INFINITE PATHS IN LOCALLY FINITE GRAPHS AND IN THEIR SPANNING TREES

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To the memory of Jiří Sedláček

Abstract. The paper concerns infinite paths (in particular, the maximum number of pairwise vertex-disjoint ones) in locally finite graphs and in spanning trees of such graphs.

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A graph G is called locally finite if every vertex of G has a finite degree. Obviously every finite graph is also locally finite. We will treat locally finite graphs which themselves are infinite.

Let G be a connected infinite locally finite graph. It is well-known that the vertex set of G is countable and G contains at least one infinite path.

There are two types of infinite paths. A one-way infinite path is an infinite connected graph which has one vertex of degree one (initial vertex) and in which all other vertices are of degree two. A two-way infinite path is an infinite connected graph which is regular of degree two. A general symbol for a one-way (or two-way) infinite path will be W_1 (or W_2 respectively). A finite path having length n (i.e. having n edges and $n+1$ vertices) will be denoted by P_n.

We will use also the symbol of the block graph of a given graph G. Let G be a graph, let $A(G)$ be the set of all cutvertices (articulations) of G, let $B(G)$ be the set of all blocks of G. The block graph $BG(G)$ of G is the bipartite graph with vertex sets $A(G)$, $B(G)$ such that $a \in A(G)$ is adjacent to $b \in B(G)$ in $BG(G)$ if and only if a is an articulation of G belonging to the block b.

Let G be a connected infinite locally finite graph. We will study the numerical invariant $IW(G)$ which denotes the maximum number of pairwise vertex-disjoint one-way infinite paths in G. Evidently $IW(G) \geqslant 1$ and it may be even infinite (countable).

Proposition 1. *Let G be an infinite locally finite connected graph. Then $IW(G) = 1$ if and only if G contains no two-way infinite path.*

Proof. A two-way infinite path is the union of two edge-disjoint one-way infinite paths and thus evidently it is also the union of two vertex-disjoint ones with one edge added. □

As usual, a circuit in a graph G is a subgraph of G which is finite, connected and regular of degree 2.

We recall the definition of a block of a graph which will be used here similarly as in the case of finite graphs. Let $\circ$ be a binary relation on the set $E(G)$ of edges of G such that $e_1 \circ e_2$ if and only if either $e_1 = e_2$, or there exists a circuit in G which contains both e_1 and e_2. The relation $\circ$ is an equivalence relation on $E(G)$. A subgraph B of G whose edge is one class $\circ$ and whose vertex set is the set of all end vertices of these edges is a block of G.

Now we shall study a special type of infinite graphs, namely the graph consisting of infinitely many blocks, each of which is finite. We will call them finite-block graphs, shortly FB-graphs.

Theorem 1. *Let G be an infinite locally finite FB-graph. The graph G contains no two-way infinite path if and only if its block graph $BG(G)$ contains no two-way infinite path.*

Proof. Suppose that G contains a two-way infinite path W_2. Then there exists a two-way infinite sequence $\ldots, B_{-2} \ldots, B_{-2}, B_{-1}, B_0, B_1, B_2, \ldots$ of G such that the intersection of W_2 with B_n for each integer n is a finite path D_n and each path D_n is immediately followed by D_{n+1} in W_2. Now we denote each block B_n by b_n and the articulation between b_n and b_{n+1} by a_n; we have a two-way infinite path in $BG(G)$ with the vertices

$$\ldots, a_{-2}, b_{-1}, a_{-1}, b_0, a_0, b_1, a_1, b_2, \ldots$$

On the other hand, let W_2' be a two-way infinite path in $BG(G)$ with the vertices

$$\ldots, b'_{-2}, a'_{-2}, b'_{-1}, a'_{-1}, b'_0, a'_0, b'_1, a'_1, \ldots$$

Each block b'_n is a connected graph, therefore there exists a finite path D'_k in it connecting a'_{n-1} with a'_n. The union of the paths D'_n is a two-way infinite path W_2 in G. □

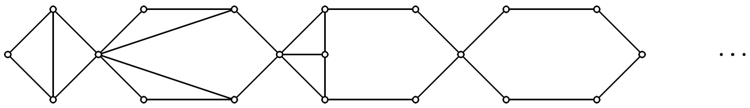

Figure 1

An example of an FB-graph without a two-way infinite path is in Fig. 1.

Proposition 2. *Let G be a connected infinite locally finite graph. Then each edge of G belongs to a one-way infinite path in G.*

P r o o f. Let e be an edge of G, let W_1 be a one-way infinite path in G. As G is connected, there exists a finite path D in G containing e and a vertex of W_1. The union of D and W_1 is the required path. $\square$

Theorem 2. *Let G be a connected infinite locally finite graph. Let B be a block of G. Then either all edges of B belong to two-way infinite paths in G, or none does.*

P r o o f. Consider the relation $\circ$ and let e be an edge of B. Let B contain an edge f belonging to a two-way infinite path W_2 in G. Then $e \circ f$. If $e = f$, the assertion is true. Otherwise there exists a circuit D in B which contains both e and f. Let D_0 be a finite path in D which is a subpath of D, contains e and is edge-disjoint with W_2. If e belonged to W_2, then the assertion would be true. Let u, v be the end vertices of D_0. If we omit the subpath of W_2 connecting u and v and replace it by D_0, we obtain a two-way infinite path in G containing e. $\square$

R e m a r k. Let again G be a connected infinite locally finite graph. The subgraph of G formed by all edges which belong to two-way infinite paths is connected. On the other hand, all other edges may be deleted without changing the structure of two-way infinite paths.

Now we turn our attention to spanning trees.

Theorem 3. *Let G be a connected infinite locally finite graph. Then there exists a spanning tree T of G such that $IW(T) = IW(G)$.*

P r o o f. Let $IW(G) = p$. Let $D_1, \ldots, D_p$ be pairwise vertex-disjoint one-way infinite paths in G. The tree T will be constructed in several steps. In the first step we have the forest F_0 whose connected components are $D_1, \ldots, D_p$ and isolated vertices. In the second step a tree T_0 is obtained from F_0 in such a way that for any path D_k with $k \geqslant 2$ a finite path connecting a vertex of D_k with a vertex of D_1 is chosen and added to the forest. If some circuits occur, edges are deleted where it is necessary. At the end of this step a tree T_0 is obtained. Further trees $T_1, T_2, \ldots$

are formed in such a way that a connected component of F_0 distinct from the trees T_i already constructed is chosen and one of its vertices is connected by a finite path with a vertex of T_i with maximal i from those which have been already constructed. The last tree from such trees is then required to fullfil $|V(T_i)| = |V(G)|$. □

There may exist a spanning tree T of G such that $IW(T) < IW(G)$. In particular, this occurs with graphs which have a one-way Hamiltonian or a two-way Hamiltonian path. A two-way Hamiltonian path is an analogue of a Hamiltonian circuit; it is regular of degree 2 (and obviously infinite). A one-way infinite Hamiltonian path is an analogue of a finite Hamiltonian path; it has one vertex of degree 1 and all others of degree 2.

E x a m p l e 1. For any positive integer k the direct product $P_k \times W_1$ has a one-way infinite Hamiltonian path, while $IW(P_k \times W_1) = k + 1$.

E x a m p l e 2. The direct product $W_1 \times W_1$ has a one-way infinite Hamiltonian path, while $IW(W_1 \times W_1)$ is infinite.

Both paths mentioned are seen in Fig. 2 and in Fig. 3.

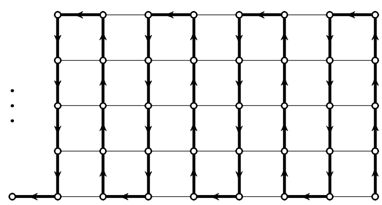

Figure 2

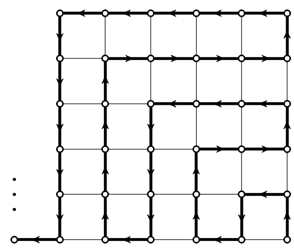

Figure 3

Using the ideas of proof of Theorem 1, the following two theorems may be proved.

Theorem 4. *Let G be a connected infinite locally finite FB-graph. The graph G contains a two-way infinite Hamiltonian path H_2 if and only if its block graph $BG(G)$ contains a two-way infinite path H_2' with a sequence of vertices*

$$\ldots, a_{-2}, b_{-1}, a_{-1}, b_0, a_0, b_1, a_1, \ldots$$

such that in each block b_n a finite Hamiltonian path connecting a_{n-1} and a_n exists (for each integer n).

Theorem 5. *Let G be a connected infinite locally finite FB graph. The graph G contains a one-way infinite Hamiltonian path H_1 if and only if its block graph*

$BG(G)$ contains a one-way infinite path H_1' with the sequence of vertices

$$b_0, a_0, b_1, a_1, b_2, a_2, b_3, a_3, \ldots$$

such that in each block b_n for a positive integer n there exists a finite Hamiltonian path connecting a_{n-1} and a_n and in the block b_0 there exists a finite Hamiltonian path ending in a_0.

At the end we shall prove a formula for $IW(T)$, where T is a tree.

Theorem 6. *Let T be a finite or infinite locally finite tree. For each positive integer k let d denote the number of vertices of T of degree k. Suppose that $\sum\limits_{k=1}^{\infty}(k-2)d_k$ is finite. Then*

$$IW(T) = 2 + \sum_{k=1}^{\infty}(k-2)d_k.$$

P r o o f. We shall do the proof by induction with respect to $IW(T)$. First let $IW(T) = 0$. Then T is a locally finite tree without infinite paths and therefore it is finite. Denote $D(T) = (k-2)d_k$. We have $D(T) = \sum\limits_{k=1}^{\infty} kd_k - 2\sum\limits_{k=1}^{\infty} d_k$; both the sums on the right-hand side are finite. The sum $\sum\limits_{k=1}^{\infty} kd_k$ is the sum of degrees of all vertices of T. Let n be the number of vertices of T; then the number of edges is $n-1$. Hence $\sum\limits_{k=1}^{\infty} kd_k = 2n-2$. The sum $\sum\limits_{k=1}^{\infty} d_k = n$ and $D(T) = -2$, hence $2 + D(T) = 0 = IW(T)$. Now suppose that the assertion is true for $IW(T) = p \geqslant 0$ and let T be a locally finite tree with $D(T)$ finite and with $IW(T) = p+1$. Let W be a one-way infinite path in T. For $k = 2$ we have $(k-2)d_k = 0$ and thus $D(T) = -d_1 + \sum\limits_{k=3}^{\infty}(k-2)d_k$. As this number is finite, both d_1 and $\sum\limits_{k=1}^{\infty}(k-2)d_k$ must be finite and thus d_k is finite for all $k \neq 2$. In particular, in W there are only finitely many vertices having degrees different from 2 in T. There exists a one-way infinite subpath W' of W, all of whose vertices have degree 2 in T. Let u be the initial vertex of W'. Let T' be the tree obtained from T by deleting all vertices and edges of W' except u. Then $IW(T') = IW(T) - 1 = p$. If d_k' denotes the number of vertices of degree k in T', then $d_1' = d_1' + 1$ and $d_k' = d_k$ for $k \geqslant 3$. By the induction hypothesis we have $2 + D(T') = 2 - d_1' + \sum\limits_{k=3}^{\infty}(k-2)d_k' = IW(T') = IW(T) - 1$ and thus $2 + D(T) = 2 - d_1 - 1 + \sum\limits_{k=3}^{\infty}(k-2)d_k = 1 + D(T') = IW(T') + 1 = IW(T)$. $\square$

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