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ON SUCCESSIVE APPROXIMATIONS FOR SOLVING THE CAUCHY PROBLEM FOR A SYSTEM OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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In the present note, we consider a succesive approximations method of construction of the solution of the Cauchy problem

$$dx(t) = dA(t) \cdot x(t) + dq(t), \tag{1}$$
$$x(t_0) = c_0, \tag{2}$$

where $t_0 \in [a,b]$, $c_0 \in R^n$, $A : [a,b] \to R^{n\times n}$ and $q : [a,b] \to R^n$ are a matrix-function and a vector-function with bounded variation components, respectively.

The following notation and definitions will be used: $R =]-\infty, +\infty[$, $[a,b]$ $(a, b \in R)$ is a closed segment, $R^{n\times m}$ is the set of all real $n\times m$-matrices $X = (x_{ik})_{i,k=1}^{n,m}$; If $X \in R^{n\times n}$, then $\det(X)$ is the determinant of X, I_n is the identity $n \times n$-matrix; $R^n = R^{n\times 1}$ is the set of all real column n-vectors $x = (x_i)_{i=1}^n$.

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions $X = (x_{ik})_{i,k=1}^{n,m} : [a,b] \to R^{n\times m}$ such that every its component x_{ik} has bounded total variation on $[a,b]$; $X(t-) = (x_{ik}(t-))_{i,k=1}^{n,m}$ and $X(t+) = (x_{ik}(t+))_{i,k=1}^{n,m}$ are the left and the right limits of X at the point $t \in [a,b]$ $(X(a-) = X(a),\ X(b+) = X(b))$, $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

If $g : [a,b] \to R$ is a nondecreasing function, $x : [a,b] \to R$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s),$$

where $\int_{]s,t[} x(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to the function g (if $s = t$, then $\int_s^t x(\tau)dg(\tau) = 0$).

If $g_j : [a,b] \to R$ $(j = 1,2)$ are nondecreasing functions, $g = g_1 - g_2$ and $x : [a,b] \to R^n$, then

$$\int_s^t x(\tau)dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)dg_2(\tau) \quad \text{for} \quad a \le s \le t \le b.$$

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If $G = (g_{ik})_{i,k=1}^{n} \in \mathrm{BV}([a,b], R^{n\times n})$, $x = (x_k)_{k=1}^{n} \in \mathrm{BV}([a,b], R^n)$, then

$$\int_s^t dG(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^{n} \int_s^t x_k(\tau)\, dg_{ik}(\tau)\Big)_{i=1}^{n} \quad \text{for} \quad a \le s \le t \le b.$$

A vector-function $x \in \mathrm{BV}([a,b], R^n)$ is said to be a solution of the problem (1),(2) if it satisfies the condition (2) and

$$x(t) = x(s) + \int_s^t dA(\tau)\cdot x(\tau) + q(t) - q(s) \quad \text{for} \quad a \le s < t \le b.$$

Theorem. *Let*

$$\det\big(I_n + (-1)^j d_j A(t)\big) \ne 0 \quad \textit{for} \quad (-1)^j (t - t_0) < 0 \;\; (j = 1, 2).$$

Then the problem (1),(2) *has a unique solution* x *and*

$$\lim_{k\to+\infty} x_k(t) = x(t) \quad \textit{uniformly on} \;\; [a,b],$$

where

$$x_k(t_0) = c_0 \quad (k = 0, 1, \dots),$$
$$x_0(t) = \big(I_n + (-1)^j d_j A(t)\big)^{-1} c_0 \quad \textit{for} \quad (-1)^j (t - t_0) < 0 \;\; (j = 1, 2)$$

and

$$x_k(t) = \big(I_n + (-1)^j d_j A(t)\big)^{-1} \Big[c_0 + \int_{t_0}^t dA(\tau)\cdot x_{k-1}(\tau) + $$
$$+ (-1)^j d_j A(t)\cdot x_{k-1}(t) + q(t) - q(t_0)\Big]$$
$$\textit{for} \quad (-1)^j (t - t_0) < 0 \quad (j = 1, 2; \;\; k = 1, 2, \dots).$$

Note that the unique solvability of the problem (1),(2) is proved in [1].

References

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