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CONTINUOUS DEPENDENCE OF THE SOLUTION OF A CLASS OF NEUTRAL DIFFERENTIAL EQUATIONS ON THE INITIAL DATA AND ON THE RIGHT-HAND SIDE

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Let $J=[a,b]$ be a finite interval, $\mathbb{R}^n$ be an n-dimensional Euclidean space, $O\subset\mathbb{R}^n$ an open set, $\eta:J\to\mathbb{R}^1$ and $\tau:J\to\mathbb{R}^1$ be continuously differentiable functions satisfying respectively the conditions: $\eta(t)<t$, $\dot\eta(t)>0$; $\tau(t)\le t$, $\dot\tau(t)>0$; moreover, let $L_1(J,\mathbb{R}^1_+)$ be the space of summable functions $m:J\to\mathbb{R}^1_+$, $\mathbb{R}^1_+=[0,+\infty)$, $\Delta(J,\mathbb{R}^{n\times n})$ be the space of piecewise continuous $n\times n$ matrix functions $C:J\to\mathbb{R}^{n\times n}$ with a finite number of points of discontinuity of the first kind, $\|C\|=\sup_{t\in J}|C(t)|$, $C^1(J_1,O)$ be the space of continuously differentiable functions $\varphi:J_1\to O$, $J_1=[\tau,b]$, $\tau=\min\{\eta(a),\tau(a)\}$, for which $\|\varphi\|=|\varphi(a)|+\max_{t\in J_1}|\dot\varphi(t)|$, and let E_f be the space of the functions $f: J\times O^2\to\mathbb{R}^n$ satisfying the following conditions:

(1) the function $f(\cdot,x,y):J\to\mathbb{R}^n$ is measurable for every $(x,y)\in O^2$:

(2) for any compactum $K\subset O$ and any function $f\in E_f$ there exist $m_{f,K}(\cdot)$, $L_{f,K}(\cdot)\in L_1(J,\mathbb{R}^1_+)$ such that

$$|f(t,x,y)|\le m_{f,K}(t),\quad \forall(t,x,y)\in J\times K^2,$$
$$\Big|f(t,x',y')-f(t,x'',y')\Big|\le L_{f,K}(t)\Big(|x'-x''|+|y'-y''|\Big),\forall(t,x',x'',y',y'')\in J\times K^4.$$

Introduce the sets:

$$V_1(K,\delta)=\Big\{f\in E_f:\ \max_{(t',t'',x,y)\in J^2\times K^2}\Big|\int_{t'}^{t''}f(t,x,y)\,dt\Big|\le\delta\Big\},$$
$$V_2(K,\alpha)=\Big\{f\in E_f:\ \int_J\Big[m_{f,K}(t)+L_{f,K}(t)\Big]\,dt\le\alpha\Big\},$$
$$W(K,\delta,\alpha)=V_1(K,\delta)\cap V_2(K,\alpha),$$

where $K\subset O$ is a compact set, $\delta>0$ and $\alpha>0$ are arbitrary numbers.

To every element $\sigma=(t_0,x_0,\varphi,C,f)\in\Sigma=J\times O\times C^1(J_1,O)\times\Delta(J,\mathbb{R}^{n\times n})\times E_f$ we assign the neutral differential equation

$$\dot x(t)=C(t)\dot{(x)}(\eta(t))+f\Big(t,x(t),x(\tau(t))\Big)\tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\quad t\in[\tau_0,t_0),\quad x(t_0)=x_0,\tag{2}$$

where $\tau_0=\min\{\eta(t_0),\tau(t_0)\}$.

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Definition. Every function $x(t;\sigma)\in O$ defined on the interval $[\tau_0,t_1]\subset(\tau,b]$ will be called a solution corresponding to the element $\sigma\in\Sigma$ if it satisfies on $[\tau_0,t_0]$ the condition (2), is absolutely continuous on $[t_0,t_1]$ and almost everywhere satisfies the equation (1).

Theorem. *Let $\widetilde{x}(t)=x(t;\widetilde{\sigma})$, $t\in[\widetilde{t}_0,\widetilde{t}_1]$, be a solution corresponding to the element $\widetilde{\sigma}=(\widetilde{t}_0,\widetilde{x}_0,\widetilde{\varphi},\widetilde{C},\widetilde{f})\in\Sigma$ and let the compactum $K_1\subset O$ contain a neighborhood of the set $K_0=\{\widetilde{x}(t):\ t\in[\widetilde{\tau}_0,\widetilde{t}_1]\}$, where $\widetilde{\tau}_0=\min\{\eta(\widetilde{t}_0),\tau(\widetilde{t}_0)\}$. Then for any $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that to every element*

$$\sigma\in V(\widetilde{\sigma},K_1,\delta,\alpha_0)=V(\widetilde{t}_0,\delta)\times V(\widetilde{x}_0,\delta)\times V(\widetilde{\varphi},\delta)\times V(\widetilde{C},\delta)\times W(K_1,\delta,\alpha_0)$$

there corresponds the solution $x(t;\sigma)$ defined on the interval $[\tau_0,\widetilde{t}_0+\delta]$. Moreover, if $\sigma_i=(t_0^i,x_0^i,\varphi_i,C_i,f_i)\in V(\widetilde{\sigma},K_1,\delta,\alpha_0)$, $i=1,2$, then

$$\Big|x(t;\sigma_1)-x(t;\sigma_2)\Big|\le\varepsilon,\quad t\in[\overline{t}_0,\widetilde{t}_1+\delta],$$

where $\overline{t}_0=\max(t_0^1,t_0^2)$, $\alpha_0>0$ is a fixed number.

Here $V(\widetilde{t}_0,\delta)$, $V(\widetilde{x}_0,\delta)$, $V(\widetilde{\varphi}_0,\delta)$, $V(\widetilde{C},\delta)$ are δ-neighborhoods of the points $\widetilde{t}_0$, $\widetilde{x}_0$, $\widetilde{\varphi}$, $\widetilde{C}$ in the spaces $\mathbb{R}^1$, $\mathbb{R}^n$, $C^1(J_1,O)$, $\Delta(J,\mathbb{R}^{n\times n})$, respectively.

The above formulated theorem is an analogue of a theorem stated in [1], [2] (see also [3] and [4]). This theorem can be proved by the method described in [2].

In conclusion it should be noted that if the right-hand side of the equation (1) depends nonlinearly on $\dot{x}(\eta(t))$, then the theorem is, generally speaking, invalid. The appropriate example is given in [3].

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