ON FUZZY NEARLY C-COMPACTNESS IN FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper the concept of fuzzy nearly C-compactness is introduced in fuzzy topological spaces and fuzzy bitopological spaces. Several characterizations and some interesting properties of these spaces are discussed. The properties of fuzzy almost continuous and fuzzy almost open functions are also discussed.

Keywords: fuzzy nearly C-compact, fuzzy almost continuous, fuzzy almost open(closed), pairwise fuzzy nearly C-compact, pairwise fuzzy almost continuous, pairwise fuzzy almost open, pairwise fuzzy continuous

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1. Introduction

C. L. Chang introduced and developed the concept of fuzzy topological spaces based on the concept of a fuzzy set introduced by Zadeh in [12]. Since then, various important notions in the classical topology such as compactness have been extended to fuzzy topological spaces. The concept of nearly C-compactness in general topology was introduced and studied in [11]. The purpose of this paper is to introduce and study the concept of nearly C-compactness in fuzzy setting. Section 2 deals with preliminaries, Section 3 deals with the concept of fuzzy nearly C-compactness and some of the characterizations in fuzzy topological spaces, and Section 4 deals with the properties of fuzzy almost continuous, fuzzy almost open functions [4]. Section 5 deals with the concept of fuzzy nearly C-compactness in fuzzy bitopological spaces.

2. Preliminaries

Let (X,T) be any fuzzy topological space [3]. Let λ be any fuzzy set in X. We define the closure of λ to be $\bigwedge\{\mu;\ \mu \geqslant \lambda, \mu$ is fuzzy closed$\}$ and interior of λ to be $\bigvee\{\sigma;\ \sigma \leqslant \lambda,\ \sigma$ is fuzzy open$\}$. The interior of the fuzzy set λ and the closure of the fuzzy set λ in X will be denoted by $\operatorname{Int}\lambda$ and $\operatorname{Cl}\lambda$, respectively.

A fuzzy set λ in X is said to be *fuzzy regular open* if $\operatorname{Int}(\operatorname{Cl}\lambda) = \lambda$ and fuzzy regular closed if $\operatorname{Cl}(\operatorname{Int}\lambda) = \lambda$. A fuzzy set is fuzzy regular open if its complement is *fuzzy regular closed*. Let $\mathscr{U} = \{\lambda_\alpha\}_{\alpha\in\Delta}$ be a family of members from T. Then $\mathscr{U}$ is called a *cover* of X if $\bigvee\limits_{\alpha\in\Delta} \lambda_\alpha = 1$, and a subfamily of $\mathscr{U}$ having a similar property is called a *subcover* of $\mathscr{U}$. A fuzzy topological space (X,T) is said to be *fuzzy compact* [9] if every cover of X by members of T has a finite subcover. Further x_α shall denote a fuzzy point [8] with a support x and value α $(0 < \alpha \leqslant 1)$. For a fuzzy set λ in X, we write $x_\alpha \in \lambda$ provided $\alpha \leqslant \lambda(x)$.

A *fuzzy filter base* on X is a non-empty collection $\mathfrak{F}$ of fuzzy sets on X satisfying the conditions

(i) $0 \notin \mathfrak{F}$; where 0 stands for empty fuzzy set;
(ii) $\lambda_1, \lambda_2 \in \mathfrak{F} \Rightarrow \lambda_1 \wedge \lambda_2 \in \mathfrak{F}$;
(iii) $\lambda \subset \mu \in \mathfrak{F} \Rightarrow \lambda \in \mathfrak{F}$.

For any two fuzzy sets λ, θ in a fuzzy topological space (X,T), we define the sum of them, denoted by $\lambda + \theta$, as follows: $(\lambda+\theta)(x) = \lambda(x) + \theta(x)$.

3. Fuzzy nearly C-compactness in fuzzy topological spaces

The concept of *nearly C-compactness* was introduced in [11]. It is defined as follows.

Definition 1 [11]**.** A topological space is said to be *nearly C-compact* if given a regular closed set A and an open cover $\mathscr{U}$ of A there exists a finite subfamily $\{O_i;\ i = 1,2,\dots,n\}$ of $\mathscr{U}$ such that $A \subset \bigcup\limits_{i=1}^{n} \operatorname{Cl}(O_i)$.

So, we are now prepared to introduce the following definition.

Definition 2. Let (X,T) be a fuzzy topological space. (X,T) is said to be *fuzzy nearly C-compact* if for any ordinary subset A of X, $A \neq X$ such that $\mathscr{X}_A$ (the characteristic function of $A \subset X$) is a proper fuzzy regular closed set and for each fuzzy open cover of $\{\lambda_\alpha;\ \alpha \in \Delta\}$ of $\mathscr{X}_A$ there exists a finite subfamily $\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_n}$ such that $\mathscr{X}_A \leqslant \bigvee\limits_{i=1}^{n} \operatorname{Cl}(\lambda_{\alpha_i})$.

From this definition it is clear that fuzzy compactness implies fuzzy nearly C-compactness. However, the converse is not true as the following example shows:

Example 1. Let $X = \{a, b\}$, $T = \{0, 1, f_n\}$ where $f_n \colon X = \{a, b\} \to [0, 1]$ is such that $f_n(X) = 1 - 1/n, \forall x \in X$. The only possible non-empty subsets of X are $A_1 = \{a\}$ and $A_2 = \{b\}$. Further, since $\operatorname{Cl}\operatorname{Int} \mathscr{X}_{A_1} = 0 \neq \mathscr{X}_{A_1}$ and $\operatorname{Cl}\operatorname{Int} \mathscr{X}_{A_2} = 0 \neq \mathscr{X}_{A_2}$, it follows that $\mathscr{X}_{A_1}$ and $\mathscr{X}_{A_2}$ are not fuzzy regular closed. So vacuously (X, T) is fuzzy nearly C-compact. Now we claim that (X, T) is not fuzzy compact. Indeed, $\bigvee\limits_{n=1}^{\infty} f_n = 1$ shows that $\{f_n\}_{n=1}^{\infty}$ is a fuzzy open cover of 1_X but for every finite integer, say n_0, we have $\bigvee\limits_{n=1}^{n_0} f_n < 1$ and therefore $\{f_n\}_{n=1}^{\infty}$ has no finite subcover for 1_X. That is, (X, T) is *not fuzzy compact.*

Proposition 1. *In a fuzzy topological space (X, T) the following assertions are equivalent.*

(a) *X is fuzzy nearly C-compact.*

(b) *For each subset $A \subset X$ such that $\mathscr{X}_A$ is proper fuzzy regular closed and for each fuzzy regular open cover $\mathscr{U} = \{\lambda_\alpha\}_{\alpha \in \Delta}$ of $\mathscr{X}_A$ there exists a finite subfamily $\lambda_{\alpha_1}, \lambda_{\alpha_2}, \ldots, \lambda_{\alpha_n}$ of $\mathscr{U}$ such that $\mathscr{X}_A \leqslant \bigvee\limits_{i=1}^{n} \operatorname{Cl}(\lambda_{\alpha_i})$.*

(c) *For each subset $A \subset X$ such that $\mathscr{X}_A$ is a proper fuzzy regular closed set and for each family $\mathfrak{F} = \{\mu_\alpha\}_{\alpha \in \Delta}$ of non-zero fuzzy regular closed sets such that $\Big(\bigwedge\limits_{\alpha \in \Delta} \mu_\alpha\Big) \wedge \mathscr{X}_A = 0$, there exists a finite subfamily $\mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_n}$ of $\mathfrak{F}$ such that $\Big\{\bigwedge\limits_{i=1}^{n} \operatorname{Int}(\mu_{\alpha_i})\Big\} \wedge \mathscr{X}_A = 0$.*

(d) *For any subset $A \subset X$ such that $\mathscr{X}_A$ is a proper fuzzy regular closed set and for each family $\mathfrak{F} = \{\mu_\alpha\}_{\alpha \in \Delta}$ of fuzzy regular closed sets, if for each finite subfamily $\mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_n}$ of $\mathfrak{F}$ we have $\Big[\bigwedge\limits_{i=1}^{n} \operatorname{Int}(\mu_{\alpha_i})\Big] \wedge \mathscr{X}_A \neq 0$, then $\Big(\bigwedge\limits_{\alpha \in \Delta} \mu_\alpha\Big) \wedge \mathscr{X}_A \neq 0$.*

Proof. (a) $\Rightarrow$ (b) follows easily from Definition 2.

(b) $\Rightarrow$ (a) Suppose (b) holds. Let A be any subset of X such that $\mathscr{X}_A$ is proper fuzzy regular closed. Let $\mathscr{U} = \{\lambda_\alpha\}_{\alpha \in \Delta}$ be a fuzzy open cover of $\mathscr{X}_A$. Then $\{\operatorname{Int}(\operatorname{Cl} \lambda_\alpha)\}_{\alpha \in \Delta}$ will be a fuzzy regular open cover of $\mathscr{X}_A$. Then by (b), there exists a finite subfamily $\{\operatorname{Int}(\operatorname{Cl} \lambda_{\alpha_i})\}_{i=1}^{n}$ such that

$$\mathscr{X}_A \leqslant \bigvee_{i=1}^{n} \operatorname{Cl}\{\operatorname{Int}(\operatorname{Cl} \lambda_{\alpha_i})\} = \bigvee_{i=1}^{n} \operatorname{Cl}(\lambda_{\alpha_i}).$$

This proves (b) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c) Let $A \subset X$ be such that $\mathscr{X}_A$ is proper fuzzy regular closed. Let $\mathfrak{F} = \{\mu_\alpha\}_{\alpha\in\Delta}$ be a family of non-zero fuzzy regular closed sets of the space X such that $(\bigwedge \mu_\alpha)_{\alpha\in\Delta} \wedge \mathscr{X}_A = 0$ for each proper fuzzy regular closed set $\mathscr{X}_A$ of X.

Then $\mathscr{U} = \{1-\mu_\alpha\}_{\alpha\in\Delta}$ is a fuzzy regular open cover of the fuzzy regular closed set $\mathscr{X}_A$ and therefore there exists a finite subfamily $\{\lambda_{\alpha_i} = 1-\mu_{\alpha_i};\ i = 1,2,\ldots,n\}$ of $\mathscr{U}$ such that $\mathscr{X}_A \leqslant \bigvee_{i=1}^{n} \mathrm{Cl}(\lambda_{\alpha_i})$. Now for each α_i we have

$$\mathrm{Int}(\mu_{\alpha_i}) = \mathrm{Int}(1-\lambda_{\alpha_i}) = 1-\mathrm{Cl}(1-(1-\lambda_{\alpha_i})) = 1-\mathrm{Cl}(\lambda_{\alpha_i}).$$

Therefore $\bigwedge_{i=1}^{n} \mathrm{Int}(\mu_{\alpha_i}) = 1 - \bigvee_{i=1}^{n}\{\mathrm{Cl}\,\lambda_{\alpha_i}\} \leqslant 1-\mathscr{X}_A$ and so

$$\left[\bigwedge_{i=1}^{n} \mathrm{Int}(\mu_{\alpha_i})\right] \wedge \mathscr{X}_A \leqslant (1-\mathscr{X}_A)\wedge\mathscr{X}_A = \mathscr{X}_{X-A}\wedge\mathscr{X}_A = 0.$$

This proves (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (b) Let $\mathscr{U} = \{\lambda_\alpha\}_{\alpha\in\Delta}$ be a fuzzy regular open cover of the proper fuzzy regular closed set $\mathscr{X}_A$ of the space X where $A \subset X$. Now $\mathscr{X}_A \leqslant \bigvee_{\alpha\in\Delta} \lambda_\alpha$ implies that $1-\mathscr{X}_A \geqslant \left(1-\bigvee_{\alpha\in\Delta}\lambda_\alpha\right)$ and

$$\bigwedge_{\alpha\in\Delta}(1-\lambda_\alpha)\wedge\mathscr{X}_A = \left(1-\bigvee_{\alpha\in\Delta}\lambda_\alpha\right)\wedge\mathscr{X}_A \leqslant (1-\mathscr{X}_A)\wedge\mathscr{X}_A = \mathscr{X}_{X-A}\wedge\mathscr{X}_A = 0.$$

So $(1-\lambda_\alpha)$ is a family of fuzzy regular closed sets such that $\bigwedge_{\alpha\in\Delta}(1-\lambda_\alpha)\wedge\mathscr{X}_A = 0$ and so by (c) there exists a finite subfamily $\{1-\lambda_{\alpha_1}, 1-\lambda_{\alpha_2},\ldots,1-\lambda_{\alpha_n}\}$ such that

$$\left[\bigwedge_{i=1}^{n}\{\mathrm{Int}(1-\lambda_{\alpha_i})\}\right]\wedge\mathscr{X}_A = 0.$$

Now it follows that

$$\mathscr{X}_A \leqslant \bigvee_{i=1}^{n}\{1-\mathrm{Int}(1-\lambda_{\alpha_i})\}$$

but for each α_i we have

$$\mathrm{Int}(1-\lambda_{\alpha_i}) = 1-\mathrm{Cl}(1-(1-\lambda_{\alpha_i})) = 1-\mathrm{Cl}(\lambda_{\alpha_i}).$$

Therefore we conclude that

$$\mathscr{X}_A \leqslant \bigvee_{i=1}^{n}\mathrm{Cl}(\lambda_{\alpha_i}).$$

This proves (c) $\Rightarrow$ (b).

(c) $\Rightarrow$ (d) Let $A \subset X$ be such that $\mathscr{X}_A$ is a proper fuzzy regular closed set and suppose that $\{\mu_\alpha\}_{\alpha\in\Delta}$ is a family of fuzzy regular closed sets such that for every finite family $\mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_n}$, $\bigwedge\limits_{i=1}^{n} \mathrm{Int}(\mu_{\alpha_i}) \wedge \mathscr{X}_A \neq 0$. We want to show that $\bigwedge\limits_{\alpha\in\Delta} \mu_\alpha \wedge \mathscr{X}_A \neq 0$. If we suppose that $\bigwedge\limits_{\alpha\in\Delta} \mu_\alpha \wedge \mathscr{X}_A = 0$, then by assumption (c) there exists a finite family $\mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_m}$ such that $\bigwedge\limits_{i=1}^{m} \mathrm{Int}\,\mu_{\alpha_i} \wedge \mathscr{X}_A = 0$, which is a contradiction. Hence $\Big(\bigwedge\limits_{\alpha\in\Delta} \mu_\alpha\Big) \wedge \mathscr{X}_A = 0$. This proves (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (c) Let $A \subset X$ be such that $\mathscr{X}_A$ is a proper fuzzy regular closed set. Let $\{\mu_\alpha\}_{\alpha\in\Delta}$ be a family of fuzzy regular closed sets in X such that $\bigwedge\limits_{\alpha\in\Delta} \mu_\alpha \wedge \mathscr{X}_A = 0$. We have to show that there exists a finite integer (say) n_0 such that $\bigwedge\limits_{i=1}^{n_0} \mathrm{Int}\,\mu_{\alpha_i} \wedge \mathscr{X}_A = 0$. Suppose now that for every finite integer n_0 we have $\bigwedge\limits_{i=1}^{n_0} \mathrm{Int}\,\mu_{\alpha_i} \wedge \mathscr{X}_A \neq 0$. Then by assumption (d) we have $\bigwedge\limits_{\alpha\in\Delta} \mathrm{Int}(\mu_\alpha) \wedge \mathscr{X}_A \neq 0$. Therefore $0 \neq \bigwedge\limits_{\alpha\in\Delta} \mathrm{Int}(\mu_\alpha) \wedge \mathscr{X}_A \leqslant \bigwedge\limits_{\alpha\in\Delta} \mu_\alpha \wedge \mathscr{X}_A$, which is a contradiction. Hence there exists a finite integer n_0 such that $\bigwedge\limits_{i=1}^{n_0} \mathrm{Int}\,\mu_{\alpha_i} \wedge \mathscr{X}_A = 0$. This proves (d) $\Rightarrow$ (c). □

Proposition 2. *For any fuzzy topological space (X, T), the following assertions are equivalent.*

(a) *X is fuzzy nearly C-compact.*

(b) *If $A \subset X$ is such that $\mathscr{X}_A$ is a proper fuzzy regular closed and $\mathfrak{F}$ is a family of fuzzy regular closed sets of X such that $\mathscr{X}_A \leqslant \Big(1 - \bigwedge\limits_{\lambda\in\mathfrak{F}} \lambda\Big)$, then there exists a finite number of elements of $\mathfrak{F}$, say $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that $\mathscr{X}_A \leqslant 1 - \bigwedge\limits_{i=1}^{n} \mathrm{Int}\,\lambda_i$.*

P r o o f. (a) $\Rightarrow$ (b) Suppose X is fuzzy nearly C-compact. Let $A \subset X$ be such that $\mathscr{X}_A$ is a proper fuzzy regular closed set. Let $\mathfrak{F}$ be a family of fuzzy regular closed sets of X such that $\mathscr{X}_A \leqslant \Big(1 - \bigwedge\limits_{\lambda\in\mathfrak{F}} \lambda\Big) = \bigvee\limits_{\lambda\in\mathfrak{F}} (1-\lambda)$. Clearly $\mathscr{U} = \{1-\lambda\}_{\lambda\in\mathfrak{F}}$ is a fuzzy regular open cover of $\mathscr{X}_A$. Hence by assumption (a) there exists a finite number of elements (say) $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathscr{X}_A \leqslant \bigvee\limits_{i=1}^{n} \mathrm{Cl}\,\lambda_i$. Therefore $\bigwedge\limits_{i=1}^{n} \mathrm{Int}\,\lambda_i = 1 - \bigvee\limits_{i=1}^{n} \mathrm{Cl}\,\lambda_i \leqslant 1 - \mathscr{X}_A$. That is, $\mathscr{X}_A \leqslant 1 - \bigwedge\limits_{i=1}^{n} \mathrm{Int}\,\lambda_i$. This proves (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) Let $A \subset X$ be such that $\mathscr{X}_A$ is a proper fuzzy regular closed set in X. Let $\mathfrak{F}$ be a family of fuzzy regular open sets of X such that $\mathscr{X}_A \leqslant \bigvee\limits_{\lambda\in\mathfrak{F}} \lambda$. Put $\mathscr{U} = \{1-\lambda\}_{\lambda\in\mathfrak{F}}$. Then $\mathscr{U}$ is clearly a family of fuzzy regular closed sets of X such that $\mathscr{X}_A \leqslant \bigvee_{\lambda\in\mathfrak{F}} \lambda = \bigvee_{\lambda\in\mathfrak{F}} [1-(1-\lambda)] = 1 - \bigwedge_{\lambda\in\mathfrak{F}} (1-\lambda)$. Hence by assumption (b) there exists a finite number of elements, say $1-\lambda_1, 1-\lambda_2, \ldots, 1-\lambda_n$, such that $\mathscr{X}_A \leqslant 1 - \bigwedge_{i=1}^{n} \operatorname{Int}(1-\lambda_i) = \bigvee_{i=1}^{n} (1-\operatorname{Int}(1-\lambda_i)) = \bigvee_{i=1}^{n} \operatorname{Cl}\lambda_i$. This proves (b) $\Rightarrow$ (a). □

4. Properties of fuzzy almost continuous and fuzzy almost open functions

A fuzzy almost continuous mapping was defined and studied by Azad [1]. In [9], such a mapping is called an A-fuzzy almost continuous mapping. In [10] the concept of an almost fuzzy open function was introduced and in [9] the same map was called almost fuzzy open in the sense of Nanda (a.f.o.N. in short).

In this paper we call the former fuzzy almost continuous and the latter fuzzy almost open. In this section we further investigate the properties of the above mappings.

Definition 3. Let (X,T) and (Y,S) be any two fuzzy topological spaces. A mapping $f\colon (X,T) \to (Y,S)$ is said to be fuzzy almost continuous if the inverse image of every fuzzy regular open (closed) set is fuzzy open (closed).

Definition 4. Let (X,T) and (Y,S) be any two fuzzy topological spaces. A mapping $f\colon (X,T) \to (Y,S)$ is said to be fuzzy almost open (closed) if the image of every fuzzy regular open (closed) set is fuzzy open (closed).

Example 2. Let $X = \{a,b,c\}$. Define $T_1 = \{0,1,\lambda\}$ and $T_2 = \{0,1,\mu\}$ where $\lambda(a) = 0$, $\lambda(b) = \frac{2}{3}$, $\lambda(c) = \frac{1}{2}$ and $\mu(a) = 1$, $\mu(b) = 0$, $\mu(c) = 0$. Let $f\colon (X,T_1) \to (X,T_2)$ be the identity mapping. In (X,T_2) the only non-zero fuzzy regular open set is 1 and $f^{-1}(1) = 1$ shows that f is fuzzy almost continuous. This example is found in [9]. Now let $g\colon (X,T_2) \to (X,T_1)$ be the identity mapping, the only non-zero fuzzy regular open set in (X,T_2) is 1 and $f(1) = 1$ implies that f is fuzzy almost open.

The following result proved as Theorem 3.5 in [9] is used in the proof of some of the subsequent propositions and so it is given below for convenience of the reader.

Result [Theorem 3.5 in [9]]. If a mapping $f\colon X \to Y$ is A-fuzzy almost continuous and a.f.o.N., then

(a) the inverse image $f^{-1}(A)$ of each fuzzy regular open set A of Y is a fuzzy regular open set in X;

(b) the inverse image $f^{-1}(B)$ of each fuzzy regular closed set B of Y is a fuzzy regular closed set in X.

Proposition 3. *Let $f\colon X \to Y$ be a fuzzy almost open (almost closed) map of a space X onto a space Y and let $g\colon Y \to Z$. If $g \circ f$ is a fuzzy almost-continuous and fuzzy almost open map then g is fuzzy almost continuous.*

Proof. First let us assume f is fuzzy almost closed. Let λ be a fuzzy regular closed subset of Z. Then $(g \circ f)^{-1}(\lambda) = f^{-1}(g^{-1}(\lambda))$. Then by Theorem 3.5 of [9], $f^{-1}[g^{-1}(\lambda)]$ is a fuzzy regular closed set in X. Since f is fuzzy almost closed and surjective, $f[f^{-1}(g^{-1}(\lambda))] = g^{-1}(\lambda)$ is fuzzy closed in Y. Thus we have shown that g is fuzzy almost continuous. The proof is similar when f is fuzzy almost open. □

Proposition 4. *Assume that $f\colon X \to Y$ is fuzzy almost continuous and let $g\colon Y \to Z$ be a fuzzy almost continuous and fuzzy almost open map. Then $g \circ f$ is fuzzy almost continuous.*

Proof. Let λ be a fuzzy regular open set in Z. Then by Theorem 3.5 of [9], $g^{-1}(\lambda)$ is fuzzy regular open in Y and $(g \circ f)^{-1}(\lambda) = f^{-1}[g^{-1}(\lambda)]$ is fuzzy open in X. This proves that $g \circ f$ is fuzzy almost continuous. □

Proposition 5. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ and suppose that $(g \circ f)$ is a fuzzy almost open (fuzzy almost closed) map. If f is fuzzy almost continuous and fuzzy almost open surjection, then g is a fuzzy almost open (almost closed) map.*

Proof. Let λ be any fuzzy regular open set in Y. Then by Theorem 3.5 of [9], $f^{-1}(\lambda)$ is fuzzy regular open in X. Since $g \circ f$ is fuzzy almost open, $g \circ f[f^{-1}(\lambda)] = g(ff^{-1}(\lambda)) = g(\lambda)$ is fuzzy open in Z. This proves that g is fuzzy almost open. The proof is similar when f is fuzzy almost closed. □

Lemma 1. *Let $f\colon X \to Y$ be any fuzzy almost open map. Given any $\lambda \in I^Y$ and any fuzzy regular closed set μ containing $f^{-1}(\lambda)$, there exists a fuzzy closed set $\theta \geqslant \lambda$ such that $f^{-1}(\theta) \leqslant \mu$.*

Proof. Let $\lambda \in I^Y$ and let μ be any fuzzy regular closed set such that $f^{-1}(\lambda) \leqslant \mu$. Since f is fuzzy almost open and $1 - \mu$ is a fuzzy regular open set, it follows that $\theta = 1_Y - f(1_X - \mu)$ is a fuzzy closed set in Y and $\theta = 1_Y - f(1_X - \mu) \geqslant 1_Y - f\left(1_X - f^{-1}(\lambda)\right) \geqslant \lambda$. Now $1_X - \mu \leqslant f^{-1}f[1_X - \mu]$, which implies that $1_X - (1_X - \mu) \geqslant 1_X - f^{-1}f[1_X - \mu]$. That is, $\mu \geqslant 1_X - f^{-1}(f[1_X - \mu])$. We conclude that $f^{-1}(\theta) = f^{-1}(1_Y - f(1_X - \mu)) = 1_X - f^{-1}(f(1_X - \mu)) \leqslant \mu$. □

Lemma 2. *Let $f\colon X \to Y$ be a fuzzy almost continuous and fuzzy almost open map and let λ be any fuzzy closed subset of Y. Then*
(a) $\mathrm{Cl}[f^{-1}(\mathrm{Int}\,\lambda)] = f^{-1}[\mathrm{Cl}(\mathrm{Int}\,\lambda)]$.
(b) *If λ is a fuzzy regular closed subset, then* $\mathrm{Cl}[f^{-1}(\mathrm{Int}\,\lambda)] = f^{-1}(\lambda)$.

Proof. (a) Let λ be any fuzzy closed set of Y. Assume $\mathrm{Int}\lambda \neq 0$ (since there is nothing to prove if $\mathrm{Int}\lambda = 0$). Hence $\mathrm{Int}\lambda$ is fuzzy regular open. Since f is fuzzy almost continuous, $f^{-1}(\mathrm{Int}\lambda)$ is open in X. Also $\mathrm{Cl}\left[f^{-1}(\mathrm{Int}\lambda)\right]$ is a fuzzy regular closed set in X containing $f^{-1}[\mathrm{Int}\lambda]$. Therefore by Lemma 1 there exists a fuzzy closed set $\theta \geqslant \mathrm{Int}\lambda$ such that $f^{-1}[\mathrm{Int}\lambda] \leqslant f^{-1}(\theta) \leqslant \mathrm{Cl}\left[f^{-1}(\mathrm{Int}\lambda)\right]$. This implies that $f^{-1}[\mathrm{Cl}(\mathrm{Int}\lambda)] \leqslant \mathrm{Cl}\left[f^{-1}(\mathrm{Int}\lambda)\right]$. Now since f is almost continuous, $f^{-1}[\mathrm{Cl}(\mathrm{Int}\lambda)]$ is closed in X and so $f^{-1}[(\mathrm{Int}\lambda)] \leqslant f^{-1}[\mathrm{Cl}(\mathrm{Int}\lambda)]$. Therefore

$$\mathrm{Cl}[f^{-1}(\mathrm{Int}\,\lambda)] \leqslant \mathrm{Cl}[f^{-1}(\mathrm{Cl}(\mathrm{Int}\,\lambda))] = f^{-1}[\mathrm{Cl}(\mathrm{Int}\,\lambda)] \leqslant \mathrm{Cl}[f^{-1}(\mathrm{Int}\,\lambda)].$$

This implies that $\mathrm{Cl}\left[f^{-1}(\mathrm{Int}\,\lambda)\right] = f^{-1}[\mathrm{Cl}(\mathrm{Int}\,\lambda)]$. This proves (a).

(b) If λ is a fuzzy regular closed subset, then $\mathrm{Cl}[(\mathrm{Int}\lambda)] = \lambda$. Using this in (a) we get $\mathrm{Cl}\left[f^{-1}(\mathrm{Int}\lambda)\right] = f^{-1}(\lambda)$. This proves (b). □

Proposition 6. *Let $f\colon X \to Y$ be a fuzzy almost continuous and fuzzy almost open map. If μ is a fuzzy open set in Y, then*
(a) $\mathrm{Int}\left[\mathrm{Cl}f^{-1}[\mathrm{Int}(\mathrm{Cl}\mu)]\right] = \mathrm{Int}\left[\mathrm{Cl}f^{-1}(\mathrm{Cl}\mu)\right] = \mathrm{Int}\left[\ f^{-1}(\mathrm{Cl}\mu)\right]$.
(b) $f^{-1}[\mathrm{Int}(\mathrm{Cl}\mu)] = \mathrm{Int}\left[f^{-1}(\mathrm{Cl}\mu)\right]$.
(c) *If μ is fuzzy regular open set in Y, then* $\mathrm{Int}f^{-1}\left[f^{-1}(\mathrm{Cl}\mu)\right] = f^{-1}(\mu)$.

Proof. (a) Let μ be a fuzzy open set in Y. Since $\mathrm{Cl}(\mu)$ is a fuzzy regular closed set in Y, $f^{-1}(\mathrm{Cl}\,\mu)$ is fuzzy closed in X. In fact, by Lemma 2 we have $\mathrm{Cl}\,f^{-1}(\mathrm{Int}\,\mathrm{Cl}\,\mu) = f^{-1}(\mathrm{Cl}(\mu))$. Therefore we have $\mathrm{Int}\,\mathrm{Cl}[f^{-1}(\mathrm{Int}\,\mathrm{Cl}\,\mu)] = \mathrm{Int}\,\mathrm{Cl}[f^{-1}(\mathrm{Cl}\,\mu)] = \mathrm{Int}[f^{-1}(\mathrm{Cl}\,\mu)]$. This proves (a). □

(b) By Theorem 3.5 in [9] we have that $f^{-1}[\mathrm{Int}\,\mathrm{Cl}(\mu)]$ is a fuzzy regular open set in X. Thus $\mathrm{Int}\,\mathrm{Cl}[f^{-1}(\mathrm{Cl}(\mu))] = f^{-1}(\mathrm{Int}\,\mathrm{Cl}(\mu)) = \mathrm{Int}[f^{-1}\,\mathrm{Cl}(\mu)]$. This proves (b).

(c) Since μ is a fuzzy regular open set, $\mu = \mathrm{Int}\,\mathrm{Cl}(\mu)$. Hence it follows that $f^{-1}(\mathrm{Int}\,\mathrm{Cl}(\mu)) = f^{-1}(\mu) = \mathrm{Int}[f^{-1}\,\mathrm{Cl}(\mu)]$. This proves (c).

Proposition 7. *The image of a fuzzy nearly C-compact space under a fuzzy almost continuous and fuzzy almost open mapping is fuzzy nearly C-compact.*

Proof. Let $f\colon X \to Y$ be a fuzzy almost continuous and fuzzy almost open mapping from a fuzzy nearly C-compact space X onto Y. We have to show that Y is also fuzzy nearly C-compact. Let $A \subset Y$ be any subset of Y such that $\mathscr{X}_A$ is fuzzy regular closed in Y. Let $\mathscr{U} = \{\lambda_i\}_{i\in\Delta}$ be a fuzzy regular open cover of $\mathscr{X}_A$ in Y.

Since f is fuzzy almost continuous and fuzzy almost open, by Theorem 3.5 in [9], $f^{-1}(\mathscr{X}_A)$ is a fuzzy regular closed subset of X and $\left\{f^{-1}(\lambda_i)\right\}_{i\in\Delta}$ is a fuzzy regular open cover of $f^{-1}(\mathscr{X}_A)$ in X. Since X is fuzzy nearly C-compact, there exists a finite subfamily $\left\{f^{-1}(\lambda_i);\ i = 1, 2, \ldots, n\right\}$ such that

$$f^{-1}(\mathscr{X}_A) \leqslant \bigvee_{i=1}^{n} \mathrm{Cl}\left\{f^{-1}(\lambda_i)\right\} \leqslant \bigvee_{i=1}^{n} \left\{f^{-1}(\mathrm{Cl}\,\lambda_i)\right\}.$$

That is, $\mathscr{X}_A \leqslant \bigvee_{i=1}^{n} \left\{\mathrm{Cl}(\lambda_i)\right\}$. This proves that Y is fuzzy nearly C-compact. □

5. Fuzzy nearly C-compactness in fuzzy bitopological spaces

The concept of fuzzy bitopological spaces was introduced in [5] and subsequently further studied by various authors [2], [6]. In [5] the definition of fuzzy bitopological space was given as follows:

Definition 5. A fuzzy bitopological space is an ordered triple (X, T_1, T_2) where T_1 and T_2 are fuzzy topologies on X.

The concept of parirwise C-compactness for bitopological spaces was introduced in [7] as follows:

Definition 6. (X, T_1, T_2) is said to be (1,2) C-*compact* if for every proper T_1-closed subset $A \subset X$ and every T_2-open cover $\mathscr{U}$ of A, there exists a finite subcollection of $\mathscr{U}$ the T_2 -closure of whose members covers A. (X, T_1, T_2) is said to be *pairwise C-compact* if it is both (1,2) C-compact and (2,1) C-compact.

So now we are prepared to introduce the following

Definition 7. (X, T_1, T_2) is said to be $(1, 2)$ *fuzzy nearly C-compact* if for every set $A \subset X$ such that $\mathscr{X}_A$ is a proper T_1-fuzzy regular closed set and for every T_2-fuzzy open cover $\mathscr{U}$ of $\mathscr{X}_A$, there exists a finite subcollection of $\mathscr{U}$, (say) $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathscr{X}_A \leqslant \bigvee_{i=1}^{n} \mathrm{Cl}_{T_2}(\lambda_i)$. Then (X, T_1, T_2) is said to be *pairwise fuzzy nearly C-compact* if it is both (1,2) fuzzy nearly C-compact and (2,1) fuzzy nearly C-compact.

Definition 8. A mapping $f\colon (X, T_1, T_2) \to (Y, S_1, S_2)$ is said to be *pairwise fuzzy almost continuous (pairwise fuzzy almost open, pairwise fuzzy continuous)* if the induced mappings $f\colon (X, T_1) \to (Y, S_1)$ and $f\colon (X, T_2) \to (Y, S_2)$ are fuzzy almost continuous (fuzzy almost open, fuzzy continuous [3]).

Proposition 8. *Every pairwise fuzzy continuous and pairwise fuzzy almost open image of a pairwise fuzzy nearly C-compact space is pairwise fuzzy nearly C-compact.*

Proof. Let $f\colon (X,T_1,T_2)\to(Y,S_1,S_2)$ be any pairwise fuzzy almost continuous and pairwise fuzzy almost open onto mapping. Assume (X,T_1,T_2) is pairwise fuzzy nearly C-compact. We want to show that (Y,S_1,S_2) is pairwise fuzzy nearly C-compact.

Let $A\subset Y$ be such that $\mathscr{X}_A$ is a proper S_1-fuzzy regular closed set and let $\mathscr{U}$ be a S_2-fuzzy open cover of $\mathscr{X}_A$. Since f is fuzzy almost continuous and fuzzy almost open, $f^{-1}(\mathscr{X}_A)$ is T_1-fuzzy regular closed by Theorem 3.5 in [9] and $\{f^{-1}(\mu);\ \mu\in\mathscr{U}\}$ is a T_2-fuzzy open cover of $f^{-1}(\mathscr{X}_A)$. Since (X,T_1,T_2) is pairwise fuzzy nearly C-compact, there exists a finite subcollection $\{f^{-1}(\mu_k);\ k=1,2,\dots,n\}$ such that $f^{-1}(\mathscr{X}_A)\leqslant\bigvee\limits_{k=1}^{n}\operatorname{Cl}_{T_2}f^{-1}(\mu_k)$. Hence,

$$\begin{aligned}\mathscr{X}_A=ff^{-1}(\mathscr{X}_A)&\leqslant\bigvee_{k=1}^{n}f[\operatorname{Cl}_{T_2}f^{-1}(\mu_k)]\\&\leqslant\bigvee_{k=1}^{n}\operatorname{Cl}_{S_2}\left(ff^{-1}(\mu_k)\right)\\&\leqslant\bigvee_{k=1}^{n}\operatorname{Cl}_{S_2}(\mu_k)\,.\end{aligned}$$

This proves that Y is (1,2) fuzzy nearly C-compact. Similarly we can show that Y is also (2,1) fuzzy nearly C-compact. Thus we have shown that (Y,S_1,S_2) is pairwise fuzzy nearly C-compact. □

Proposition 9. *Let (X,T_1,T_2) be any pairwise fuzzy nearly C-compact space. Then if $A\subset X$ is such that $\mathscr{X}_A$ is proper T_i-fuzzy regular closed and $\mathfrak{F}$ is a family of T_j-fuzzy closed subsets of X such that $\bigwedge\{\lambda\wedge\mathscr{X}_A;\ \lambda\in\mathfrak{F}\}=0$, there exists a finite number of elements, say $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ of $\mathfrak{F}$ such that*

$$\bigwedge_{k=1}^{n}\{\operatorname{Int}_{T_j}\lambda_k\wedge\mathscr{X}_A\}=0,\quad i\neq j,\ i,j=1,2.$$

Proof. Suppose (X,T_1,T_2) is fuzzy pairwise nearly C-compact. Let $A\subset X$ be such that $\mathscr{X}_A$ is proper T_i-fuzzy regular closed and $\mathfrak{F}$ is a family of T_j-fuzzy closed sets of X such that $\bigwedge\{\lambda\wedge\mathscr{X}_A;\ \lambda\in\mathfrak{F}\}=0$. Now $\bigwedge\limits_{\lambda\in\mathfrak{F}}\{\lambda\wedge\mathscr{X}_A\}=0\Rightarrow\bigwedge\limits_{\lambda\in\mathfrak{F}}(\lambda)\leqslant 1-\mathscr{X}_A\Rightarrow\mathscr{X}_A\leqslant 1-\bigwedge\limits_{\lambda\in\mathfrak{F}}\lambda=\bigvee\{1-\lambda;\ \lambda\in\mathfrak{F}\}$. So $\{1-\lambda;\ \lambda\in\mathfrak{F}\}$ is a T_j-fuzzy open cover of $\mathscr{X}_A$ which is T_i-fuzzy regular closed and hence by assumption we have a finite collection, say $1-\lambda_1, 1-\lambda_2, \ldots, 1-\lambda_n$ such that

$$\mathscr{X}_A \leqslant \bigvee_{k=1}^{n} \operatorname*{Cl}_{T_j}(1-\lambda_k) = \bigvee_{k=1}^{n} (1-\operatorname{Int}_{T_j}\lambda_k) = 1 - \bigwedge_{k=1}^{n} \operatorname{Int}_{T_j}\lambda_k.$$

This implies $\bigwedge\limits_{k=1}^{n} \operatorname{Int}_{T_j}\lambda_k \leqslant 1-\mathscr{X}_A = \mathscr{X}_{X-A}$. Therefore $\mathscr{X}_A \bigwedge \Big(\bigwedge\limits_{k=1}^{n} \operatorname{Int}_{T_j}\lambda_k\Big) \leqslant \mathscr{X}_A \wedge \mathscr{X}_{X-A} = 0 \Rightarrow \bigwedge\limits_{k=1}^{n} \big(\operatorname{Int}_{T_j}\lambda_k \wedge \mathscr{X}_A\big) = 0$. This proves the proposition. □

Definition 9. A T_j-fuzzy filter base $\mathfrak{F}$ is said to be T_{ij}-*regular adherent convergent* if every T_i-regular open neighbourhood of the T_j-adherent set of $\mathfrak{F}$ contains an element of $\mathfrak{F}, i \neq j, i, j = 1, 2$, where we define the T_j-*adherent set* of $\mathfrak{F}$ to be $\bigwedge\{\operatorname{Cl}_{T_j}\lambda;\ \lambda \in \mathfrak{F}\}$.

Proposition 10. *If (X, T_1, T_2) is pairwise fuzzy nearly C-compact then every T_j-open filter base is T_{ij}-regular adherent convergent ($i \neq j, i, j = 1, 2$).*

Proof. Suppose (X, T_1, T_2) is pairwise fuzzy nearly C-compact and let $\mathfrak{F}$ be any T_j-open filter base with λ the T_j -adherent set of $\mathfrak{F}$. Let σ be any T_i-regular open neighbourhood of λ. So we have $\lambda = \bigwedge\{\operatorname{Cl}_{T_j}\mu;\ \mu \in \mathfrak{F}\}$ and $\lambda \leqslant \sigma$ and $1-\sigma$ is T_i-fuzzy regular closed. Now $1-\sigma \leqslant 1-\lambda = 1-\bigwedge\{\operatorname{Cl}_{T_j}\mu;\ \mu \in \mathfrak{F}\} = \bigvee\{1-\operatorname{Cl}_{T_j}\mu;\ \mu \in \mathfrak{F}\}$. Therefore $\{1-\operatorname{Cl}_{T_j}\mu;\ \mu \in \mathfrak{F}\}$ is a T_j-open cover of the fuzzy regular closed set $1-\sigma$. Since (X, T_1, T_2) is pairwise fuzzy nearly C-compact, we can find a subcollection, say $\{1-\operatorname{Cl}_{T_j}\mu_k;\ k = 1, 2, \ldots, n\}$ such that $1-\sigma \leqslant \bigvee\limits_{k=1}^{n} \operatorname{Cl}_{T_j}\big(1-\operatorname{Cl}_{T_j}\mu_k\big) = \bigvee\limits_{k=1}^{n} \big[\big(1-\operatorname{Int}_{T_j}\big(\operatorname{Cl}_{T_j}\mu_k\big)\big)\big] = 1 - \bigwedge\limits_{k=1}^{n} \operatorname{Int}_{T_j}\big(\operatorname{Cl}_{T_j}\mu_k\big) \Rightarrow \bigwedge\limits_{k=1}^{n} \operatorname{Int}_{T_j}\big(\operatorname{Cl}_{T_j}\mu_k\big) \leqslant \sigma$. Further, $\mu_k \leqslant \operatorname{Int}_{T_j}\operatorname{Cl}_{T_j}\mu_k$ for $k = 1, 2, \ldots, n$. Therefore $\bigwedge\limits_{k=1}^{n} \mu_k \leqslant \operatorname{Int}_{T_j}\operatorname{Cl}_{T_j}\mu_k$. Hence $\bigwedge\limits_{k=1}^{n} \mu_k \leqslant \sigma \Rightarrow \sigma \in \mathfrak{F}$. Hence the proposition follows. □

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