ON BELLMAN SYSTEMS WITHOUT ZERO ORDER TERM IN THE CONTEXT OF RISK SENSITIVE DIFFERENTIAL GAMES

A. Bensoussan, Paris, J. Frehse, Bonn

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. Bellman systems corresponding to stochastic differential games arising from a cost functional which models risk aspects are considered. Here it leads to diagonal elliptic systems without zero order term so that no simple L^∞-estimate is available.

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1. Introduction

Bellman systems of the type, say

$$
\begin{aligned}
-\frac{1}{2}\Delta u_\nu + \alpha u_\nu &= H_\nu(x, Du),\ \nu = 1, \dots, N \\
u_\nu|_{\partial O} &= 0
\end{aligned}
\tag{1.1}
$$

with certain structure conditions on H and quadratic cost functionals have been studied in order to solve stochastic games.

For instance, in [1] the authors solved the differential game

$$
\begin{aligned}
\mathrm{d}y &= \Big(g(y(t)) + \sum_{\mu=1}^{N} v_\mu(t)\Big)\,\mathrm{d}t + \mathrm{d}w(t), \\
y(0) &= x,\ x \in \mathbb{R}^n,
\end{aligned}
\tag{1.2}
$$

where $v_1(\cdot), \dots, v_N(\cdot)$ are controls at the disposal of N decision makers. In (1.2), $w(t)$ is a Wiener process in $\mathbb{R}^n$, and $y_{x,v} = y(\cdot)$ is the solution of an Ito stochastic differential equation. Let O be an open smooth bounded domain of $\mathbb{R}^n$, and let

$$\tau = \inf\{t;\ y_{x,v}(t) \notin O\} \tag{1.3}$$

be the first exit time of the process $y_{x,v}(t)$ outside O. Using the notation

$$v(t) = (v_\nu(t),\ v^\nu(t)) \tag{1.4}$$

where v^ν represents the vector of all components which are different from v_ν and

$$\bar{v}_\nu(t) = \sum_{\mu \neq \nu} v_\mu(t),\ \ \nu, \mu = 1, \dots, N, \tag{1.5}$$

we consider the cost function of the player ν, given by

$$\begin{aligned} J_\nu(x, v(\cdot)) &= J_\nu(x, v_\nu(\cdot),\ v^\nu(\cdot)) \\ &= \mathbb{E} \int_0^\tau \mathrm{e}^{-\alpha t} \big(f_\nu(y_{x,v}(t)) + \tfrac{1}{2}|v_\nu(t)|^2 + \theta v_\nu(t) \cdot \bar{v}_\nu(t) \big)\,\mathrm{d}t. \end{aligned} \tag{1.6}$$

A Nash point of the functionals $J_\nu(x, v(\cdot))$ is a control $\hat{v}(\cdot)$ such that

$$J_\nu(x, \hat{v}_\nu(\cdot),\ \hat{v}^\nu(\cdot)) \leqslant J_\nu(x, v_\nu(\cdot),\ \hat{v}^\nu(\cdot)),\ \ \nu = 1, \dots, N \tag{1.7}$$

for any admissible control $v(\cdot) = (v_1(\cdot), \dots, v_N(\cdot))$. Defining a function

$$L_\nu(v, p) = \frac{1}{2}|v_\nu|^2 + \theta v_\nu \cdot \bar{v}_\nu + p_\nu \cdot \sum_\mu v_\mu \tag{1.8}$$

where $p = (p_1, \dots, p_N) \in \mathbb{R}^{nN}$, $v = (v_1, \dots, v_N) \in \mathbb{R}^{nN}$ and considering a Nash point $\hat{v}_1(p), \dots, \hat{v}_N(p)$ of the functions (1.8) (the definition is similar to (1.7), but it is pointwise in x), then setting

$$L_\nu(p) = L_\nu(\hat{v}(p), p) \tag{1.9}$$

it is proved that the functions

$$u_\nu(x) = J_\nu(x, \hat{v}(\cdot)) \tag{1.10}$$

are solutions of the system of partial differential equations

$$-\frac{1}{2}\Delta u_\nu - g(\cdot) \cdot Du_\nu + \alpha u_\nu = f_\nu + L_\nu(Du) \tag{1.11}$$

which is of the form (1.1) with

$$H_\nu(x,p) = L_\nu(p) + f_\nu(x) + g(x)\cdot p_\nu. \tag{1.12}$$

Note that the discount factor $\mathrm{e}^{-\alpha t}$ gives the 0-order term αu_ν in the Bellman system (1.11).

This term helps very much in obtaining L^∞-estimates, via Maximum Principle type of argument.

In recent years, there has been a rising interest in taking into consideration *risk* aspects in the cost functions. One convenient way of modelling risk is to consider the cost functions (instead of (1.6))

$$\begin{aligned} J_\nu^\delta(x,v(\cdot)) &= J_\nu^\delta(x,v_\nu(\cdot),v^\nu(\cdot)) \\ &= \frac{1}{\delta}\log\mathbb{E}\exp\delta\Big[\int_0^\tau \Big(f_\nu(y_{x,v}(t)) + \frac{1}{2}|v_\nu(t)|^2 + \theta v_\nu(t)\cdot\bar{v}_\nu(t)\Big)\,\mathrm{d}t\Big] \end{aligned} \tag{1.13}$$

where δ is called the risk factor ($\delta > 0$ represents an aversion to risk, $\delta < 0$ represents an attraction to risk).

Note that in the integral $\int_0^\tau$, there is no discount factor any more.

The reason for omitting the discount factor is that Nash points of functionals of the type (1.13) are amenable to systems of partial differential equations similar to (1.11). Introducing the discount factor leads unfortunately to parabolic systems and not to elliptic ones.

If $\hat{v}(\cdot)$ is a Nash point for (1.13), then

$$u_\nu(x) = J_\nu(x,\hat{v}(\cdot))$$

is a solution of the system

$$-\frac{1}{2}\Delta u_\nu - g(x)\cdot Du_\nu = \frac{\delta}{2}|Du_\nu|^2 + f_\nu(x) + L_\nu(Du) \tag{1.14}$$

and thus we are led to systems of the type

$$\begin{aligned} -\frac{1}{2}\Delta u_\nu &= H_\nu(x,Du) \\ u_\nu|_{\partial O} &= 0. \end{aligned} \tag{1.15}$$

One of the main difficulties is to recover L^∞-estimates. In this note we present some cases where the L^∞-estimate is available. In particular, we show that the drift $g(x)$ can have an influential role in obtaining these estimates.

2.1. Assumptions and model. We consider here the system

$$
\begin{aligned}
-\frac{1}{2}\Delta u_\nu - g(x)\cdot Du_\nu &= H_\nu(x, Du) \\
u_\nu|_{\partial O} &= 0,
\end{aligned} \tag{2.1}
$$

where $H_\nu(x,p)$ are Carathéodory functions, $g \in W^{1,\infty}(O)$ with the following assumptions

$$\sum_\nu H_\nu(x,p) \geqslant -\lambda \qquad \forall x, p, \tag{2.2}$$

$$H_\nu(x,p) \leqslant \lambda_\nu + \lambda_\nu^0 |p_\nu|^2. \tag{2.3}$$

If Γ is an $N \times N$-matrix and if we set

$$H_\nu^\Gamma(x,p) = (\Gamma H)_\nu(x, \Gamma^{-1}p) \tag{2.4}$$

where $H(x,p)$ represents the vector $(H_1(x,p), \ldots, H_N(x,p))$ then we assume that

$$\text{there exists a matrix } \Gamma \text{ such that } H_\nu^\Gamma(x,p) = Q(x,p)\cdot p_\nu + H_\nu^0(x,p) \tag{2.5}$$

with

$$|Q(x,p)| \leqslant k + K|p|, \tag{2.6}$$

$$|H_\nu^0(x,p)| \leqslant k_\nu + K_\nu \sum_{\mu \leqslant \nu} |p_\mu|^2. \tag{2.7}$$

The assumptions (2.5), (2.6), (2.7) represent the *special structure assumption* (note that this special structure may not be available on the original H_ν but only after a linear manipulation represented by the matrix Γ).

An additional smallness condition on the product $\lambda_\nu \lambda_\nu^0$ is assumed, namely

$$4\lambda_\nu\lambda_\nu^0 < k_0 + \inf \operatorname{div} g \tag{2.8}$$

where k_0 is the constant arising in the Poincaré inequality

$$k_0 \int_O \varphi^2 \,\mathrm{d}x \leqslant \int_O |D\varphi|^2 \,\mathrm{d}x \qquad \forall \varphi \in H_0^1(O). \tag{2.9}$$

2.2. Statement of the results.

Theorem 2.1. *Assuming O to be smooth bounded and $H_\nu(x,p)$, Carathéodory functions satisfying* (2.2), (2.3), (2.5), (2.6), (2.7) *as well as* (2.8) *there exists a solution u of* (2.1) *such that $u \in (W^{2,s}(O))^N$, for every $2 \leqslant s < \infty$.*

3. Proof of the L^∞-estimate

We will not give the complete proof of Theorem 2.1, nevertheless, we will show some details how the L^∞-estimates are obtained.

Write

$$\tilde{u} = \sum u_\nu,$$

then adding up the equations (2.1) we have

$$-\frac{1}{2}\Delta\tilde{u} - gD\tilde{u} \geqslant -\lambda. \tag{3.1}$$

For any point ξ of O consider the Green function

$$\begin{gathered} -\frac{1}{2}\Delta G^\xi + \operatorname{div}(gG^\xi) = \delta(x-\xi) \\ G^\xi|_{\partial O} = 0. \end{gathered} \tag{3.2}$$

We test (3.1) with $\tilde{u}^- G^\xi$ obtaining, from the definition of the Green function,

$$\frac{1}{4}\int_O D(\tilde{u}^-)^2 DG^\xi \,\mathrm{d}x + \frac{1}{2}(\tilde{u}^-(\xi))^2 \leqslant \lambda \int_O \tilde{u}^- G^\xi \,\mathrm{d}x. \tag{3.3}$$

Suppose now ξ is a point where $\tilde{u}^-$ reaches a positive maximum (necessarily in O), then we get

$$\|\tilde{u}^-\|_\infty \leqslant 2\lambda \int_O G^\xi \,\mathrm{d}x \leqslant C$$

so that we have proved the first L^∞-estimate

$$\sum u_\nu \geqslant -C. \tag{3.4}$$

Next, we introduce the function $E_\nu = \exp 2\lambda_\nu^0 u_\nu$. We can check from (2.1) and assumption (2.3) that

$$-\frac{1}{2}\Delta E_\nu - gDE_\nu \leqslant 2\lambda_\nu\lambda_\nu^0 E_\nu. \tag{3.5}$$

Testing (3.5) with $(E_\nu - 1)^+$, which vanishes on the boundary, yields

$$\begin{aligned} &\int_O |D(E_\nu-1)^+|^2 \,\mathrm{d}x + \int_O \operatorname{div} g(E_\nu-1)^{+2} \,\mathrm{d}x \\ &\qquad \leqslant 4\lambda_\nu\lambda_\nu^0 \int_O (E_\nu-1)^{+2} \,\mathrm{d}x + 4\lambda_\nu\lambda_\nu^0 \int_O (E_\nu-1)^+ \,\mathrm{d}x \end{aligned}$$

and from Poincaré's inequality we obtain

$$\int\limits_O (k_0 + \operatorname{div} g)(E_\nu - 1)^{+2}\,\mathrm{d}x \leqslant 4\lambda_\nu\lambda_\nu^0 \int\limits_O (E_\nu - 1)^{+2}\,\mathrm{d}x + 4\lambda_\nu\lambda_\nu^0 \int\limits_O (E_\nu - 1)^{+}\,\mathrm{d}x.$$

Thanks to the smallness condition (2.8), we deduce easily

$$\int\limits_O E^2\,\mathrm{d}x \leqslant C. \tag{3.6}$$

Using this knowledge we are going to check that E is in L^∞, without using anymore the smallness condition. For that purpose, we test again (3.5) with $E_\nu G^\xi$, using the Green function (3.2). We obtain

$$\frac{1}{2}\big(E_\nu^2(\xi) - 1\big) \leqslant 2\lambda_\nu\lambda_\nu^0 \int\limits_O E_\nu^2 G^\xi\,\mathrm{d}x, \tag{3.7}$$

hence, taking ξ as a point of maximum of E_ν^2,

$$\|E_\nu^2\|_\infty \leqslant 1 + 4\lambda_\nu\lambda_\nu^0 L^2 \int\limits_O G^\xi\,\mathrm{d}x + 4\lambda_\nu\lambda_\nu^0 \|E_\nu^2\|_\infty \int_{\{E_\nu > L\}} G^\xi\,\mathrm{d}x \qquad \forall L. \tag{3.8}$$

But from (3.6) one has

$$\operatorname{Meas}\{E_\nu > L\} \leqslant \frac{C}{L^2},$$

and thus

$$\int_{\{E_\nu > L\}} G^\xi\,\mathrm{d}x \leqslant C\|G^\xi\|_{L^q}\frac{1}{L^{2q}}.$$

So by picking L sufficiently large, we can make the coefficient of $\|E_\nu^2\|_\infty$ on the right hand side of (3.8) as small as we wish, in particular, strictly smaller than 1. So (3.8) yields an estimate on $\|E_\nu^2\|_\infty$.

Remark 3.1. We see from (2.8) that, if $\inf \operatorname{div} g$ is large, the limitation on the product $\lambda_\nu\lambda_\nu^0$ is not so restrictive. The role of the drift, as a way to soften some restrictions, has already been investigated by H. Nagaï [2]. Furthermore, if $\lambda_\nu \leqslant 0$, λ_ν^0 may be "large".

References

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Authors' addresses: *A. Bensoussan*, CNES, 2, Place Maurice Quentin, 75039 Paris CEDEX 01, France; *J. Frehse*, Institut für Angewandte Mathematik, Universität Bonn, Beringstr. 6, 53115 Bonn, Germany, e-mail: aaa@iam.uni-bonn.de.