibounded with respect to $\bar{u}$. Suppose

$$\begin{cases} \{u_n\} \subset D(T) & \\ u_n \to u \in Y & \text{for } \sigma(Y, Z_0) \\ T(u_n) \to \chi \in Z & \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T(u_n) \rangle & \leqslant \langle u, \chi \rangle. \end{cases}$$

Since T_1 and T_2 are quasibounded with respect to $\bar{u}$, we may deduce that

$$T_1(u_n) \to \chi_1 \in Z \text{ and } T_2(u_n) \to \chi_2 \in Z \qquad \text{for } \sigma(Z, Y_0)$$

for a subsequence with $\chi = \chi_1 + \chi_2$. Since $T_2 \in (QM)$, we have

$$\limsup \langle u_n, T_1(u_n) \rangle \leqslant \langle u, \chi_1 \rangle.$$

In view of $T_1 \in (S_m)$, we get $u \in D(T_1)$, $\chi_1 = T_1(u)$, $\langle u_n, T_1(u_n) \rangle \to \langle u, \chi_1 \rangle$ and $u_n \to u$ modularly in Y. Hence

$$\limsup \langle u_n, T_2(u_n) \rangle \leqslant \langle u, \chi_2 \rangle$$

implying, on account of $T_2 \in (QM_m)$, that $u \in D(T_2)$, $T_2(u) = \chi_2$ and

$$\langle u_n, T_2(u_n) \rangle \to \langle u, \chi_2 \rangle.$$

□

Let Ω be an open and bounded subset in $\mathbb{R}^N$ with the segment property and denote

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^m L_M(\Omega) & W^{-m} L_{\overline{M}}(\Omega) \\ W_0^m E_M(\Omega) & W^{-m} E_{\overline{M}}(\Omega) \end{pmatrix}.$$

We shall consider differential operators in divergence form

$$A^{(1)}u(x) = \sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha a_\alpha(x, u, \nabla u, \dots, \nabla^m u), \qquad x \in \Omega \tag{4.1}$$

$$A^{(0)}u(x) = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha a_\alpha(x, u, \nabla u, \dots, \nabla^m u), \qquad x \in \Omega \tag{4.2}$$

and the corresponding mappings $T_1 \colon D(T_1) \to Z$ and $T_0 \colon D(T_0) \to Z$ defined by the formulas

$$\langle v, T_1(u) \rangle = \int_\Omega \sum_{|\alpha|=m} a_\alpha\big(x, \xi(u)\big) D^\alpha v \,\mathrm{d}x, \qquad u \in D(T_1), v \in Y$$

and

$$\langle v, T_0(u) \rangle = \int_\Omega \sum_{|\alpha|<m} a_\alpha\big(x, \xi(u)\big) D^\alpha v \,\mathrm{d}x, \qquad u \in D(T_0), v \in Y$$

where

$$D(T_1) = \big\{ u \in Y \colon a_\alpha\big(x, \xi(u)\big) \in L_{\overline{M}}(\Omega) \text{ for } |\alpha| = m \big\}$$
$$D(T_0) = \big\{ u \in Y \colon a_\alpha\big(x, \xi(u)\big) \in L_{\overline{M}}(\Omega) \text{ for } |\alpha| < m \big\}.$$

We use the following notations: If $\xi = \{\xi_\alpha \colon |\alpha| \leqslant m\} \in \mathbb{R}^{N_0}$ is an m-jet, then $\zeta = \{\xi_\alpha \colon |\alpha| = m\} \in \mathbb{R}^{N_1}$ denotes its top order part and $\eta = \{\xi_\alpha \colon |\alpha| < m\} \in \mathbb{R}^{N_2}$ its lower order part. For a differentiable function u, $\xi(u)$ denotes $\{D^\alpha u \colon |\alpha| \leqslant m\}$. Now we introduce the conditions on the differential operators $A^{(0)}$ and $A^{(1)}$ which give mappings T_0 and T_1 the properties described in Definition 3.1.

(A_1) Each $a_\alpha(x, \xi) \colon \Omega \times \mathbb{R}^{N_0} \to \mathbb{R}$ is measurable for any fixed $\xi \in \mathbb{R}^{N_0}$ and continuous in ξ for a.e. fixed x

(A_2) There exist constants $c_1, c_2 > 0$ and functions k_α in $E_{\overline{M}}(\Omega)$ for all $|\alpha| = m$, $k_\alpha \in L_{\overline{M}}(\Omega)$ for $|\alpha| < m$ and an N-function $P << M$ such that for a.e. x in Ω and all ξ in $\mathbb{R}^{N_0}$

$$|a_\alpha(x, \xi)| \leqslant k_\alpha(x) + c_1 \sum_{|\beta|=m} \overline{M}^{-1}\big(M(c_2 \xi_\beta)\big) + c_1 \sum_{|\beta|<m} \overline{P}^{-1}\big(M(c_2 \xi_\beta)\big)$$

if $|\alpha| = m$,

$$|a_\alpha(x,\xi)| \leqslant k_\alpha(x) + c_1 \sum_{|\beta|=m} \overline{M}^{-1}\bigl(P(c_2\xi_\beta)\bigr) + c_1 \sum_{|\beta|<m} \overline{M}^{-1}\bigl(M(c_2\xi_\beta)\bigr)$$

if $|\alpha| < m$.

(A_3) For a.e. x in Ω, all η in $\mathbb{R}^{N_2}$, ζ and ζ' in $\mathbb{R}^{N_1}$ with $\zeta \neq \zeta'$,

$$\sum_{|\alpha|=m} \bigl(a_\alpha(x,\eta,\zeta) - a_\alpha(x,\eta,\zeta')\bigr)\bigl(\zeta_\alpha - \zeta'_\alpha\bigr) > 0$$

$(A_3)_e$ For a.e. x in Ω, all η in $\mathbb{R}^{N_2}$, ζ and ζ' in $\mathbb{R}^{N_1}$,

$$\sum_{|\alpha|=m} \bigl(a_\alpha(x,\eta,\zeta) - a_\alpha(x,\eta,\zeta')\bigr)\bigl(\zeta_\alpha - \zeta'_\alpha\bigr) \geqslant 0$$

(A_4) There exist functions $b_\alpha(x)$ in $E_{\overline{M}}(\Omega)$ for $|\alpha| = m$, $b(x)$ in $L^1(\Omega)$, constants $d_1, d_2 > 0$ and some fixed element $\varphi \in W_0^m E_M(\Omega)$ such that

$$\sum_{|\alpha|=m} a_\alpha\bigl(x,\xi\bigr)\bigl(\xi_\alpha - D^\alpha\varphi(x)\bigr) \geqslant d_1 \sum_{|\alpha|=m} M(d_2\xi_\alpha) - \sum_{|\alpha|=m} b_\alpha(x)\xi_\alpha - b(x)$$

for a.e. x in Ω and all ξ in $\mathbb{R}^{N_0}$.

These conditions are generalizations of the classical Leray-Lions conditions to Orlicz-Sobolev space setting (cf. [9, 17, 18]).

We shall study first the properties of T_0.

Proposition 4.1. *If* (A_1) *and* (A_2) *hold, then* T_0 *is finitely continuous,* $D(T_0) = Y$, T_0 *is bounded and belongs to* (QM_m).

P r o o f. It is proved ([9]) that if $\{u_n\}$ remains bounded in Y, then $\{a_\alpha\bigl(x,\xi(u_n)\bigr)\}$ remains bounded in $L_{\overline{M}}(\Omega)$ for $|\alpha| < m$, which proves that $D(T_0) = Y$ and T_0 is bounded. Finite continuity follows as in [6].

Suppose $u_n \to u$ in Y for $\sigma(Y, Z_0)$ and $T_0(u_n) \to \chi$ in Z for $\sigma(Z, Y_0)$. By compact embedding, $D^\alpha u_n \to D^\alpha u$ in $E_M(\Omega)$ for $|\alpha| < m$. Since $\bigl\{a_\alpha\bigl(x,\xi(u_n)\bigr)\bigr\}$ is bounded in $L_{\overline{M}}(\Omega)$ for $|\alpha| < m$, we may assume

$$a_\alpha(x,\xi(u_n)) \to h_\alpha \in L_{\overline{M}}(\Omega) \qquad \text{for } \sigma(L_{\overline{M}}, E_M)$$

for a subsequence. Clearly

$$\langle \varphi, \chi\rangle = \sum_{|\alpha|<m} \int_\Omega h_\alpha\, D^\alpha\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in Y$$

and

$$\langle u_n, T_0(u_n)\rangle = \sum_{|\alpha|<m} \int_\Omega a_\alpha(x, \xi(u_n)) D^\alpha u_n \, \mathrm{d}x \to \sum_{|\alpha|<m} \int_\Omega h_\alpha \, D^\alpha u \, \mathrm{d}x = \langle u, \chi\rangle$$

proving that T_0 is quasimonotone. If $u_n \to u$ in Y modularly in the above, then $a_\alpha(x, \xi(u_n)) \to a_\alpha(x, \xi(u))$ a.e. for a subsequence implying $h_\alpha = a_\alpha(x, \xi(u))$. Hence T_0 satisfies the condition (M_m) and the proof is complete. □

For the operator T_1 we adopt the following continuity and boundedness property from [9].

Proposition 4.2. *If* (A_1), (A_2) *and* $(\mathrm{A}_3)_e$ *hold, then* T_1 *is finitely continuous and quasibounded with respect to any* $\bar{u} \in Y_0$.

Next we have the following extensions of the previous results of [6, 9] for the mapping T_1.

Theorem 4.3.

a) *If* (A_1), (A_2) *and* $(\mathrm{A}_3)_e$ *hold, then* T_1 *is pseudomonotone.*

b) *If* (A_1), (A_2), (A_3) *and* (A_4) *hold, then* T_1 *is of class* (S_m).

Proof. First we prove part a). Suppose (A_1), (A_2) and $(\mathrm{A}_3)_e$ hold and

$$\begin{cases} \{u_n\} \subset D(T_1) & \\ u_n \to u \in Y & \text{for } \sigma(Y, Z_0) \\ T_1(u_n) \to \chi \in Z & \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T_1(u_n)\rangle & \leqslant \langle u, \chi\rangle. \end{cases}$$

By the argument used in the proof of [9, Proposition 5.1], we may assume that $\{a_\alpha\big(x, \xi(u_n)\big)\}$ remains bounded in $L_{\overline{M}}(\Omega)$. Consequently,

$$a_\alpha\big(x, \xi(u_n)\big) \to h_\alpha \in L_{\overline{M}}(\Omega) \qquad \text{for } \sigma(L_{\overline{M}}, E_M)$$

for a subsequence and

$$\langle \varphi, \chi\rangle = \sum_{|\alpha|=m} \int_\Omega h_\alpha \, D^\alpha \varphi \, \mathrm{d}x \qquad \text{for all } \varphi \in Y_0. \tag{4.3}$$

By $\sigma(Y, Z)$ density of Y_0 in Y, (4.3) holds for all $\varphi \in Y$. Next we prove that $a_\alpha(x, \xi(u)) = h_\alpha$ a.e. in Ω for all $|\alpha| = m$. By the compact embedding, $D^\beta u_n \to D^\beta u$ in $E_M(\Omega)$ for $|\beta| < m$. The condition $(\mathrm{A}_3)_e$ implies

$$\sum_{|\alpha|=m} \int_\Omega \big(a_\alpha(x, \eta(u_n), \bar{v}) - a_\alpha(x, \xi(u_n))\big)\big(v_\alpha - D^\alpha u_n\big) \, \mathrm{d}x \geqslant 0$$

for all $\bar{v} = (v_\alpha) \in \big(L^\infty(\Omega)\big)^{N_1}$. Therefore

$$\begin{aligned} \langle u_n, T_1(u_n)\rangle \geqslant \sum_{|\alpha|=m} \int_\Omega a_\alpha\big(x, \xi(u_n)\big) v_\alpha \,\mathrm{d}x \\ + \sum_{|\alpha|=m} \int_\Omega a_\alpha\big(x, \eta(u_n), \bar{v}\big)\big(D^\alpha u_n - v_\alpha\big)\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

The condition (A_2) and the compact embedding imply

$$a_\alpha\big(x, \eta(u_n), \bar{v}\big) \to a_\alpha\big(x, \eta(u), \bar{v}\big)$$

in $E_{\overline{M}}(\Omega)$ (see [6]). Hence

$$\langle u, \chi\rangle \geqslant \sum_{|\alpha|=m} \int_\Omega h_\alpha v_\alpha \,\mathrm{d}x + \sum_{|\alpha|=m} \int_\Omega a_\alpha\big(x, \eta(u), \bar{v}\big)\big(D^\alpha u - v_\alpha\big)\,\mathrm{d}x$$

and consequently

$$\sum_{|\alpha|=m} \int_\Omega \big(a_\alpha(x, \eta(u), \bar{v}) - h_\alpha\big)\big(v_\alpha - D^\alpha u\big)\,\mathrm{d}x \geqslant 0 \tag{4.5}$$

for $\bar{v} = (v_\alpha) \in \big(L^\infty(\Omega)\big)^{N_1}$. Let $0 < j < i$ be arbitrary integers and $t > 0$. Denote

$$\Omega_i = \big\{x \in \Omega\colon\ |D^\alpha u(x)| \leqslant i \ \text{ a.e. in } \Omega \text{ for all } |\alpha| = m\big\}$$

and

$$\bar{v} = (\nabla u)\chi_{\Omega_i} + t\overline{w}\chi_{\Omega_j},$$

where $\overline{w} \in (L^\infty(\Omega))^{N_1}$ is arbitrary. By (4.5),

$$\begin{aligned} -\sum_{|\alpha|=m} \int_{\Omega\setminus\Omega_i} a_\alpha(x, \eta(u), \bar{0}) D^\alpha u \,\mathrm{d}x \\ + t \sum_{|\alpha|=m} \int_{\Omega_j} \big(a_\alpha(x, \eta(u), \zeta(u) + t\overline{w}) - h_\alpha\big) w_\alpha \,\mathrm{d}x \geqslant 0. \end{aligned}$$

Letting $i \to \infty$ and dividing by t, we get

$$\sum_{|\alpha|=m} \int_{\Omega_j} \big(a_\alpha(x, \eta(u), \zeta(u) + t\overline{w}) - h_\alpha\big) w_\alpha \,\mathrm{d}x \geqslant 0.$$

Since $D^\alpha u + tw_\alpha \to D^\alpha u$ in $L^\infty(\Omega_j)$, when $t \to 0^+$, we have

$$a_\alpha(x, \eta(u), \zeta(u) + t\overline{w}) \to a_\alpha(x, \eta(u), \zeta(u))$$

in $E_{\overline{M}}(\Omega_j)$. Consequently,

$$\sum_{|\alpha|=m} \int_{\Omega_j} \big(a_\alpha(x, \xi(u)) - h_\alpha\big) w_\alpha \,\mathrm{d}x \geqslant 0$$

for all $\overline{w} \in (L^\infty(\Omega))^{N_1}$ implying

$$a_\alpha(x, \xi(u)) = h_\alpha \qquad \text{a.e. in } \Omega_j \tag{4.6}$$

for all $|\alpha| = m$. Since j was arbitrary, (4.6) holds a.e. in Ω. Therefore $u \in D(T_1)$ and $\chi = T_1(u)$. Substituting $\bar{v} = \zeta(u)\chi_{\Omega_i}$ into (4.4) we get

$$\begin{aligned}\langle u_n, T_1(u_n)\rangle \geqslant &\sum_{|\alpha|=m} \int_\Omega a_\alpha\big(x, \xi(u_n)\big)\big(D^\alpha u\big)\chi_{\Omega_i} \,\mathrm{d}x \\ &+ \sum_{|\alpha|=m} \int_\Omega a_\alpha\big(x, \eta(u_n), \zeta(u_n)\chi_{\Omega_i}\big)\big(D^\alpha u_n - (D^\alpha u)\chi_{\Omega_i}\big) \,\mathrm{d}x\end{aligned}$$

implying

$$\begin{aligned}\liminf \langle u_n, T_1(u_n)\rangle \geqslant &\sum_{|\alpha|=m} \int_{\Omega_i} a_\alpha\big(x, \xi(u)\big) D^\alpha u \,\mathrm{d}x + \sum_{|\alpha|=m} \int_{\Omega\setminus\Omega_i} a_\alpha(x, u, \bar{0}) D^\alpha u \,\mathrm{d}x \\ &\to \sum_{|\alpha|=m} \int_\Omega a_\alpha\big(x, \xi(u)\big) D^\alpha u \,\mathrm{d}x,\end{aligned}$$

when $i \to \infty$. Hence $\langle u_n, T_1(u_n)\rangle \to \langle u, T_1(u)\rangle$ and the proof of part a) is complete.

To prove part b), suppose (A_1), (A_2), (A_3) and (A_4) hold and

$$\begin{cases} \{u_n\} \subset D(T_1) & \\ u_n \to u \in Y & \text{for } \sigma(Y, Z_0) \\ T_1(u_n) \to \chi \in Z & \text{for } \sigma(Z, Y_0) \\ \limsup \langle u_n, T_1(u_n)\rangle & \leqslant \langle u, \chi\rangle. \end{cases}$$

By the previous part, $u \in D(T_1), \chi = T_1(u)$ and $\langle u_n, T_1(u_n)\rangle \to \langle u, \chi\rangle$. As above, $D^\alpha u_n \to D^\alpha u$ in $E_M(\Omega)$ for $|\alpha| < m$. In view of strict inequality in (A_3), we may deduce as in [15] that $D^\alpha u_n \to D^\alpha u$ a.e. for $|\alpha| = m$, for a subsequence. This implies $D^\alpha u_n \to D^\alpha u$ in measure for the original sequence. By (A_2) and (A_3),

$$\begin{aligned} f_n &:= \sum_{|\alpha|=m} a_\alpha\big(x, \xi(u_n)\big) D^\alpha u_n \geqslant \sum_{|\alpha|=m} a_\alpha\big(x, \eta(u_n), \bar{0}\big) D^\alpha u_n \\ &\geqslant - \sum_{|\alpha|=m} \Big(|k_\alpha(x) D^\alpha u_n| + c_1 \sum_{|\beta|<m} \overline{P}^{-1}\big(M(c_2 D^\beta u_n)\big) D^\alpha u_n \Big). \end{aligned}$$

By compact embedding and Lemma 2.2, the right hand side converges in $L^1(\Omega)$. Denoting $f = \sum_{|\alpha|=m} a_\alpha(x, \xi(u)) D^\alpha u$ we get for some $h \in L^1(\Omega)$ that $f_n \geqslant -h$, $f_n \to f$ a.e. in Ω and

$$\int_\Omega f_n \, \mathrm{d}x \to \int_\Omega f \, \mathrm{d}x,$$

for a subsequence. Using the result of [11, p. 208], $f_n \to f$ in $L^1(\Omega)$ for a subsequence, and hence, by standard contradiction argument, $f_n \to f$ in $L^1(\Omega)$ also for the original sequence. By condition (A_4),

$$\begin{aligned} d_1 \sum_{|\alpha|=m} M(d_2 D^\alpha u_n) \leqslant & \sum_{|\alpha|=m} a_\alpha\big(x, \xi(u_n)\big)\big(D^\alpha u_n - D^\alpha \varphi(x)\big) \\ & + \sum_{|\alpha|=m} b_\alpha(x) D^\alpha u_n + b(x). \end{aligned}$$

Using Lemma 2.2. we conclude that the right hand side of the inequality above converges in $L^1(\Omega)$. Therefore $D^\alpha u_n \to D^\alpha u$ in $L_M(\Omega)$ modularly for $|\alpha| = m$, by Lemma 2.2 (iii). □

Remark 4.4. If $\lim_{t\to\infty} M(ct)/M(t) = \infty$ for some $c > 1$, then any bounded sequence in $L_M(\Omega)$ which converges a.e. converges also modularly and hence we may remove condition (A_4) from Theorem 4.4 b) (see [20]). Note also that if M and $\overline{M}$ satisfy the $\triangle_2$-condition, then we may choose $P = M$ in condition (A_2).

5. Solvability results for equations

We shall close this paper by solvability and almost solvability results for monotone-like mappings in the complementary systems of Orlicz-Sobolev spaces. We adopt first a well-known existence result for pseudomonotone mappings in a complementary system from [9].

Theorem 5.1. *Let $(Y, Y_0; Z, Z_0)$ be a complementary system with Y_0 and Z_0 separable. Let $T\colon Y_0 \subset D(T) \subset Y \to Z$ be pseudomonotone. Assume that the following conditions hold with respect to some elements $\bar{u} \in Y_0$ and $f \in Z_0$:*

(i) *T is finitely continuous*
(ii) *T is quasibounded with respect to $\bar{u}$*
(iii) *$\langle u-\bar{u}, T(u)-f\rangle > 0$ when $u \in D(T)$ has sufficiently large norm in Y.*
Then $f \in T(D(T))$, i.e., the equation $T(u)=f$ is solvable.

Let Ω be an open and bounded subset in $\mathbb{R}^N$ with the segment property and denote the complementary system of Orlicz-Sobolev spaces by

$$\begin{pmatrix} Y & Z \\ Y_0 & Z_0 \end{pmatrix} = \begin{pmatrix} W_0^m L_M(\Omega) & W^{-m} L_{\overline{M}}(\Omega) \\ W_0^m E_M(\Omega) & W^{-m} E_{\overline{M}}(\Omega) \end{pmatrix}.$$

For this complementary system we have the following generalization.

Theorem 5.2. *Let $T\colon Y_0 \subset D(T) \subset Y \to Z$ belong to class (QM_m). Assume the following conditions hold with respect to some elements $\bar{u} \in Y_0$ and $f \in Z_0$:*
(i) *T is finitely continuous*
(ii) *T is quasibounded with respect to $\bar{u}$*
(iii) *$\langle u-\bar{u}, T(u)-f\rangle \geqslant 0$ when $u \in D(T)$ has sufficiently large norm in Y.*
Then $f \in \overline{T(D(T))}$, i.e., the equation $T(u)=f$ is almost solvable.

P r o o f. Define a mapping $\widehat{T}\colon Y_0 \subset D(\widehat{T}) \subset Y \to Z$ by

$$\widehat{T}(u) = T(u+\bar{u})$$

with $D(\widehat{T}) = D(T) - \bar{u}$. It is straightforward to check that also the mapping $\widehat{T}$ belongs to class (QM_m). Moreover, $\widehat{T}$ satisfies the following conditions:
($\hat{\text{i}}$) $\widehat{T}$ is finitely continuous
($\widehat{\text{ii}}$) $\widehat{T}$ is quasibounded with respect to 0
($\widehat{\text{iii}}$) $\langle u, \widehat{T}(u)-f\rangle \geqslant 0$ when $u \in D(\widehat{T})$ has sufficiently large norm in Y.
Define $J_n\colon D(J_n) \to Z$ by

$$\langle v, J_n(u)\rangle = \tfrac{1}{n} \sum_{|\alpha|=m} \int_\Omega \overline{M}^{-1}\big(M(\tfrac{1}{n} D^\alpha u)\big) D^\alpha v \,\mathrm{d}x \qquad \text{for } v \in Y \tag{5.1}$$

with

$$D(J_n) = \big\{u \in Y \mid \overline{M}^{-1}\big(M(\tfrac{1}{n} D^\alpha u)\big) \in L_{\overline{M}}(\Omega) \text{ for all } |\alpha| = m\big\}.$$

We can apply Theorem 4.2 and 4.3 to conclude that $J_n \in (S_m)$, J_n is finitely continuous and quasibounded with respect to any $\bar{v} \in Y_0$. According to Theorem 3.2, the mapping $T_n = J_n + \widehat{T}$ with $D(T_n) = D(J_n) \cap D(\widehat{T})$ belongs to class (S_m) and is quasibounded with respect to 0. In particular, T_n is pseudomonotone and satisfies the conditions (i) and (ii) of Theorem 5.1. To prove (iii) with respect 0 and f, we note that $\langle u, J_n(u)\rangle > 0$ for all $u \in D(J_n)$ with $u \neq 0$. By $(\widehat{\widehat{\mathrm{iii}}})$,

$$\langle u, T_n(u) - f\rangle = \langle u, \widehat{T}(u) - f\rangle + \langle u, J_n(u)\rangle > 0,$$

when $u \in D(T_n)$ has sufficiently large norm in Y. By Theorem 5.1, there exists $u_n \in D(T_n)$ such that

$$J_n(u_n) + \widehat{T}(u_n) = f$$

for any n. Therefore

$$\langle u_n, \widehat{T}(u_n) - f\rangle = -\langle u_n, J_n(u_n)\rangle < 0 \qquad \text{whenever } u_n \neq 0.$$

In view of $(\widehat{\widehat{\mathrm{iii}}})$, $\{u_n\}$ remains bounded in Y. Consequently, we may conclude from (5.1) that $\|J_n(u_n)\|_Z \to 0$ and $\widehat{T}(u_n) = T(u_n + \bar{u}) \to f$ in Z strongly, when $n \to \infty$. Therefore f belongs to the norm-closure of $T(D(T))$. □

Remark 5.3. Let Ω be an open bounded subset in $\mathbb{R}^N$. To indicate the application of our solvability results we consider a boundary value problem

$$\begin{cases} A^{(1)}u(x) + A^{(0)}u(x) = h(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

where $A^{(1)}$ and $A^{(0)}$ are differential operators in divergence form defined by (4.1) and (4.2), respectively. We assume that the coefficient functions a_α satisfy the conditions (A_1) and (A_2) for all $|\alpha| \leqslant m$ and h is a given function in $E_{\overline{M}}(\Omega)$. We also assume that the conditions (A_4) holds implying the condition (iii) of Theorem 5.1 is true for $\bar{u} = \varphi$ and for any $f \in W^{-m}E_{\overline{M}}(\Omega)$ (see [9]). Applying Theorem 5.1 and 5.2 we obtain following results for the existence of weak solution of (5.2).

(a) If $A^{(1)}$ satisfies (A_3), then (5.2) is solvable for any $h \in E_{\overline{M}}(\Omega)$
(b) If $A^{(1)}$ satisfies $(A_3)_e$, then (5.2) is almost solvable for any $h \in E_{\overline{M}}(\Omega)$
(c) If $A^{(1)}$ satisfies $(A_3)_e$ and $A^{(0)}$ has the form

$$A^{(0)}u(x) = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha a_\alpha(x, u, \nabla u, \ldots, \nabla^{m-1}u),$$

then T_0 and $T_1 + T_0$ are pseudomonotone. Hence (5.2) is solvable for any $h \in E_{\overline{M}}(\Omega)$.

References

[1] *Adams, R.*: Sobolev spaces. Academic Press, New York, 1975.

[2] *Berkovits, J.; Mustonen, V.*: On topological degree for mappings of monotone type. Nonlinear Anal. TMA *10* (1986), 1373–1383.

[3] *Browder, F. E.*: Fixed point theory and nonlinear problems. Bull. Amer. Math. Soc. *9* (1983), 1–39.

[4] *Donaldson, T.*: Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces. J. Differential Equations *10* (1971), 507–528.

[5] *Donaldson, T.; Trudinger, N. S.*: Orlicz-Sobolev spaces and imbedding theorems. J. Functional Analysis *8* (1971), 52–75.

[6] *Gossez, J.-P.*: Nonlinear elliptic boundary value prolems for equations with rapidly (or slowly) increasing coefficients. Trans. Am. Math. Soc. *190* (1974), 163–205.

[7] *Gossez, J.-P.*: Orlicz spaces and nonlinear elliptic boundary value problems. Nonlinear Analysis, Function Spaces and Applications, Teubner-Texte zur Mathematik. 1979, pp. 59–94.

[8] *Gossez, J.-P.*: Some approximation properties in Orlicz-Sobolev spaces. Studia Math. *74* (1982), 17–24.

[9] *Gossez, J.-P.; Mustonen, V.*: Variational inequalities in Orlicz-Sobolev spaces. Nonlinear Anal. *11* (1987), 379–392.

[10] *Hess, P.*: On nonlinear mappings of monotone type with respect to two Banach spaces. J. Math. Pures Appl. *52* (1973), 13–26.

[11] *Hewitt, E.; Stromberg, K.*: Real and abstract analysis. Springer-Verlag, Berlin, 1965.

[12] *Kittilä, A.*: On the topological degree for a class of mappings of monotone type and applications to strongly nonlinear elliptic problems. Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertationes *91* (1994).

[13] *Krasnosel'skii, M.; Rutickii, J.*: Convex functions and Orlicz spaces. P. Noordhoff Ltd., Groningen, 1961.

[14] *Kufner, A.; John, O.; Fučík, S.*: Function spaces. Academia, Praha, 1977.

[15] *Landes, R.*: On Galerkin's method in the existence theory of quasilinear elliptic equations. J. Funct. Anal. *39* (1983), 123–148.

[16] *Landes, R.; Mustonen, V.*: On pseudomonotone operators and nonlinear noncoercive variational problems on unbounded domains. Math. Ann. *248* (1980), 241–246.

[17] *Landes, R.; Mustonen, V.*: Pseudo-monotone mappings in Orlicz-Sobolev spaces and nonlinear boundary value problem on unbounded domains. J. Math. Anal. Appl. *88* (1982), 25–36.

[18] *Leray, J.; Lions, J. L.*: Quelques résultats de Višik sur des problémes elliptiques non linéaires par les méthodes de Minty-Browder. Bul. Soc. Math. France *93* (1965), 97–107.

[19] *Skrypnik, I.*: Nonlinear higher order elliptic equations. Naukova Dumka, Kiev, 1973.

[20] *Tienari, M.*: A degree theory for a class of mappings of monotone type in Orlicz-Sobolev spaces. Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertationes *97* (1994).

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