

ON MULTIFUNCTIONS WITH CLOSED GRAPHS

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Abstract. The set of points of upper semicontinuity of multi-valued mappings with a closed graph is studied. A topology on the space of multi-valued mappings with a closed graph is introduced.

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1. INTRODUCTION

In what follows let X, Y be Hausdorff topological spaces. A (multi-valued) mapping $F: Y \rightarrow X$ is said to be upper semicontinuous (USC) at y_0 , if for every open set O which includes $F(y_0)$ there exists a neighbourhood V of y_0 such that $O \supset F(V) (= \bigcup\{F(y) : y \in V\})$.

The outer part of F at y_0 is the mapping $\tilde{F}: Y \rightarrow X$ given by $\tilde{F}(y) = F(y) \setminus F(y_0)$ (see [4]). By the (outer) active boundary of F at y_0 ($\text{Frac } F(y_0)$) we understand

$$\text{Frac } F(y_0) = \bigcap \{\text{Cl } \tilde{F}(V) : V \in \mathcal{B}(y_0)\},$$

where $\mathcal{B}(y_0)$ stands for a neighbourhood base at y_0 (see [3]).

A mapping $F: Y \rightarrow X$ is said to be c -upper semicontinuous at y_0 [1] if for every open set V containing $F(y_0)$ and having a compact complement, there is an open neighbourhood U of y_0 such that $F(U) \subset V$. We say that $F: Y \rightarrow X$ is c -upper semicontinuous if it is c -upper semicontinuous at every $y \in Y$.

As usual, we say that $F: Y \rightarrow X$ has a closed graph if $G(F) = \{(y, x) : x \in F(y); y \in Y\}$ is a closed subset of $Y \times X$ with the product topology. Of course if $F: Y \rightarrow X$ has a closed graph then F is c -upper semicontinuous and also $\text{Frac } F(y) \subset$

$F(y)$ for every $y \in Y$. We will see later that in general the opposite is not true. Let us mention here also papers [7], [8], [9], [11], which deal with the above properties of multifunctions.

Let (X, \mathcal{U}) be a uniform space. We say that a multi-valued mapping F is totally bounded at y_0 , if to every $U \in \mathcal{U}$ there corresponds a neighbourhood V of y_0 and a finite set C such that $F(V \setminus \{y_0\}) \subset U[C]$, where $U[C] = \bigcup \{U[c] : c \in C\}$.

Combining Theorems 8.1 and 10.1 from [4] we obtain the following result:

Theorem A. *Let Y be first countable at y_0 , let (X, \mathcal{U}) be a complete uniform space and $F: Y \rightarrow X$ a multi-valued mapping. Suppose that $F(y_0)$ is closed. Then F is USC at y_0 if and only if (i) $\text{Frac } F(y_0) \subset F(y_0)$ and (ii) the outer part of F at y_0 is totally bounded at y_0 .*

1. It is easy to see that one implication of the above theorem holds under weaker conditions.

Lemma 1.1. *Let (X, \mathcal{U}) be a complete uniform space, Y a topological space and $F: Y \rightarrow X$ a multi-valued mapping. If $\text{Frac } F(y_0) \subset F(y_0)$ and the outer part of F at y_0 is totally bounded at y_0 then F is USC at y_0 .*

Proof. Suppose F is not USC at y_0 . There is an open set $O \subset X$ such that for every neighbourhood V of y_0 we have $F(V) \setminus O \neq \emptyset$. Let $\mathcal{B}(y_0)$ denote the family of all open neighbourhoods of y_0 . For every $V \in \mathcal{B}(y_0)$ choose $x_V \in F(V) \setminus O$. Now the set $\bigcap_{V \in \mathcal{B}(y_0)} \overline{\{x_G : G \subset V\}}$ is nonempty, since (X, \mathcal{U}) is complete and for every $U \in \mathcal{U}$ there is $V \in \mathcal{B}(y_0)$ such that $\overline{\{x_G : G \subset V\}} \subset U[C]$, where C is a finite set in X . □

The completeness of X in the previous lemma is essential.

Proposition 1.2. *Suppose (X, \mathcal{U}) is not complete. Then there are a topological space Y , a point $y_0 \in Y$ and a mapping $F: Y \rightarrow X$ such that*

- (i) $\text{Frac } F(y_0) \subset F(y_0)$,
- (ii) the outer part of F at y_0 is totally bounded at y_0 ,
- (iii) F is not USC at y_0 .

Proof. There is a Cauchy net $\{x_\alpha : \alpha \in A\}$ with no cluster point in X . Let X^* be a completion of X . There is a point $x \in X^*$ such that $\{x_\alpha : \alpha \in A\}$ converges to x . Put $Y = \{x_\alpha : \alpha \in A\} \cup \{x\}$ and for each $\alpha \in A$ put $V_\alpha = \{x_\beta : \beta \geq \alpha\} \cup \{x\}$. The system $\tau = \{K \subset Y : x \notin K\} \cup \{V_\alpha : \alpha \in A\}$ is a base of a topology \mathcal{G} . Define a mapping $F: (Y, \mathcal{G}) \rightarrow X$ as follows: $F(x_\alpha) = x_\alpha$ for each $\alpha \in A$ and $F(x) = x_0$, where x_0 is a point from X . It is easy to see that F is totally bounded at x . (Let

U be a symmetric element from \mathcal{U} . There is $\alpha \in A$ such that $(x_\alpha, x_\beta) \in U$ for each $\beta \geq \alpha$. Thus V_α is a neighbourhood of x for which $F(V_\alpha \setminus \{x\}) \subset U[x_\alpha]$.

Since $\{x_\alpha : \alpha \in A\}$ has no cluster point in X , $\text{Frac } F(x) = \emptyset$. But F is not upper semicontinuous. (There are an open set V in X and $\alpha \in A$ such that $x_0 \in V$ and $x_\beta \notin V$ for each $\beta \geq \alpha$. Thus the set $\{z \in Y : F(z) \subset V\}$ is not open in Y). \square

Proposition 1.3. *Let X, Y be topological spaces. Then $F: Y \rightarrow X$ has a closed graph if and only if for every $y \in Y$ the following conditions are satisfied:*

- (i) $\text{Frac } F(y) \subset F(y)$,
- (ii) $F(y)$ is a closed set in X .

Proof. Of course the “only if” implication is trivial. To prove the other one, we will suppose that (i) and (ii) are satisfied for all $y \in Y$ and F does not have a closed graph. Thus there is $(y_0, x) \in \overline{G(F)} \setminus G(F)$. We claim that $x \in \text{Frac } F(y_0)$. Let $V \in \mathcal{B}(y_0)$ and $G \in \mathcal{B}(x)$ with $G \subset X \setminus F(y_0)$, where $\mathcal{B}(y_0)$, and $\mathcal{B}(x)$ are the families of all neighbourhoods of y_0, x , respectively. Since $(y_0, x) \in \overline{G(F)}$ we have $(V \times G) \cap G(F) \neq \emptyset$. Choose $(v, g) \in V \times G$ and $(v, g) \in G(F)$; $g \in G$ implies that $g \notin F(y_0)$, i.e. $g \in \tilde{F}(v)$. Thus $x \in \text{Frac } F(y_0)$ and, by the assumption, $x \in F(y_0)$, a contradiction. \square

Proposition 1.4. *Let X a locally compact space and Y be a topological space. Let $F: Y \rightarrow X$ be c -upper semicontinuous with closed values. Then F has a closed graph.*

Proposition 1.5. *Let X, Y be first countable topological spaces. Let $F: Y \rightarrow X$ be c -upper semicontinuous with closed values. Then F has a closed graph.*

Proof. Suppose the implication is not true. Thus there is $(y, x) \in \overline{G(F)} \setminus G(F)$; i.e. $x \notin F(y)$. Since X, Y are first countable, there are sequences $\{y_n\}_n$ in Y and $\{x_n\}_n$ in X such that $\{(y_n, x_n)\}_n$ converges to (y, x) and $(y_n, x_n) \in G(F)$ for every $n \in \mathbb{N}$. The openness of $X \setminus F(y)$ and the convergence of $\{x_n\}_n$ to x imply that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin F(y)$ for every $n \geq n_0$. Now $K = \{x_n : n \geq n_0\} \cup \{x\}$ is a compact set in X with $F(y) \cap K = \emptyset$. The c -upper semicontinuity of F at y implies that there is an open neighbourhood V of y with $F(V) \cap K = \emptyset$, a contradiction. \square

The following example shows that the first countability of X in the above proposition is essential.

Example 1.6. *For every $n \in \mathbb{N}$ let $\{y_k^n\}_k$ be a sequence of different points from $(1/(n+1), 1/n)$ convergent to $1/(n+1)$. Consider $Y = \{0\} \cup \{y_k^n : k, n \in \mathbb{N}\}$ and let τ be the inherited euclidean topology on Y . Let further \mathcal{G} be the following topology*

on Y : all points y_k^n are isolated. For every $n \in \mathbb{N}$ put $E_n = \{y_k^n : k \in \mathbb{N}\}$. The basic neighbourhoods of 0 in \mathcal{G} are sets obtained from Y by removing finitely many E_n 's and a finite number of points in all the remaining E_n 's. It is easy to verify that the function $f: (Y, \tau) \rightarrow (Y, \mathcal{G})$ defined by $f(0) = 1$ and $f(y) = y$ for all $y \neq 0$ is c -upper semicontinuous, since compact sets in (Y, \mathcal{G}) are finite, but the graph of f is not closed since $(0, 0) \in \overline{G(f)} \setminus G(f)$.

2. Let (X, \mathcal{U}) be a uniform space. The smallest cardinal number of the form $|\mathcal{B}|$, where \mathcal{B} is a base for \mathcal{U} , is called the weight of the uniformity \mathcal{U} and is denoted by $w(\mathcal{U})$. ($|\mathcal{B}|$ denotes the cardinality of \mathcal{B}).

A system is a synonym for an indexed family. If m is a cardinal number, then an m -system is a system whose index set has cardinality m .

A subset G of a space Y is said to be a $G(m)$ -subset of Y , if it is the intersection of an open m -system in Y (see [6]), where an open family of a space Y is a family consisting of open subsets of Y .

Now we state the main theorem:

Theorem 2.1. *Let (X, \mathcal{U}) a complete uniform space and Y be a topological space. Let $F: Y \rightarrow X$ be a multi-valued mapping with totally bounded values and such that $\text{Frac } F(y) \subset F(y)$ for every $y \in Y$. Then the set of points of upper semicontinuity of F is a (possibly empty) $G(m)$ -subset of Y , where $m = w(\mathcal{U})$.*

Proof. Put $\Omega = \{y \in Y : F \text{ is totally bounded at } y\}$. By Lemma 1.1, F is USC at each point from Ω . Now let $y \in Y$ be such that F is USC at y . Let $U \in \mathcal{U}$. There is an element U_1 from \mathcal{U} such that $U_1 \odot U_1 \subset U$. The upper semicontinuity of F at y implies that there is a neighbourhood V of y such that $F(V) \subset U_1[F(y)]$. The total boundedness of $F(y)$ implies that there is a finite subset K of X such that $F(y) \subset U_1[K]$, thus $F(V) \subset U[K]$, i.e. $y \in \Omega$.

Now let \mathcal{B} be a base for \mathcal{U} such that $|\mathcal{B}| = w(\mathcal{U})$. For each $B \in \mathcal{B}$ put $G_B = \{y \in Y : \text{there are } V \in \mathcal{B}(y) \text{ and a finite set } K \subset X \text{ such that } F(V) \subset B[K]\}$, where $\mathcal{B}(y)$ denotes a family of all neighbourhoods of y . It is easy to verify that G_B is open for each $B \in \mathcal{B}$. Thus $\Omega = \bigcap \{G_B : B \in \mathcal{B}\}$, i.e. the set of points of upper semicontinuity of F is a $G(m)$ -subset of Y , where $m = w(\mathcal{U})$. \square

Corollary 2.2. *Let (X, ρ) be a complete metric space and Y a topological space. Let $F: Y \rightarrow X$ be a multi-valued mapping with totally bounded values and such that $\text{Frac } F(y) \subset F(y)$ for every $y \in Y$. Then the set of points of upper semicontinuity of F is a (possibly empty) G_δ -subset of Y .*

Recall that a (completely regular) space is metric topologically complete if its topology admits a complete metric.

Theorem 2.3. *Let (X, ϱ) be a metric space. X is metric topologically complete if and only if for each topological space Y and for each compact-valued mapping $F: Y \rightarrow X$ with a closed graph the set of points of upper semicontinuity of F is a G_δ -subset of Y .*

Proof. \Rightarrow Suppose that a metric space (X, ϱ) is metric topologically complete and $F: Y \rightarrow X$ is a compact-valued mapping with a closed graph. There is a complete metric d in X topologically equivalent to ϱ . It is easy to see that the multi-valued mapping $F: Y \rightarrow (X, d)$ satisfies the assumptions of Corollary 2.2. Thus the set of points of the upper semicontinuity of $F: Y \rightarrow (X, \varrho)$ is a G_δ -subset of Y .

\Leftarrow Let (X^*, ϱ^*) be a completion of (X, ϱ) . Put $Y = (X^*, \varrho^*)$. Choose $y_0 \in X$ and define the multi-valued mapping $F: Y \rightarrow X$ as follows: $F(y) = \{y, y_0\}$ if $y \in X$ and $F(y) = \{y_0\}$ otherwise.

Then the set of points of upper semicontinuity of F is the set X . (Let $y \notin X$. There are open sets U, V in Y such that $y \in U$, $y_0 \in V$ and $U \cap V = \emptyset$. For each $z \in U \cap X$, we have $F(z) \cap (X \setminus V) \neq \emptyset$, i.e. F is not USC at $y \notin X$). It is easy to verify that F is a compact-valued mapping with a closed graph. By assumption X is a G_δ -subset of $Y = (X^*, \varrho^*)$, i.e. X is metric topologically complete. \square

3. In this part we assume that the range space of multi-valued mappings possesses a complete m -system of open coverings.

A centered family is a family of sets having the finite intersection property.

Definition 3.1 (See [6]). A system $\{\mathcal{B}_i: i \in I\}$ of open coverings of a space X is said to be complete if the following condition is satisfied: If \mathcal{U} is an open centered family in X such that $\mathcal{U} \cap \mathcal{B}_i \neq \emptyset$ for each $i \in I$ then $\bigcap_{U \in \mathcal{U}} \overline{U} \neq \emptyset$.

It follows from Theorem 2.8 in [6] that if X is a Čech-complete space then X possesses a complete countable system of open coverings of X .

Proposition 3.2 (See [6]). *Let $\{\mathcal{B}_i: i \in I\}$ be a complete system of open coverings of a regular space X . Suppose that \mathcal{M} is a centered family of subsets of X such that for each $i \in I$ there exists an $M \in \mathcal{M}$ and a finite subfamily \mathcal{U}_i of \mathcal{B}_i which covers M . Then $\bigcap_{M \in \mathcal{M}} \overline{M} \neq \emptyset$.*

Theorem 3.3. *Let X, Y be topological spaces and let X be a regular space with a complete m -system of open coverings. Let $F: Y \rightarrow X$ be a compact-valued mapping with a closed graph. The set of points of upper semicontinuity of F is a (possibly empty) $G(m)$ -subset of Y .*

Proof. First denote by $\Omega(F)$ the set of points of Y at which F is upper semicontinuous. Let $\{\mathcal{B}_i: i \in I\}$ be a complete m -system of open coverings of X .

For each $i \in I$ put $G_i = \{y \in Y : \text{there are } V \in \mathcal{B}(y) \text{ and a finite subfamily } \mathcal{U}_i \text{ of } \mathcal{B}_i \text{ which covers } F(V)\}$.

Then of course $G(i)$ is open for every $i \in I$. Now we prove that $\Omega(F) = \bigcap_{i \in I} G(i)$. Let $y \in \Omega(F)$. Let $i \in I$. The compactness of $F(y)$ implies that there is a finite subfamily \mathcal{U}_i of \mathcal{B}_i such that $F(y) \subset \bigcup_{U \in \mathcal{U}_i} U$. The upper semicontinuity of F at y implies that there is $V \in \mathcal{B}(y)$ such that $F(V) \subset \bigcup_{U \in \mathcal{U}_i} U$; i.e. $y \in G_i$. Thus we have proved that $\Omega(F) \subset \bigcap_{i \in I} G(i)$.

Now let $y \in \bigcap_{i \in I} G(i)$ and suppose that F is not upper semicontinuous at y . There is an open set H in Y such that $F(y) \subset H$ and $F(V) \cap (X \setminus H) \neq \emptyset$ for every $V \in \mathcal{B}(y)$.

For every $V \in \mathcal{B}(y)$ choose $y_V \in V$ and $x_V \in F(y_V) \setminus H$. For every $V \in \mathcal{B}(y)$ put $M(V) = \overline{\{x_U : U \subset V\}}$. By Proposition 3.2 we have $\bigcap_{V \in \mathcal{B}(y)} M(V) \neq \emptyset$; choose a point x from this intersection. Then of course $x \notin H$ and it is easy to verify that $(y, x) \in \overline{G(F)}$; i.e. $(y, x) \in G(F)$, a contradiction. □

Corollary 3.4 (See [10]). *Let X, Y be topological spaces and let X be Čech-complete. Let $F: Y \rightarrow X$ be a compact-valued mapping with a closed graph. Then the set of points of upper semicontinuity of F is a G_δ subset of Y .*

Corollary 3.5. *Let X, Y be topological spaces and let X be locally compact. Let $F: Y \rightarrow X$ be a compact-valued mapping with a closed graph. Then the set of points of upper semicontinuity of F is open.*

Notice that even if Y is compact and X locally compact hemicompact, the set of points of upper semicontinuity of a compact-valued mapping $F: Y \rightarrow X$ with a closed graph can be empty. (See Example 2 in [1]).

4. Denote by $M(Y, X)$ the space of all closed-valued mappings F from Y to X such that $F(y) \neq \emptyset$ for every $y \in Y$. If we equip the space of all nonempty closed subsets $CL(X)$ of X with a uniformity, we can consider the uniform topology on compacta on $M(Y, X)$. Of course we have many possibilities to do this; if X is a metric space, we can take the Hausdorff metric on $CL(X)$, or a uniformity of the Wijsman topology, if X is a locally compact space, then also the Fell topology is uniformizable [2]. In this paper we will consider a uniformity of the Fell topology.

For a subset A of X put

$$A^- = \{B \in CL(X) : B \cap A \neq \emptyset\} \text{ and } A^+ = \{B \in CL(X) : B \subset A\}.$$

Definition 4.1 (See [2]). Let X be a Hausdorff topological space. The Fell topology τ_F on $CL(X)$ has as a subbase all sets of the form V^- , where V is a nonempty open subset of X , plus all sets of the form W^+ , where W is a nonempty open subset of X with a compact complement.

It is known [2] that if X is a locally compact space, then the Fell topology on $CL(X)$ is uniformizable and the following family of sets forms a base for a uniformity \mathcal{F} compatible with the Fell topology on $CL(X)$. Let \mathcal{U} be a compatible uniformity on X and let $K(X)$ stand for the family of all nonempty compact subsets of X . For $K \in K(X)$ and $U \in \mathcal{U}$, write

$$[K, U] = \{(A_1, A_2) \in CL(X) \times CL(X) : A_1 \cap K \subset U[A_2] \text{ and } A_2 \cap K \subset U[A_1]\}.$$

Then the family $\mathcal{F} = \{[K, U] : K \in K(X), U \in \mathcal{U}\}$ forms a base for a uniformity compatible with the Fell topology on $CL(X)$.

Now the basic open sets for the uniform topology on compacta τ_K on $M(Y, X)$ are the sets

$$\langle F, A, [K, U] \rangle = \{G \in M(Y, X) : (G(y), F(y)) \in [K, U] \text{ for all } y \in A\},$$

where $F \in M(Y, X)$, $A \in K(Y)$, $K \in K(X)$, $U \in \mathcal{U}$.

Theorem 4.2. *Let X, Y be locally compact Hausdorff topological spaces. Then the space $U(Y, X)$ of multi-valued mappings with closed graphs is a closed subset of $(M(Y, X), \tau_K)$.*

Proof. We show that $M(Y, X) \setminus U(Y, X)$ is open in τ_K . Since X is a locally compact Hausdorff topological space, by Proposition 1.4 the notion of a multi-valued mapping with a closed graph is equivalent to the notion of a c -upper semicontinuous multifunction.

Let $F \in M(Y, X) \setminus U(Y, X)$. There is a point $p \in Y$ such that F is not c -upper semicontinuous at p . Let A be a compact set in Y with $p \in \text{Int}A$. There is $K \in K(X)$ such that $F(p) \in (K^c)^+$ and $F(V) \cap K \neq \emptyset$ for every $V \in \mathcal{B}(p)$, where $\mathcal{B}(p)$ stands for the family of all neighbourhoods of p .

For every $V \in \mathcal{B}(p)$ such that $V \subset A$ choose $y_V \in V$ and $x_V \in F(y_V) \cap K$. The compactness of K implies that there is a cluster point $x \in K$ of $\{x_V : V \in \mathcal{B}(p), V \subset A\}$. Let $H \in \mathcal{U}$ be such that $H[F(p)] \cap H[K] = \emptyset$ and $\overline{H[K]}$ is compact. Let $H_1 \in \mathcal{U}$ be such that $H_1 \circ H_1 \subset H$. We claim that the set $L = \{G \in M(Y, X) : (G(y), F(y)) \in [\overline{H_1[K]}, H_1] \text{ for every } y \in A\} \cap \{G \in M(Y, X) : (G(y), F(y)) \in [K, H_1] \text{ for every } y \in A\}$ is a subset of $M(Y, X) \setminus U(Y, X)$.

Let $G \in L$. Then $G(p) \cap \overline{H_1[K]} = \emptyset$ and for each $V \in \mathcal{B}(p)$ with $V \subset A$ there is $z_v \in G(y_V) \cap H_1[K]$; i.e. G is not c -upper semicontinuous at p . \square

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