BIFURCATIONS FOR A PROBLEM WITH JUMPING NONLINEARITIES

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Abstract. A bifurcation problem for the equation

$$\Delta u + \lambda u - \alpha u^{+} + \beta u^{-} + g(\lambda, u) = 0$$

in a bounded domain in $\mathbb{R}^N$ with mixed boundary conditions, given nonnegative functions $\alpha, \beta \in L_\infty$ and a small perturbation g is considered. The existence of a global bifurcation between two given simple eigenvalues $\lambda^{(1)}, \lambda^{(2)}$ of the Laplacian is proved under some assumptions about the supports of the functions α, β. These assumptions are given by the character of the eigenfunctions of the Laplacian corresponding to $\lambda^{(1)}, \lambda^{(2)}$.

Keywords: nonlinearizable elliptic equations, jumping nonlinearities, global bifurcation, half-eigenvalue

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Introduction

Consider a boundary value problem

$$\text{(BP)} \qquad \begin{aligned} \Delta u + \lambda u - \alpha u^{+} + \beta u^{-} + g(\lambda, u) = 0 \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D, \ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N. \end{aligned}$$

We assume that Ω is a bounded domain in $\mathbb{R}^N$ with a Lipschitzian boundary $\partial\Omega$, Γ_D, Γ_N are disjoint subsets of $\partial\Omega$, $\operatorname{meas}(\partial\Omega \setminus (\Gamma_D \cup \Gamma_N)) = 0$, $\operatorname{meas} \Gamma_D > 0$. Further,

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$\alpha, \beta \in L_\infty(\Omega)$ are nonnegative functions, λ is a real bifurcation parameter, u^+ and u^- denotes the positive and the negative part of u, respectively (i.e. $u = u^+ - u^-$). Finally, $g\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a small perturbation satisfying standard growth conditions, that means

$$\text{(SP)} \qquad \lim_{|\xi|\to 0} \frac{g(\lambda,\xi)}{|\xi|} = 0 \text{ uniformly on compact } \lambda\text{-intervals,}$$

$$\text{(GC)} \qquad \begin{cases} |g(\lambda,\xi)| \leqslant C_\lambda(1+|\xi|^{q-1}) \text{ for all } \xi \in \mathbb{R} \text{ with } C_\lambda > 0, \\ C_\lambda \text{ bounded on compact } \lambda\text{-intervals, and } q \geqslant 1 \\ \text{ or } 1 \leqslant q < \frac{2N}{N-2} \text{ if } N \leqslant 2 \text{ or } N \geqslant 3, \text{ respectively.} \end{cases}$$

We will show that if $\lambda^{(1)}, \lambda^{(2)}$ are two different simple eigenvalues of the problem

$$\text{(EP)} \qquad \begin{aligned} \Delta u + \lambda u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \end{aligned}$$

then there is a global bifurcation of nontrivial solutions to (BP) between $\lambda^{(1)}$ and $\lambda^{(2)}$ under some assumptions about the supports of the functions α, β. These assumptions will be given by the character of the eigenfunctions of (EP) corresponding to $\lambda^{(1)}, \lambda^{(2)}$. Notice that it will be essential that α and β have supports in disjoint regions. The eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$ need not be immediately subsequent. A similar result for variational inequalities (the existence of bifurcation points of a variational inequality lying between certain eigenvalues of the corresponding equation) was proved by the second author in [9] on the basis of the penalty method and generalized by P. Quittner [13] by using a simpler approach based on a jump of the Leray-Schauder degree of a suitable mapping. We will use the same approach for our proof.

Our assumptions about supports of α, β can seem unusual. In fact our aim is to show what kind of results can be obtained by applying in a simple natural way the techniques developed for variational inequalities to problems with jumping nonlinearities. Our problem can be understood as a penalty problem corresponding to a variational inequality on the cone K introduced in Notation 2.3 below (cf. [9]) and at that moment our assumptions are quite natural.

The main result (Theorem 1.1) is not formulated in the whole possible generality. For instance, an analogue for the case of multiple eigenvalues $\lambda^{(1)}$, $\lambda^{(2)}$ can be proved and the Laplacian can be replaced by a general elliptic operator.

The problem (BP) for one-dimensional case was studied in [2], [3] and [16]. Relations to these results are explained in Remark 1.3. A certain relation to the Fučík spectrum is also mentioned (Remark 1.2).

Let us note that the terms αu^+ and βu^- can describe a sink and a source in the region where $\alpha > 0$ and $\beta > 0$ which is switched on at a given x only if $u(x) > 0$

and $u(x) < 0$, respectively. From the point of view of applications, a more natural situation is when the threshold value at which the switching on/of takes place is a positive number but can be shifted to zero.

In [5], the corresponding result for reaction-diffusion systems with jumping nonlinearities is given. This paper belongs to a series of results concerning bifurcation of spatial patterns in reaction-diffusion systems with unilateral boundary conditions—see e.g. [6] for a survey of recent results of this type.

1. Main results

Solutions of all problems considered will be understood in the weak sense. Denote $\mathbb{V} = \{u \in W^{1,2}(\Omega);\ u = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}$ and introduce the inner product

$$\langle u, v\rangle = \int_\Omega \sum_{i=1}^N u_{x_i} v_{x_i}\,\mathrm{d}x$$

generating a norm $\|\cdot\|$ equivalent in $\mathbb{V}$ to the standard $W^{1,2}(\Omega)$ norm under our assumption $\operatorname{meas}\Gamma_D > 0$. Define operators $A, P, N\colon\ \mathbb{V} \to \mathbb{V}$ and $G\colon\ \mathbb{R}\times\mathbb{V} \to \mathbb{V}$ as follows:

$$\langle Au, \varphi\rangle = \int_\Omega u(x)\varphi(x)\,\mathrm{d}x,\ \ \langle G(\lambda, u), \varphi\rangle = \int_\Omega g(\lambda, u(x))\varphi(x)\,\mathrm{d}x,$$
$$\langle P(u), \varphi\rangle = \int_\Omega \alpha(x)u^+(x)\varphi(x)\,\mathrm{d}x,\ \ \langle N(u), \varphi\rangle = \int_\Omega \beta(x)u^-(x)\varphi(x)\,\mathrm{d}x$$
$$\text{for all } u, \varphi \in \mathbb{V}.$$

Remark 1.1. The operator A is linear, symmetric and positive (i.e. $\langle Au, u\rangle > 0$ for all $u \in \mathbb{V},\ \|u\| \neq 0$), the operators P, N are positively homogeneous (i.e. $P(tu) = tP(u),\ N(tu) = tN(u)$ for every $t > 0,\ u \in \mathbb{V}$). It follows from the compactness of the embedding $W^{1,2}(\Omega) \subset L_2(\Omega)$ that the operators A, P, N are completely continuous. The assumption (GC) together with the compactness of the embedding $W^{1,2}(\Omega) \subset L_q(\Omega)$ and the Nemyckii theorem imply that also G is completely continuous. Further,

$$\langle P(u), u\rangle \geqslant 0,\ \ \langle N(u), u\rangle \leqslant 0 \text{ for every } u \in \mathbb{V}, \tag{1.1}$$

$$\lim_{\|u\|\to 0} \frac{G(\lambda, u)}{\|u\|} \to 0 \text{ uniformly on compact } \lambda\text{-intervals} \tag{1.2}$$

(see Appendix in [10] for the detailed proof of the last assertion).

The equations

$$u - \lambda Au + P(u) - N(u) - G(\lambda, u) = 0 \tag{1.3}$$

and

$$u - \lambda Au = 0 \tag{1.4}$$

are weak formulations of (BP) and (EP), respectively. We will use also the "homogenized" equation

$$u - \lambda Au + P(u) - N(u) = 0. \tag{1.5}$$

Of course, the equation (1.5) is nonlinear again, our problem cannot be linearized.

N o t a t i o n 1.1. Let us denote

$\sigma = \{\lambda \in \mathbb{R};\ \exists\, u \in \mathbb{V}, \|u\| \neq 0, (1.4) \text{ holds}\}$—the set of all eigenvalues of (EP),

$\sigma_J = \{\lambda \in \mathbb{R};\ \exists\, u \in \mathbb{V}, \|u\| \neq 0, (1.5) \text{ holds}\}$—the set of all eigenvalues (or half-eigenvalues—see Remark 1.3) of the problem

$$\begin{aligned} &\Delta u + \lambda u - \alpha u^+ + \beta u^- = 0 \text{ in } \Omega \\ &u = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N. \end{aligned} \tag{1.6}$$

A parameter λ_0 is called a *bifurcation point* of the problem (BP) (in the weak sense), i.e. of (1.3), if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times \mathbb{V}$ satisfying (1.3) such that $0 \neq \|u_n\| \to 0$, $\lambda_n \to \lambda_0$. However, Theorem 1.1 below ensures a bifurcation in a stronger sense.

For any $a < b$, let us denote by $\mathcal{S}_{a,b}$ the component of the set

$$\overline{\{(\lambda, u) \in \mathbb{R} \times \mathbb{V};\ \|u\| \neq 0, (1.3) \text{ holds}\}} \cup ([a, b] \times \{0\})$$

containing $[a, b] \times \{0\}$.

O b s e r v a t i o n 1.1. Analogously to the case of standard linearizable problems, if $\lambda_0 \in \mathbb{R}$ is a bifurcation point of the problem (BP) then $\lambda_0 \in \sigma_J$.

Theorem 1.1. *Let $\lambda^{(1)}, \lambda^{(2)} \in \sigma$ be simple, $\lambda^{(1)} < \lambda^{(2)}$, let* (SP), (GC) *hold. Assume that eigenfunctions $u^{(1)}$ and $u^{(2)}$ of* (EP) *corresponding to $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively, can be chosen such that there are measurable subsets Ω_{0+}, Ω_{0-} satisfying*

$$\begin{cases} \operatorname{meas}(\Omega_{0+} \cup \Omega_{0-}) > 0, \\ \alpha(x) > 0 \text{ a.e. in } \Omega_{0-},\ \alpha(x) = 0 \text{ a.e. in } \Omega \setminus \Omega_{0-}, \\ \beta(x) > 0 \text{ a.e. in } \Omega_{0+},\ \beta(x) = 0 \text{ a.e. in } \Omega \setminus \Omega_{0+}, \end{cases} \tag{1.7}$$

$$\overline{\Omega_{0+}} \subset \Omega_{j+},\ \overline{\Omega_{0-}} \subset \Omega_{j-},\ j = 1, 2, \tag{1.8}$$

where $\Omega_{j+} = \{x \in \Omega;\ u^{(j)}(x) > 0\}$, $\Omega_{j-} = \{x \in \Omega;\ u^{(j)}(x) < 0\}$ *for* $j = 1, 2$. *Then there exist* a, b *such that* $\lambda^{(1)} < a < b < \lambda^{(2)}$ *and at least one of the following conditions is fulfilled:*

(i) $\mathcal{S}_{a,b}$ *is unbounded in* $\mathbb{R} \times \mathbb{V}$,

(ii) $\mathcal{S}_{a,b} \cap [(\mathbb{R} \setminus (\lambda^{(1)}, \lambda^{(2)})) \times \{0\}] \neq \emptyset$.

In particular, there exists at least one bifurcation point $\lambda^{(0)} \in (\lambda^{(1)}, \lambda^{(2)}) \cap \sigma_J$ *of* (BP).

Let us emphasize that $\lambda^{(1)}, \lambda^{(2)}$ need not be immediately subsequent eigenvalues. Further, let us note that the assumption (1.8) implies $u^{(j)}(x) \geqslant \varepsilon$ in $\overline{\Omega_{0+}}$, $u^{(j)}(x) \leqslant -\varepsilon$ in $\overline{\Omega_{0-}}$ (with some $\varepsilon > 0$). This means in the case $N = 1$ and $N \geqslant 2$ that $u^{(j)}$ lies in the interior or in a certain pseudointerior, respectively, of the cone K defined in Notation 2.3 below. (In the terminology introduced in [9], $\lambda^{(1)}, \lambda^{(2)}$ are interior eigenvalues.) The importance of such eigenvalues in connection with variational inequalities was shown in [9] for the cones with nonempty interior and in [13], [14] for the general case. Clearly $\lambda^{(1)}, \lambda^{(2)} \in \sigma \cap \sigma_J$ because $u^{(j)}$ satisfy also (1.6) under the assumptions of Theorem 1.1. Moreover, it can be shown by using Lemmas 2.2, 2.5, 2.6 below that $\lambda^{(1)}, \lambda^{(2)}$ are global bifurcation points of (BP) in the sense of P. H. Rabinowitz (cf. [14] where assertions of such type for variational inequalities are proved).

Corollary 1.1. *If the assumptions of Theorem* 1.1 *are fulfilled then for any couple of positive* μ, ν *there is at least one bifurcation point* $\lambda_{\mu,\nu} \in (\lambda^{(1)}, \lambda^{(2)})$ *of* (BP) *with* α, β *replaced by* $\mu\alpha, \nu\beta$. *In particular, the problem*

$$(1.9)\qquad \begin{aligned} \Delta u + \lambda u - \mu\alpha u^+ + \nu\beta u^- &= 0 \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_N \end{aligned}$$

has a nontrivial solution for $\lambda = \lambda_{\mu,\nu}$ *(cf. Observation* 1.1*). This is a consequence of the fact that if the functions* α, β *fulfil the assumption* (1.7) *then also* $\mu\alpha, \nu\beta$ *with arbitrary positive reals* μ, ν *satisfy this assumption, and of Observation* 1.1 *(used with* α, β *replaced by* $\mu\alpha, \nu\beta$*).*

R e m a r k 1.2. For any $c \in \mathbb{R}$ let us set $B_c u = -\Delta u + cu$ and consider the problem

$$(1.10)\qquad \begin{aligned} B_c u &= \mu a u^+ - \nu b u^- \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_N \end{aligned}$$

with given functions $a, b \in L_\infty(\Omega)$ nonnegative a.e. in Ω. Following [16], we can introduce the generalized Fučík spectrum of the operator B_c corresponding to the functions a, b as the set

$$\Sigma(B_c; a, b) = \{[\mu, \nu] \in \mathbb{R}^2 ;\ (1.10) \text{ has a nontrivial solution}\}.$$

For $a \equiv b \equiv 1$, this is the standard Fučík spectrum studied for the first time in [7], [4]. The set $\Sigma(B_c; a, b)$ was considered in [16] for the one-dimensional case, the general Sturm-Liouville operator and a, b positive a.e. It is proved there that it consists of a collection of C^1 curves having basic geometrical properties similar to those of the curves forming the standard Fučík spectrum. Let us realize that the nonnegative functions α, β in (1.9) have opposite signs than a, b in (1.10). Hence, in terms of the generalized Fučík spectrum, the second part of Corollary 1.1 says that if the assumptions of Theorem 1.1 are fulfilled then for any couple of positive μ, ν there is $\lambda_{\mu,\nu} \in (\lambda^{(1)}, \lambda^{(2)})$ such that $[-\mu, -\nu] \in \Sigma(B_{-\lambda_{\mu,\nu}}; \alpha, \beta)$.

Let us note that the generalized Fučík spectrum with indefinite weights for the p-Laplacian is considered (from a different point of view) in [1].

Remark 1.3. Let us consider (BP) in the case $N = 1$, $\Omega = (0, \pi)$, i.e.

$$(1.11) \qquad \begin{aligned} &u'' + \lambda u - \alpha u^+ + \beta u^- + g(\lambda, u) = 0 \text{ in } (0, \pi) \\ &c_{00}u(0) + c_{01}u'(0) = 0,\ c_{10}u(\pi) + c_{11}u'(\pi) = 0 \end{aligned}$$

where $c_{ij} \in \{0, 1\}$, $c_{i0}c_{i1} = 0$, $c_{i0} + c_{i1} = 1$, $i, j = 0, 1$, $c_{00} + c_{10} > 0$, and α, $\beta \in L_\infty(0, \pi)$ are given. The corresponding homogenized problem is

$$(1.12) \qquad \begin{aligned} &u'' + \lambda u - \alpha u^+ + \beta u^- = 0 \text{ in } (0, \pi) \\ &c_{00}u(0) + c_{01}u'(0) = 0,\ c_{10}u(\pi) + c_{11}u'(\pi) = 0. \end{aligned}$$

Such problems are studied in [2], [3], [16] where a general Sturm-Liouville operator is considered instead of u''. A half-eigenvalue of the problem (1.12) is introduced as λ for which (1.12) has a nontrivial solution, i.e. $\lambda \in \sigma_J$ in our notation. Let S_k^+ and S_k^- be the sets of all functions u from $W^{2,2}(0, \pi)$ having exactly $k - 1$ zeros in $(0, \pi)$, all zeros in $(0, \pi)$ being simple, and which are positive and negative, respectively, in a right neighbourhood of 0. Let us denote by λ_k and u_k, $k = 1, 2, \ldots$, the eigenvalues and the corresponding normed eigenfunctions of the problem

$$(1.13) \qquad \begin{aligned} &u'' + \lambda u = 0 \text{ in } (0, \pi) \\ &c_{00}u(0) + c_{01}u'(0) = 0,\ c_{10}u(\pi) + c_{11}u'(\pi) = 0. \end{aligned}$$

We can choose $u_k \in S_k^+$. In [2] it is proved that "the term $\alpha u^+ - \beta u^-$ splits apart" the classical eigenvalue λ_k of (1.13) into two half-eigenvalues λ_k^+ and λ_k^- of (1.12) with the corresponding half-lines of nontrivial solutions $\{tu_k^+, t > 0\}$ with some $u_k^+ \in S_k^+$ and $\{tu_k^-, t > 0\}$ with $u_k^- \in S_k^-$, respectively. These half-eigenvalues λ_k^+ and λ_k^-, $k = 1, 2, \ldots$, are the only elements of σ_J. An unbounded continuum of nontrivial solutions lying in $\mathbb{R} \times S_k^+$ and in $\mathbb{R} \times S_k^-$ bifurcates at λ_k^+ and λ_k^- from the branch of trivial solutions (see [2]). Moreover,

$$\lambda_k^{\nu} < \lambda_{k'}^{\nu'} \text{ for all } k < k',\ \nu,\ \nu' \in \{+, -\} \tag{1.14}$$

(see [16], Section 5).

Now, let us consider the situation of Theorem 1.1 with two immediately subsequent eigenvalues $\lambda^{(1)} = \lambda_k$, $\lambda^{(2)} = \lambda_{k+1}$. The assumptions (1.7), (1.8) imply that the corresponding eigenfunctions $u^{(1)} = \nu u_k$, $u^{(2)} = \nu' u_{k+1}$ (with some $\nu, \nu' \in \{+, -\}$) satisfy (1.13) as well as (1.12). It follows that $\lambda_k = \lambda_k^{\nu}$, $\lambda_{k+1} = \lambda_{k+1}^{\nu'}$. Comparing the problems (1.12) and (1.13) (understood as (1.12) with $\alpha \equiv \beta \equiv 0$), it is possible to show by using the results of [16], Section 5, that $\lambda_k^{-\nu} \geqslant \lambda_k$, $\lambda_{k+1}^{-\nu'} \geqslant \lambda_{k+1}$. This together with the estimate (1.14) implies that $\lambda_k^{-\nu}$ is the only element of $\sigma_J \cap [\lambda_k, \lambda_{k+1}]$. In particular, $\lambda^0 \in (\lambda_k, \lambda_{k+1})$ from the last assertion of Theorem 1.1 is unique, it coincides with $\lambda_k^{-\nu}$ and therefore an unbounded continuum of nontrivial solutions lying in $\mathbb{R} \times S_k^{-\nu}$ bifurcates at $\lambda^{(0)}$.

However, Theorem 1.1 deals with a general N-dimensional case. It can be also understood as an assertion about splitting the eigenvalue $\lambda^{(1)}$ of (EP) into a half-eigenvalue $\lambda_+^{(1)} = \lambda^{(1)}$ and at least one half-eigenvalue $\lambda_-^{(1)} = \lambda^{(0)} \in (\lambda^{(1)}, \lambda^{(2)})$. The symbols $+, -$ have now only a formal sense because we can use no analogue of $S_k^{\pm}$ for $N > 1$. Nevertheless, at least one bifurcation point $\lambda^{(0)} \in (\lambda^{(1)}, \lambda^{(2)})$ is really associated to $\lambda^{(1)}$ under the assumptions of Theorem 1.1 in the following sense. Similar considerations as in [9] (based on a Dancer's global bifurcation theorem and Lemma 2.3 below) ensure the existence of a connected branch C_0 of triplets $[\lambda, u, \tau] \in \mathbb{R} \times \mathbb{V} \times \mathbb{R}$ satisfying

$$\begin{gathered} \Delta u + \lambda u - \tau\alpha u^+ + \tau\beta u^- = 0 \text{ in } \Omega \\ \|u\| = 1,\ u = 0 \text{ on } \Gamma_D,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N \end{gathered}$$

which starts at $[\lambda^{(1)}, -u^{(1)}, 0]$ and contains at least one point of the type $[\lambda^{(0)}, u^{(0)}, 1]$ with $\lambda^{(0)} \in (\lambda^{(1)}, \lambda^{(2)})$. Hence, $\lambda^{(0)} \in \sigma_J$, i.e. $\lambda^{(0)}$ is a half-eigenvalue of (1.6), and $\{tu^{(0)};\ t > 0\}$ is the corresponding half-line of nontrivial solutions. The branch C_0 can be understood as an association of the point (or points) $\lambda^{(0)}$ to $\lambda^{(1)}$ and as a justification of the notation $\lambda^{(0)} = \lambda_-^{(1)}$. It can be shown that $\lambda^{(0)}$ obtained by this method is also a bifurcation point of (BP) (cf. [9]).

E x a m p l e 1.1. Let $N = 2$, $\Omega = (0,a) \times (0,b)$ with some $a > b$, $\Gamma_D = \partial\Omega$, $\Gamma_N = \emptyset$. Then the problem (EP) has eigenvalues $\lambda_{k,l} = \big((\frac{k}{a})^2 + (\frac{l}{b})^2\big)\pi^2$ with the corresponding eigenfunctions $u_{k,l}(x) = \sin\frac{k\pi}{a}x \sin\frac{l\pi}{b}y$, $k, l = 1, 2, \ldots$.

First, let us take $\lambda^{(1)} = \lambda_{1,1}$, $\lambda^{(2)} = \lambda_{2,1}$. Both these eigenvalues are simple. The eigenfunctions $u^{(1)}$ and $u^{(2)}$ can be chosen as $\nu u_{1,1}$ and $\nu' u_{2,1}$ with $\nu,\ \nu' \in \{+,-\}$. The assumptions of Theorem 1.1 are fulfilled if one of the functions α, β is identically zero in Ω (i.e. either $\Omega_{0-} = \emptyset$ or $\Omega_{0+} = \emptyset$) and the other is positive on a set having the closure in $(0, \frac{a}{2}) \times (0,b)$ or in $(\frac{a}{2}, a) \times (0,b)$ (i.e. one of the conditions $\overline{\Omega_{0+}} \subset (0, \frac{a}{2}) \times (0,b)$, $\overline{\Omega_{0+}} \subset (\frac{a}{2}, a) \times (0,b)$, $\overline{\Omega_{0-}} \subset (0, \frac{a}{2}) \times (0,b)$, $\overline{\Omega_{0-}} \subset (\frac{a}{2}, a) \times (0,b)$ holds).

It is easy to describe possible supports of the functions α, β in order to ensure the assumptions (1.7), (1.8) for any given couple of eigenvalues $\lambda^{(1)} = \lambda_{i,j} < \lambda^{(2)} = \lambda_{k,l}$ and the corresponding eigenfunctions $u^{(1)} = \nu u_{i,j}$, $u^{(2)} = \nu' u_{k,l}$, $\nu, \nu' \in \{+,-\}$. In particular, if $\lambda^{(1)} = \lambda_{k,l}$, $\lambda^{(2)} = \lambda_{k+2,l}$, $u^{(1)} = u_{k,l}$, $u^{(2)} = u_{k+2,l}$ with k, l large then $\Omega_{1+} \cap \Omega_{2+}$ and $\Omega_{1-} \cap \Omega_{2-}$ consist of a large number of small rectangles located along the whole domain Ω. The functions α and β can be positive in a set Ω_{0-} and Ω_{0+}, respectively, having the closure in the union of these rectangles. Let us only recall that for the use of Theorem 1.1 we need to know that $\lambda^{(1)}, \lambda^{(2)}$ are simple. (We have already mentioned that a generalization of this result for multiple eigenvalues would be also possible.)

2. Proof of Theorem 1.1

N o t a t i o n 2.1. $B_r(w)$ will denote the ball with a radius r centered at w, $\deg(I - F, B_r(w), 0)$ will be the Leray-Schauder degree of a mapping $I - F$ with respect to $B_r(w)$ and the origin.

The proof of Theorem 1.1 is based on the following modification of the Rabinowitz global bifurcation theorem:

Theorem 2.1. *Let V be a real reflexive Banach space. Let $F\colon \mathbb{R} \times V \to V$ be completely continuous and such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Assume that $a, b \in \mathbb{R}$ are not bifurcation points of the equation*

$$u - F(\lambda, u) = 0, \tag{2.1}$$

$a < b$. Furthermore, let

$$\deg(I - F(a,\cdot),\ B_r(0), 0) \neq \deg(I - F(b,\cdot),\ B_r(0), 0),$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution. Let

$$\mathcal{S} = \overline{\{(\lambda, u) \in \mathbb{R} \times \mathbb{V};\ (\lambda, u) \text{ is a solution of } (2.1) \text{ with } \|u\| \neq 0\}} \cup ([a, b] \times \{0\}),$$

and let $\mathcal{C}$ be the connected component of $\mathcal{S}$ containing $[a, b] \times \{0\}$. Then either
(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times V$, or*
(ii) *$\mathcal{C} \cap [(\mathbb{R} \setminus [a, b]) \times \{0\}] \neq \emptyset$.*

The proof can proceed similarly as that of Theorem 3 in [15]. Cf. [11] for the formulation.

Notation 2.2. Let us denote

$$T(\lambda, u) = \lambda Au - P(u) + N(u) + G(\lambda, u),\ T_0(\lambda, u) = \lambda Au - P(u) + N(u).$$

The equations (1.3) and (1.5) can be written as $u - T(\lambda, u) = 0$ and $u - T_0(\lambda, u) = 0$, respectively. Let us remark that T, $T_0\colon \mathbb{R} \times \mathbb{V} \to \mathbb{V}$ are completely continuous operators and for any $\lambda \in \mathbb{R}$, $T_0(\lambda, \cdot)$ is a positively homogeneous operator.

Remark 2.1. For $\lambda \notin \sigma_J$, the Leray-Schauder degree $\deg(I - T_0(\lambda, \cdot), B_R(0), 0)$ is well-defined for every $R > 0$ and does not depend on R.

Lemma 2.1 (Cf. [13]). *Let $Z \subset \mathbb{R} \setminus \sigma_J$ be a compact interval. Then for every $M > 0$ there exists $R > 0$ such that if $\lambda \in Z$, $f \in \mathbb{V}$, $\|f\| < M$, $u - T_0(\lambda, u) = f$, then $\|u\| < R$.*

Proof. Assume by contradiction that there exist sequences $\lambda_n \in Z$, $f_n \in \mathbb{V}$, $\|f_n\| < M$, $\|u_n\| \to \infty$ such that

$$u_n - T_0(\lambda_n, u_n) = f_n. \tag{2.2}$$

Without loss of generality we can assume $\lambda_n \to \lambda \in Z$, $\frac{u_n}{\|u_n\|} \rightharpoonup w$ (the weak convergence). Dividing (2.2) by $\|u_n\|$ we get

$$\frac{u_n}{\|u_n\|} - T_0\Big(\lambda_n, \frac{u_n}{\|u_n\|}\Big) = \frac{f_n}{\|u_n\|}.$$

It follows by virtue of the compactness of T_0 that

$$\frac{u_n}{\|u_n\|} \to w,\ w - T_0(\lambda, w) = 0.$$

Hence, w is a non-trivial solution of the equation (1.5), that means $\lambda \in \sigma_J$. This contradicts the assumption $Z \cap \sigma_J = \emptyset$. $\square$

Corollary 2.1. *If $f \in \mathbb{V}$, $\lambda \notin \sigma_J$ then there is $R_0 > 0$ such that $\deg(I - T_0(\lambda, \cdot) - f, B_R(0), 0)$ is defined for all $R > R_0$ and does not depend on $R > R_0$. Moreover,*

$$\deg(I - T_0(\lambda, \cdot) - f, B_R(0), 0) = \deg(I - T_0(\lambda, \cdot), B_R(0), 0) \text{ for all } R > R_0.$$

Lemma 2.2 (Cf. [13], [14]). *Let $\lambda \notin \sigma_J$. Then there exists $r_0 > 0$ such that*

$$\deg(I - T(\lambda, \cdot), B_r(0), 0) = \deg(I - T_0(\lambda, \cdot), B_r(0), 0) \text{ for every } r < r_0.$$

P r o o f. Let us introduce the homotopy $S(t, u) = tT(\lambda, u) + (1 - t)T_0(\lambda, u)$. Assume by contradiction that the homotopy S is not admissible for sufficiently small r, i.e. there exist sequences $t_n \in [0, 1]$, $u_n \in \mathbb{V}$, $\|u_n\| \to 0, \|u_n\| \neq 0$, such that

$$u_n - \lambda A u_n + P(u_n) - N(u_n) + t_n G(\lambda, u_n) = 0. \tag{2.3}$$

Without loss of generality we can assume that $t_n \to t$, $w_n = \frac{u_n}{\|u_n\|} \rightharpoonup w$. Dividing (2.3) by $\|u_n\|$, using the positive homogeneity and the compactness of A, P, N together with the condition (1.2), we obtain $w_n \to w$ and

$$w - \lambda A w + P(w) - N(w) = 0.$$

Hence, w is a non-trivial solution of the equation (1.5) and $\lambda \in \sigma_J$, which is a contradiction. □

N o t a t i o n 2.3. We will denote

$$\begin{aligned} E(\lambda) &= \{u \in \mathbb{V};\ u - \lambda A u = 0\}, \\ E_J(\lambda) &= \{u \in \mathbb{V};\ u - \lambda A u + P(u) - N(u) = 0\}, \\ K &= \{u \in \mathbb{V};\ u \geqslant 0 \text{ on } \Omega_{0+}, u \leqslant 0 \text{ on } \Omega_{0-}\} \end{aligned}$$

where Ω_{0+}, Ω_{0-} are from Theorem 1.1. Clearly K is a closed convex cone with its vertex at the origin in $\mathbb{V}$.

R e m a r k 2.2. Under the assumption (1.7) we have

$$\begin{aligned} &P(u) = N(u) = 0 \text{ for all } u \in K, \\ &\langle P(v), u\rangle \leqslant 0, \langle N(v), u\rangle \geqslant 0 \text{ for all } v \in \mathbb{V},\ u \in K. \end{aligned}$$

Corollary 2.2. *For every λ we have $E_J(\lambda) \cap K = E(\lambda) \cap K$.*

Lemma 2.3 (Cf. [9], [13]). *Let* (1.7) *hold and let* $\lambda_0 \in \sigma$ *be such that*

$$\text{there exists } u_0 \in E(\lambda_0), u_0 > 0 \text{ in } \overline{\Omega_{0+}}, u_0 < 0 \text{ in } \overline{\Omega_{0-}}. \tag{2.4}$$

Then $E_J(\lambda_0) = E(\lambda_0) \cap K$.

Proof. The inclusion $E(\lambda_0) \cap K \subseteq E_J(\lambda_0)$ follows from Corollary 2.2. Let $v \in E_J(\lambda_0)$. We have

$$v - \lambda_0 Av + P(v) - N(v) = 0, \tag{2.5}$$

$$u_0 - \lambda_0 Au_0 = 0. \tag{2.6}$$

Multiplying the equation (2.5) by u_0, the equation (2.6) by v and subtracting we get $\langle P(v), u_0\rangle - \langle N(v), u_0\rangle = 0$, that means

$$\int_\Omega \alpha(x)v^+(x)u_0(x)\,\mathrm{d}x - \int_\Omega \beta(x)v^-(x)u_0(x)\,\mathrm{d}x = 0. \tag{2.7}$$

It follows that $v^+(x) = 0$ in Ω_{0-} and $v^-(x) = 0$ in Ω_{0+} because otherwise the last expression should be negative by (1.7), (2.4). That means $v \in K$ and Corollary 2.2 gives $v \in E(\lambda_0)$. □

Lemma 2.4 (Cf. [13], [14]). *Assume that* Ω_{0+}, $\Omega_{0-} \subset \Omega$ *are such that* (1.7) *holds. Let* $\lambda_0 \in \sigma$ *be simple and let* (2.4) *hold,* $\|u_0\| = 1$. *Then there exists* $\varepsilon > 0$ *such that* $\lambda \notin \sigma_J$ *for all* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon)$.

Proof. Assume by contradiction that there exist $\lambda_n \to \lambda_0$, $\lambda_n \neq \lambda_0$, $\|u_n\| = 1$, $u_n \rightharpoonup u$ such that

$$u_n - \lambda_n Au_n + P(u_n) - N(u_n) = 0.$$

The compactness of the operators A, P, N implies that $u_n \to u$ (strongly) and (2.5) holds with v replaced by u, i.e. $u \in E_J(\lambda_0) = E(\lambda_0) \cap K$ by Lemma 2.3. Further, $-u_0 \notin K$ by (2.4) and by virtue of the simplicity of λ_0 we get $u = u_0$. It follows by subtracting the last equation and (2.6) that

$$\Delta(u_0 - u_n) = \lambda_n(u_n - u_0) + (\lambda_n - \lambda_0)u_0 - \alpha u_n^+ + \beta u_n^- \quad \text{in } \Omega. \tag{2.8}$$

Using the standard L_p estimates and embedding theorems, the usual procedure (see Remark 2.3 below for details) gives $u_n \to u_0$ in $C(\overline{\Omega_{0+}} \cup \overline{\Omega_{0-}})$. We have $u_n \notin K$ for n large enough. (Otherwise we could select a subsequence $u_k \in K$, i.e. $P(u_k) = N(u_k) = 0$. It would follow $u_k - \lambda_k Au_k = 0$, that means $\lambda_k \in \sigma$, which is impossible for k large.) Hence, there exist $x_n \in \overline{\Omega_{0+}}$ or $x_n \in \overline{\Omega_{0-}}$ such that $u_n(x_n) < 0$ or $u_n(x_n) > 0$, respectively. We can assume without loss of generality that $x_n \to x_0$ where $x_0 \in \overline{\Omega_{0+}}$ or $x_0 \in \overline{\Omega_{0-}}$ and $u_0(x_0) \leqslant 0$ and $u_0(x_0) \geqslant 0$, respectively, which contradicts (2.4). □

R e m a r k 2.3. Consider the situation from the last proof, i.e. we have $u_n \to u_0$ and (2.8) holds. Set $q_0 = 2$. In the case $N > 3$, define further $q_j = \frac{q_{j-1}N}{N-2q_{j-1}}$ if $q_{j-1} < \frac{N}{2}$ and $q_j > \frac{N}{2}$ arbitrary if $q_{j-1} = \frac{N}{2}$, $j = 1, 2, \ldots, m$, where m is the first index for which $q_m > \frac{N}{2}$. In the case $N \leqslant 3$ we set $m = 0$. Let us choose subdomains Ω_j, $j = 0, 1, \ldots, m+1$ such that $\Omega_0 = \Omega$, $\Omega_{m+1} = \Omega_{0+} \cup \Omega_{0-}$, $\overline{\Omega_{j+1}} \subset \Omega_j$, $j = 0, 1, \ldots, m$. Hence, we have continuous embeddings $W^{2,q_j}(\Omega_{j+1}) \subset L_{q_{j+1}}(\Omega_{j+1})$, $j = 0, \ldots, m-1$, $W^{2,q_m}(\Omega_{m+1}) \subset C(\overline{\Omega_{m+1}})$. For $j = 0$ it follows from (2.8) (see e.g. [8], Theorems 8.8, 9.13) that $u_n - u_0 \in W^{2,q_j}(\Omega_{j+1})$ and

$$\begin{aligned} \|u_n - u_0\|_{2,q_j,\Omega_{j+1}} &\leqslant C(\|\lambda_n(u_n - u_0) \\ &\quad + (\lambda_n - \lambda_0)u_0 - \alpha u_n^+ + \beta u_n^-\|_{q_j,\Omega_j} + \|u_n - u_0\|_{q_j,\Omega_j}) \\ &\leqslant C(\lambda_n\|u_n - u_0\|_{q_j,\Omega_j} + |\lambda_n - \lambda_0|\|u_0\|_{q_j,\Omega_j} + \|\alpha u_n^+\|_{q_j,\Omega_j} \\ &\quad + \|\beta u_n^-\|_{q_j,\Omega_j} + \|u_n - u_0\|_{q_j,\Omega_j}) \end{aligned} \tag{2.9}$$

where $\|\cdot\|_{2,q,\Omega}$ and $\|\cdot\|_{q,\Omega}$ denote the norms in $W^{2,q}(\Omega)$ and $L_q(\Omega)$, respectively. We know from (1.7), (2.4) that $\alpha u_0^+ = 0$, $\beta u_0^- = 0$ and therefore

$$\begin{aligned} \|\alpha u_n^+\|_{q_j,\Omega_j} &= \|\alpha(u_n^+ - u_0^+)\|_{q_j,\Omega_j} \leqslant \|\alpha\|_{\infty,\Omega} \cdot \|u_n - u_0\|_{q_j,\Omega_j}, \\ \|\beta u_n^-\|_{q_j,\Omega_j} &= \|\beta(u_n^- - u_0^-)\|_{q_j,\Omega_j} \leqslant \|\beta\|_{\infty,\Omega} \cdot \|u_n - u_0\|_{q_j,\Omega_j}. \end{aligned}$$

In the case $N \leqslant 3$ it follows from (2.9) (with $j = 0$) by using the last estimates and the continuity of the embedding that $u_n \to u_0$ in $C(\overline{\Omega_{0+} \cup \Omega_{0-}})$. In the case $N > 3$, we use the embedding theorem and the smoothness of u_0 to get $u_n \in L_{q_{j+1}}(\Omega_{j+1})$. It follows from (2.8) that $u_n - u_0 \in W^{2,q_{j+1}}(\Omega_{j+2})$ (see e.g. [8], Lemma 9.16) and the standard L_p estimates (see e.g. [8], Theorem 9.11) give (2.9) with j replaced by $j+1$. Repeating this consideration we obtain (2.9) successively for $j = 1, \ldots, m$. It follows by using the last estimates and the continuity of the embeddings used that $u_n \to u_0$ in $W^{2,q_j}(\Omega_{j+1})$, $j = 0, \ldots, m$, $u_n \to u_0$ in $C(\overline{\Omega_{0+} \cup \Omega_{0-}})$.

Lemma 2.5 (Cf. [13], [14]). *Let* (1.7) *hold, let* $\lambda_0 \in \sigma$ *be simple and let* (2.4) *hold,* $\|u_0\| = 1$. *Then there is* $\varepsilon > 0$ *such that* $\deg(I - T_0(\lambda, \cdot), B_R, 0) = 0$ *for every* $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, $R > 0$.

P r o o f. According to Corollary 2.1 and Lemma 2.4, it is sufficient to prove that $\deg(I - T_0(\lambda, \cdot) - f, B_R, 0) = 0$ for some $f \in \mathbb{V}$ and R large enough. We will prove that the equation $u - T_0(\lambda, u) = f$ has no solution for $f = u_0$ and λ sufficiently close to λ_0, $\lambda > \lambda_0$. Assume by contradiction that there exist $\lambda_n \searrow \lambda_0$, $\lambda_n \neq \lambda_0$, $u_n \in \mathbb{V}$ such that $u_n - T_0(\lambda_n, u_n) = u_0$, i.e.

$$u_n - \lambda_n A u_n + P(u_n) - N(u_n) = u_0. \tag{2.10}$$

Multiplying the equation (2.10) by u_0, the equation (2.6) by u_n and subtracting we get

$$(\lambda_n-\lambda_0)\langle Au_n,u_0\rangle=\langle P(u_n),u_0\rangle-\langle N(u_n),u_0\rangle-\|u_0\|^2.$$

It follows from Remark 2.2 that

$$\langle P(u_n),u_0\rangle-\langle N(u_n),u_0\rangle-\|u_0\|^2\leqslant-\|u_0\|^2<0.$$

We have $\lambda_n>\lambda_0$, $\lambda_n\to\lambda_0$ and therefore $\lim\langle Au_n,u_0\rangle=-\infty$. In particular, $\|u_n\|\to\infty$. Without loss of generality we can assume $\frac{u_n}{\|u_n\|}=w_n\rightharpoonup w$. Dividing the equation (2.10) by $\|u_n\|$ we get

$$w_n-\lambda_nAw_n+P(w_n)-N(w_n)=\frac{u_0}{\|u_n\|}.$$

It follows by using the compactness of the operators A,P,N that $w_n\to w$, $\|w\|=1$,

$$w-\lambda_0Aw+P(w)-N(w)=0.$$

Lemma 2.3 ensures that $w\in E(\lambda_0)\cap K$, that means $w=u_0$. We have $\langle Aw_n,u_0\rangle<0$ for n large enough. The limiting process gives $\langle Au_0,u_0\rangle\leqslant0$ and therefore $u_0=0$ by Remark 1.1, which is a contradiction. □

Lemma 2.6 (Cf. [13], [14]). *Let $\lambda_0\in\sigma$ be simple, $\lambda_0>0$, let (2.4) hold, $\|u_0\|=1$. Then there exists $\varepsilon>0$ so that*

$$\deg(I-T_0(\lambda,\cdot),B_R(0),0)=(-1)^\zeta \text{ for every } \lambda\in(\lambda_0-\varepsilon,\lambda_0) \text{ where}$$
$$\zeta=\sum_{\substack{\nu\in\sigma\\0<\nu<\lambda_0}}n_\nu,\ n_\nu=\dim\Big(\bigcup_{p=1}^{\infty}\operatorname{Ker}(I-\nu A)^p\Big).$$

Proof. It follows from Lemma 2.4 that there is $\varepsilon>0$ such that $\lambda\notin\sigma\cup\sigma_J$ for all $\lambda\in(\lambda_0-\varepsilon,\lambda_0)$. Let us show that ε can be simultaneously chosen such that

$$\text{(2.11)}\quad \begin{aligned}&\text{for all } t\in[0,1],\ \lambda\in(\lambda_0-\varepsilon,\lambda_0),\ \text{the equation}\\ &u-\lambda Au+\Big(\frac{\lambda}{\lambda_0}-1\Big)u_0+tP(u)-tN(u)=0 \text{ has a unique solution } u_0.\end{aligned}$$

Since $P(u_0)=N(u_0)=0$ by the assumption (2.4), it is easy to see that u_0 is really a solution. For the proof of unicity, let us assume by contradiction that there exist $\lambda_n<\lambda_0$, $\lambda_n\to\lambda_0$, $t_n\in[0,1]$, $t_n\to t_0$, and $u_n\neq u_0$, such that

$$\text{(2.12)}\quad u_n-\lambda_nAu_n+\Big(\frac{\lambda_n}{\lambda_0}-1\Big)u_0+t_nP(u_n)-t_nN(u_n)=0.$$

Let us show that for n large enough we have $u_n \notin K$. Otherwise we could select a subsequence $u_k \in K$, i.e. $P(u_k) = N(u_k) = 0$, and obtain

$$u_k - \lambda_k A u_k + \Big(\frac{\lambda_k}{\lambda_0} - 1\Big) u_0 = 0.$$

This can be written as

$$(I - \lambda_k A) u_k = (I - \lambda_k A) u_0,$$

which results in $u_k = u_0$ because $\lambda_n \notin \sigma$ for n large, a contradiction.

Multiplying the equation (2.12) by u_0, we get

$$\text{(2.13)}\quad \langle u_n, u_0\rangle - \lambda_n \langle A u_n, u_0\rangle + \Big(\frac{\lambda_n}{\lambda_0} - 1\Big)\|u_0\|^2 + t_n \langle P(u_n), u_0\rangle - t_n \langle N(u_n), u_0\rangle = 0.$$

It follows from Remark 2.2 and (2.4) that $\langle P(u_n), u_0\rangle - \langle N(u_n), u_0\rangle \leqslant 0$. Hence, the equation (2.13) implies by virtue of the symmetry of A that

$$0 \leqslant \Big\langle u_n - \lambda_n A u_n + \Big(\frac{\lambda_n}{\lambda_0} - 1\Big) u_0, u_0 \Big\rangle = \Big(1 - \frac{\lambda_n}{\lambda_0}\Big)\langle u_0, u_n\rangle + \Big(\frac{\lambda_n}{\lambda_0} - 1\Big)\|u_0\|^2.$$

We have $\frac{\lambda_n}{\lambda_0} < 1$ and therefore it follows $\|u_0\|^2 \leqslant \langle u_0, u_n\rangle \leqslant \|u_0\|\|u_n\|$, that means $\|u_n\| \geqslant \|u_0\|$.

We can assume without loss of generality that $w_n = \frac{u_n}{\|u_n\|} \rightharpoonup w$. Dividing the equation (2.12) by $\|u_n\|$ we have

$$\text{(2.14)}\qquad w_n - \lambda_n A w_n + \Big(\frac{\lambda_n}{\lambda_0} - 1\Big)\frac{u_0}{\|u_n\|} + t_n P(w_n) - t_n N(w_n) = 0,$$

where $w_n \notin K$ for sufficiently large n. Further, $(\frac{\lambda_n}{\lambda_0} - 1)\frac{u_0}{\|u_n\|} \to 0$ as $n \to \infty$ and it follows from (2.14) by using the compactness of the operators A, P, N that $w_n \to w$,

$$w - \lambda_0 A w + t_0 P(w) - t_0 N(w) = 0.$$

If $t_0 \neq 0$ then $w \in E(\lambda_0) \cap K$ by Lemma 2.3 used for $E_J(\lambda_0)$ corresponding to the operators $t_0 P$, $t_0 N$ instead of P, N. However, λ_0 is a simple eigenvalue and therefore $w = u_0$. If $t_0 = 0$ then we get directly $w \in E(\lambda_0)$, we know that $\langle u_0, w_n\rangle \geqslant 0$ and this together with the simplicity of λ_0 implies again $w = u_0$. We have

$$\Delta w_n = -\lambda_n w_n - \Big(\frac{\lambda_n}{\lambda_0} - 1\Big)\frac{\Delta u_0}{\|u_n\|} + t_n \alpha w_n^+ - t_n \beta w_n^-$$

and

$$\Delta u_0 = -\lambda_0 u_0.$$

Hence, we obtain that

$$\Delta(w_n - u_0) = \lambda_n(u_0 - w_n) + (\lambda_0 - \lambda_n)u_0 + (\lambda_n - \lambda_0)\frac{u_0}{\|u_n\|} + t_n\alpha w_n^+ - t_n\beta w_n^-.$$

The same considerations as in the proof of Lemma 2.4 (cf. also Remark 2.3) give $w_n \to u_0$ in $C(\overline{\Omega_{0+} \cup \Omega_{0-}})$, $u_0(x_0) = 0$ for some $x_0 \in \overline{\Omega_{0+}} \cup \overline{\Omega_{0-}}$. This contradicts the assumption (2.4), and (2.11) is proved.

Let $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ be fixed. It follows from Corollary 2.1 and (2.11) that for $R > 0$ sufficiently large and ϱ small enough we get

$$\begin{aligned}\deg(I - T_0(\lambda,\cdot), B_R(0), 0) &= \deg\Big(I - T_0(\lambda,\cdot) + \Big(\frac{\lambda}{\lambda_0} - 1\Big)u_0, B_R(0), 0\Big) = \\ &= \deg\Big(I - T_0(\lambda,\cdot) + \Big(\frac{\lambda}{\lambda_0} - 1\Big)u_0, B_\varrho(u_0), 0\Big).\end{aligned}$$

Further, it follows from (2.11) by using the homotopy argument that

$$\deg\Big(I - T_0(\lambda,\cdot) + \Big(\frac{\lambda}{\lambda_0} - 1\Big)u_0, B_\varrho(u_0), 0\Big) = \deg\Big(I - \lambda A + \Big(\frac{\lambda}{\lambda_0} - 1\Big)u_0, B_\varrho(u_0), 0\Big).$$

Realizing that the linearization of $\lambda A - (\frac{\lambda}{\lambda_0} - 1)u_0$ at u_0 is λA, we conclude by the Leray-Schauder formula for the index of an isolated solution (see e.g. [12]) that the last degree equals $(-1)^\zeta$ where

$$\zeta = \sum_{\mu\in\sigma'} n'_\mu,\ n'_\mu = \dim\Big(\bigcup_{p=1}^{\infty} \operatorname{Ker}(\mu I - \lambda A)^p\Big) = \dim\Big(\bigcup_{p=1}^{\infty} \operatorname{Ker}(I - \frac{\lambda}{\mu}A)^p\Big),$$

$$\sigma' = \{\mu \in \mathbb{R};\ \mu > 1, \exists u \in \mathbb{V}\colon\ \|u\| \neq 0, \mu u = \lambda A u\} = \Big\{\mu \in \mathbb{R};\ \mu > 1, \frac{\lambda}{\mu} \in \sigma\Big\}.$$

Substituting $\frac{\lambda}{\mu} = \nu$ and using the fact that $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, $(\lambda_0 - \varepsilon, \lambda_0) \cap \sigma = \emptyset$ we get

$$\zeta = \sum_{\substack{\nu\in\sigma\\ 0<\nu<\lambda_0}} n_\nu,\ n_\nu = \dim\Big(\bigcup_{p=1}^{\infty} \operatorname{Ker}(I - \nu A)^p\Big),$$

and the proof is completed. □

Proof of Theorem 1.1. It follows from Lemmas 2.2, 2.4, 2.5, 2.6 that there exist a, b such that $\lambda^{(1)} < a < b < \lambda^{(2)}$, $((\lambda^{(1)}, a] \cup [b, \lambda^{(2)})) \cap \sigma_J = \emptyset$, and a constant $r_0 > 0$ such that $\deg(I - T(a,\cdot), B_r(0), 0) = \deg(I - T_0(a,\cdot), B_r(0), 0) = 0$, $\deg(I - T(b,\cdot), B_r(0), 0) = \deg(I - T_0(b,\cdot), B_r(0), 0) = (-1)^\zeta$ for every $0 < r < r_0$. At the same time, a, b are not bifurcation points of (1.3) (Observation 1.1). Hence, the hypotheses of Theorem 2.1 are fulfilled and the assertion of Theorem 1.1 about the properties of $S_{a,b}$ follows. The existence of at least one bifurcation point $\lambda_0 \in [a, b]$ of (BP) is a direct consequence.

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