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REPRESENTATION OF c_0-SEMIGROUPS OF OPERATORS BY A CHRONOLOGICAL INTEGRAL

Abstract. Right and Left chronological integrals are defined for a function of real variable whose domain of values is endowed with minimal algebraic and limiting structures. It is proved that an exponential function of a linear (possibly unbounded) operator can be represented by means of a chronological integral which preserves a number of properties of a chronological exponent.

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Key words and Phrases. monoid, Riemann's chronological integral, formula of arrow inversion, formula of integration by parts, c_0-semigroup of operators, exponent.

რეზიუმე. ნამდვილი ცლადის ფუნქციისათვის, რომლის მნიშვნელობათა სიმრავლეში მინიმალური ალგებრული და ზღვარითი სტრუქტურებია შეტანილი, განსაზღვრულია მარჯვენა და მარცხენა ქრონოლოგიური ინტეგრალები. დამტკიცებულია, რომ წრფივი (შესაძლოა შემოუსაზღვრელი) ოპერატორის ექსპონენტა შეიძლება წარმოდგენილი იქნეს ისეთი ქრონოლოგიური ინტეგრალის სახით, რომელიც ინახავს მთელს რიგს ქრონოლოგიური ექსპონენტის თვისებებისა.

1. Definition of the Riemann Chronological Integral

1.1. Auxiliary Definitions and Facts. Denote by M an abstract monoid whose algebraic structure is defined by a binary associative operation $\{(g_1, g_2) \longmapsto g_1 * g_2\} : M \times M \to M$ and by the unity e. If some $g \in M$ is invertible, then we denote its inverse by $\overline{g}$.

The limiting structure in M is determined by means of directedness es. This permits us to retain simplicity and generality. As usual, a family $G = \{g_w\}_{w\in\Omega}$ is said to be a directedness in M if $(\Omega, \geq)$ is a directed set, and $w \longmapsto g_w$ maps Ω into M. Given two directednesses $G = \{g_{w_1}\}_{w_1\in\Omega_1}$ and $F = \{f_{w_2}\}_{w_2\in\Omega_2}$ in M, G is said to be a subdirectedness of F when there exists a mapping $N : \Omega_1 \to \Omega_2$ such that:

(a) $g_{w_1} = f_{N(w_1)}$, $\forall w_1 \in \Omega_1$;

(b) for every $w_2 \in \Omega_2$ there exists $w_1 \in \Omega_1$ such that from $w \in w_1$ and $w \geq w_1$ it follows $N(w) \geq w_2$.

The limiting structure in M is defined by a system $\mathfrak{L}$ composed of the pairs (G, g) ($g \in M$, G is a directedness in M) and satisfying the following restrictions:

(i) if $G = \{g_w\}_{w\in\Omega}$ is a directedness such that $g_w = g$ for every $w \in \Omega$, then $(G, g) \in \mathfrak{L}$;

(ii) if $(G, g_1) \in \mathfrak{L}$ and $(G, g_2) \in \mathfrak{L}$, then $g_1 = g_2$;

(iii) if F is a subdirectedness in G, and $(G, g) \in \mathfrak{L}$, then $(F, g) \in \mathfrak{L}$.

In the sequel, using the conventional terminology, if $(G, g) \in \mathfrak{L}$ we will say that G converges to g and g is a limit of G.

The limiting and algebraic structures are compatible, i.e.,

$$\{(g_1, g_2) \longmapsto g_1 * g_2\} : M \times M \to M$$

is a continuous mapping. Under our notation this means that if $\{g_{w_1}\}_{w_1\in\Omega_1}$ converges to g and $\{f_{w_2}\}_{w_2\in\Omega_2}$ converges to f, then $\{g_{w_1} * f_{w_2}\}_{(w_1,w_2)\in\Omega_1\times\Omega_2}$ converges to $g * f$. $(\Omega_1 \times \Omega_2)$ denotes a directed product of the directed sets:

$$(w_1, w_2) \geq (\widetilde{w}_1, \widetilde{w}_2) \quad \text{is equivalent to} \quad (w_1 \geq \widetilde{w}_1 \quad \textit{and} \quad w_2 \geq \widetilde{w}_2).$$

1.2. Definition of the Chronological Integral and Some Examples. Let $(t, s) \longmapsto f(t, s)$ map the triangle $[a, b] \times [0, \eta]$ into M, where $[a, b] \subset \mathbb{R}$, and η is a positive number. $\Sigma(a, b)$ denotes the set of all partitions of the form $\sigma = \{a = s_0 \leq \xi_1 \leq s_1 \leq \cdots \leq \xi_n \leq s_n = b\}$, and

$$\Delta s_i = s_i - s_{i-1}, \quad |\sigma| = \max\{\Delta s_i | i = 1, \ldots, n\}.$$

$\Sigma(a, b)$ is a directed set, the relation $\sigma_1 \leq \sigma_2$ meaning $|\sigma_1| \geq |\sigma_2|$.

For a sufficiently fine partition σ we denote

$$\overset{\rightarrow}{\sum_{\sigma}} f = f(\xi_1, \Delta s_1) * f(\xi_2, \Delta s_2) * \cdots * f(\xi_n, \Delta s_n),$$

$$\overset{\leftarrow}{\sum_{\sigma}} f = f(\xi_n, \Delta s_n) * \cdots * f(\xi_2, \Delta s_2) * f(\xi_1, \Delta s_1),$$

the arrow showing the order of co-factors in the right-hand side, which is important in the non-commutative case.

Definition 1. We say that $g \in M$ is the right (left) chronological integral (or simply, integral) of the function f from a to b, and write

$$g = \int\limits_a^b f(\overrightarrow{\tau}, d\tau) \quad \left(g = \int\limits_a^b f(\overleftarrow{\tau}, d\tau)\right),$$

if for some $\sigma_0 \in \Sigma(a,b)$ the directedness

$$\left\{ \overset{\rightarrow}{\sum_{\sigma}} f \right\}_{\sigma_0 \leq \sigma \in \Sigma(a,b)} \quad \left(\left\{ \overset{\leftarrow}{\sum_{\sigma}} f \right\}_{\sigma_0 \leq \sigma \in \Sigma(a,b)} \right). \tag{1}$$

is defined correctly and converges to g. When

$$g = \int\limits_a^b f(\overrightarrow{\tau}, d\tau) \quad \left(g = \int\limits_a^b f(\overleftarrow{\tau}, d\tau)\right),$$

is invertible, we say that $\overline{g} \in M$ is the right (left) integral of the function f from b to a, and write

$$\overline{g} = \int\limits_b^a f(\overrightarrow{\tau}, d\tau) \quad \left(\overline{g} = \int\limits_b^a f(\overleftarrow{\tau}, d\tau)\right).$$

The directedness (1) is defined for every σ_0 such that $|\sigma_0| < \eta$. $\Sigma(a,b)$ is a directed set, and for every σ_1, σ_2 there exists their majorant. Therefore in Definition 1 the values of the integral do not depend on the choice of σ_0.

In what follows, the notation $\{\alpha_\nu\}_{\nu_0 \leq \nu \in \Sigma(a,b)}$ means that we consider the given directedness starting from some ν_0.

When the values of the right and of the left integrals coincide, we can omit the arrow and write $\int\limits_a^b f(\tau, d\tau)$. Such cases arise usually when M is a commutative monoid, or when the values of the function f commute, since for a sufficiently fine σ there takes place $\sum_\sigma^{\rightarrow} f = \sum_\sigma^{\leftarrow} f$, and therefore the arrow can be omitted.

Remark 1. In case of necessity (for example, when on the support M we can determine in two ways a structure of monoid consistent with the limiting structure), in our notation the binary operation will be indicated regarding to which the integral is taken

$$
{}^{\circledast}\sum_{\sigma}^{\rightarrow} f, \quad {}^{\circledast}\int_a^b f(\tau, d\tau).
$$

Example 1. (The Riemann integral). Let $t \longmapsto f(t)$ map $[a,b]$ into $\mathbb{R}$. If we consider $\mathbb{R}$ with the ordinary convergence and with the operations O, $+$, then Definition 1 for the function $\{(t,s) \longmapsto f(t)s\} : [a,b] \times \mathbb{R} \to \mathbb{R}$ provides the Riemann's integral of the function f.

Example 2. (the multiplicative integral). Let us consider the limit of products of the form

$$
\prod_{\sigma} = \exp(A(s_n)(s_n - s_{n-1})) \cdots \exp(A(s_1)(s_1 - s_0)), \tag{2}
$$

where $\sigma = \{a = s_0 \leq \cdots \leq s_n = b\}$ and $A(s)$ is a continuous function from $[a,b]$ to the space $B(E)$ of bounded linear operators in the Banach space E. The limit is taken as $|\sigma| \to 0$, is denoted as

$$
\overset{\frown}{\int_a^b} \exp(A(s)ds)
$$

and is called the multiplicative integral ([1]).

If we consider $B(E)$ with the operation of addition, $f : [a,b] \to B(E)$, and apply Definition 1, then we obtain the ordinary Riemann integral (as in the foregoing example): $\int_a^b f(\tau)d\tau = {}^{\oplus}\int_a^b f(\tau)d\tau$.

Consider $B(E)$ with the operation of composition and apply for $(t,s) \to \exp(A(t)s)$ Definition 1. Then from the existence of ${}^{\circledast}\int_a^b \exp(A)(\overleftarrow{s}\, ds)$ there follows that of $\overset{\frown}{\int_a^b} \exp(A(s)ds)$, and hence their equality.

Example 3. (T-exponent). Let $A(t)$, $t \in [a,b]$, be a piecewise-continuous family in a noncommutative Banach algebra.

By the definition ([2]), the T-exponent $U = \operatorname{Exp} \int_a^t A(s)ds$ is the solution of the Cauchy problem for the evolution equation

$$
\frac{\partial U}{\partial t} = A(t)U, \quad U|_{t=a} = 1(\text{unity of the algebra}),
$$

and the following condition is fulfilled:

$$\operatorname{Exp}\int_a^t A(s)ds = \lim_{|\sigma|\to 0} \exp(A(s_n)(s_n - s_{n-1}))\cdots\exp(A(s_1)(s_1 - s_0)).$$

The Banach algebra with its limiting structure and operations of composition and unity is a monoid such that one can apply Definition 1 to the mapping $(\tau, s) \longmapsto \exp(A(\tau)s)$. Obviously, from the existence of $\exp^{\circledast}\int_a^t A(\overleftarrow{s})ds$ there follows that of $\operatorname{Exp}\int_a^t A(s)ds$ and hence their equality (we can prove that these integrals exist simultaneously).

Example 4. Let $A(t)$, $t \in [a, b]$ be a family of unbounded linear operators on the Banach space X. Under certain conditions (see [3]), for any $a \le s \le t \le b$ and for sufficiently fine $\sigma = \{s = s_0 \le \cdots \le s_n = t\}$ the limit of products (2) is defined correctly, and there exists a strong limit $\lim\limits_{|\sigma|\to 0}\prod\limits_{\sigma} = U(s, t)$.

1.3. Algebraic Properties of Integrals. Directly from the definition we have

Proposition 1. Let for some $\eta > 0$ $f : [a, b] \times [0, \eta] \to M$ and $f(t, 0) = e$ for every t. Then $\int_t^t f(\tau, d\tau) = e$, $\forall t \in [a, b]$.

The following result is analogous to the formula of arrow inversion for the chronological exponent ([4]).

Proposition 2. (Formula of arrow inversion). Let $f : [a, b] \times [0, \eta] \to M$, $\eta > 0$, let there exist $\overline{f(t, s)}$, $\forall (t, s) \in [a, b] \times [0, \eta]$ and let for some t_1, $t_2 \in [a, b]$ there exist $\int_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)$ and $\int_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}$. Then $\exists \int_{t_2}^{t_1} f(\overrightarrow{\tau}, d\tau)$ and

$$\int_{t_2}^{t_1} f(\overrightarrow{\tau}, d\tau) = \int_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}. \tag{3}$$

Proof. For the sake of simplicity, let us consider first the case $t_1 \le t_2$. By our condition, there exists sufficiently fine $\sigma_0 \in \Sigma(t_1, t_2)$ such that $\left\{\overrightarrow{\sum_{\sigma}} f\right\}_{\sigma_0 \le \sigma \in \Sigma(t_1, t_2)}$ and $\left\{\overleftarrow{\sum_{\sigma}} \overline{f}\right\}_{\sigma_0 \le \sigma \in \Sigma(t_1, t_2)}$ converge respectively to $\int_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)$ and $\int_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}$. The binary operation is continuous in M. Therefore

$$\left\{\left(\overrightarrow{\sum_{\sigma_1}} f\right) * \left(\overleftarrow{\sum_{\sigma_2}} \overline{f}\right)\right\}_{(\sigma_0, \sigma_0) \le (\sigma_1, \sigma_2) \in \Sigma(t_1, t_2) \times \Sigma(t_1, t_2)} \tag{4}$$

converges to $\left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)\right) * \left(\int\limits_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}\right)$. Consequently, $\left\{\left(\overrightarrow{\sum_\sigma} f\right) * \left(\overleftarrow{\sum_\sigma} \overline{f}\right)\right\}_{(\sigma_0,\sigma_0)\leq(\sigma,\sigma)\in\Sigma(t_1,t_2)\times\Sigma(t_1,t_2)}$, being the subdirectedness in (4), converges to the same limit. Taking into account that for every $\sigma \geq \sigma_0$ $\left(\overrightarrow{\sum_\sigma} f\right) * \left(\overleftarrow{\sum_\sigma} \overline{f}\right) = \left(\overleftarrow{\sum_\sigma} \overline{f}\right) * \left(\overrightarrow{\sum_\sigma} f\right) = e$, we obtain $\left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)\right) * \left(\int\limits_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}\right) = e$.

Analogously we obtain that $\left(\int\limits_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}\right) * \left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)\right) = e$. Thus we have determined both sides in (3), hence (3) holds.

Let now $t_2 < t_1$. By Definition 1, from the condition of our proposition there follows the existence of the integrals

$$\int\limits_{t_2}^{t_1} f(\overrightarrow{\tau}, d\tau) = \overline{\left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)\right)}, \quad \int\limits_{t_2}^{t_1} \overline{f(\overleftarrow{\tau}, d\tau)} = \overline{\left(\int\limits_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}\right)}.$$

Applying the case considered above, we can see that

$$\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau) = \int\limits_{t_2}^{t_1} \overline{f(\overleftarrow{\tau}, d\tau)}.$$

Consequently,

$$\int\limits_{t_2}^{t_1} f(\overrightarrow{\tau}, d\tau) = \overline{\left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)\right)} = \overline{\left(\int\limits_{t_2}^{t_1} \overline{f(\overleftarrow{\tau}, d\tau)}\right)} =$$

$$= \int\limits_{t_1}^{t_2} \overline{f(\overleftarrow{\tau}, d\tau)}. \quad \blacksquare$$

Proposition 3. Let $\eta > 0$, $f : [a,b] \times [0,\eta] \to M$ and t_1, t_2, $t_3 \in [a,b]$. Then:

(a) if there exist $\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)$, $\int\limits_{t_2}^{t_3} f(\overrightarrow{\tau}, d\tau)$ and $\int\limits_{t_1}^{t_3} f(\overrightarrow{\tau}, d\tau)$, then

$$\int\limits_{t_1}^{t_3} f(\overrightarrow{\tau}, d\tau) = \left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau)\right) * \left(\int\limits_{t_2}^{t_3} f(\overrightarrow{\tau}, d\tau)\right); \tag{5}$$

(b) if there exist $\int\limits_{t_1}^{t_2} f(\overleftarrow{\tau}, d\tau)$, $\int\limits_{t_2}^{t_3} f(\overleftarrow{\tau}, d\tau)$ and $\int\limits_{t_1}^{t_3} f(\overleftarrow{\tau}, d\tau)$, then

$$\int\limits_{t_1}^{t_3} f(\overleftarrow{\tau}, d\tau) = \int\limits_{t_2}^{t_3} f(\overleftarrow{\tau}, d\tau) * \int\limits_{t_1}^{t_2} f(\overleftarrow{\tau}, d\tau).$$

Proof. Let us prove the case (a) (the case (b) can be proved analogously). Let $t_1 < t_2 < t_3$. Owing to the continuity of the binary operation in M, the directedness

$$\left\{ \left(\overrightarrow{\sum_{\sigma_1}} f \right) * \left(\overrightarrow{\sum_{\sigma_2}} f \right) \right\}_{(\sigma_1^0, \sigma_2^0) \le (\sigma_1, \sigma_2) \in \Sigma(t_1,t_2) \times \Sigma(t_2,t_3)}$$

converges to the right-hand side of (5). On the other hand, being the subdirectedness of the directedness $\left\{ \overrightarrow{\sum\limits_{\sigma}} f \right\}_{\sigma_0 \le \sigma \in \Sigma(t_1,t_3)}$, it also converges to the left-hand side of (5). Thus (5) holds.

The remaining five cases are reduced to that proven above. As an example, consider the case $t_3 < t_2 < t_1$:

$$\int\limits_{t_1}^{t_3} f(\overrightarrow{\tau}, d\tau) = \overline{\left(\int\limits_{t_3}^{t_1} f(\overrightarrow{\tau}, d\tau) \right)} =$$

$$= \overline{\left\{ \left(\int\limits_{t_3}^{t_2} f(\overrightarrow{\tau}, d\tau) \right) * \left(\int\limits_{t_2}^{t_1} f(\overrightarrow{\tau}, d\tau) \right) \right\}} =$$

$$= \overline{\left\{ \left(\int\limits_{t_2}^{t_1} f(\overrightarrow{\tau}, d\tau) \right)} * \overline{\left\{ \left(\int\limits_{t_3}^{t_2} f(\overrightarrow{\tau}, d\tau) \right)} =$$

$$= \left(\int\limits_{t_1}^{t_2} f(\overrightarrow{\tau}, d\tau) \right) * \left(\int\limits_{t_2}^{t_3} f(\overrightarrow{\tau}, d\tau) \right). \quad \blacksquare$$

Proposition 4. Let for some $\eta > 0$ $f_i : [a,b] \times [0,\eta] \to M$, $i \in \{1,2\}$, $f_1(t,s) * f_2(t,s) = f_2(t,s) * f_1(t,s)$, $\forall (t,s) \in [a,b] \times [0,\eta]$ and let $t_1, t_2 \in [a,b]$. Then:

(a) if there exist $\int\limits_{t_1}^{t_2} f_i(\overrightarrow{\tau}, d\tau)$, $i \in \{1,2\}$, then $\int\limits_{t_1}^{t_2} (f_1(\overrightarrow{\tau}, d\tau) * f_2(\overrightarrow{\tau}, d\tau))$ does exist and equals the product $\left(\int\limits_{t_1}^{t_2} f_i(\overrightarrow{\tau}, d\tau) \right) * \left(\int\limits_{t_1}^{t_2} f_{3-i}(\overrightarrow{\tau}, d\tau) \right)$, $\forall i \in \{1,2\}$.

(b) if there exist $\int\limits_{t_1}^{t_2} f_i(\overleftarrow{\tau}, d\tau)$, $i \in \{1, 2\}$, then $\int\limits_{t_1}^{t_2} (f_1(\overleftarrow{\tau}, d\tau) * f_2(\overleftarrow{\tau}, d\tau))$ does exist and equals the product $\left(\int\limits_{t_1}^{t_2} f_i(\overleftarrow{\tau}, d\tau)\right) * \left(\int\limits_{t_1}^{t_2} f_{3-i}(\overleftarrow{\tau}, d\tau)\right)$, $\forall i \in \{1, 2\}$.

Proof. We prove here only the case (a), because the case (b) can be proved similarly.

Consider the case $t_1 \le t_2$. Introduce the notation

$$\alpha_\sigma = \overrightarrow{\sum_\sigma}(f_1 * f_2), \quad \beta_{(\sigma_1,\sigma_2)} = \Big(\overrightarrow{\sum_{\sigma_1}} f_1\Big) * \Big(\overrightarrow{\sum_{\sigma_2}} f_2\Big), \quad u(\sigma) = (\sigma, \sigma),$$

that is, $u : \Sigma(t_1, t_2) \to \Sigma(t_1, t_2) \times \Sigma(t_1, t_2)$. By the condition, $\alpha_\sigma = \beta_{u(\sigma)}$, and we can easily see that for every $(\sigma_1, \sigma_2) \in \Sigma(t_1, t_2) \times \Sigma(t_1, t_2)$ $\exists \widehat{\sigma} \in \sum(t_1, t_2)$ such that $u(\sigma) \ge (\sigma_1, \sigma_2)$, $\forall \sigma \ge \widehat{\sigma}$. Hence $\{\alpha_\sigma\}_{\sigma_0 \le \sigma \in \Sigma(t_1,t_2)}$ is the subdirectedness of the directedness

$$\{\beta_{(\sigma_1,\sigma_2)}\}_{(\sigma_0,\sigma_0) \le (\sigma_1,\sigma_2) \in \Sigma(t_1,t_2) \times \Sigma(t_1,t_2)}.$$

Taking into consideration the continuity of operation and also the equality $\beta_{(\sigma_1,\sigma_2)} = \Big(\overrightarrow{\sum}_{\sigma_2} f_2\Big) * \Big(\overrightarrow{\sum}_{\sigma_1} f_1\Big)$, we get

$$\int\limits_{t_1}^{t_2} (f_1(\overrightarrow{\tau}, d\tau) * f_2(\overrightarrow{\tau}, d\tau)) = \Big(\int\limits_{t_1}^{t_2} f_i(\overrightarrow{\tau}, d\tau)\Big) * \Big(\int\limits_{t_1}^{t_2} f_{3-i}(\overrightarrow{\tau}, d\tau)\Big), \quad \forall i \in \{1, 2\}.$$

If $t_1 > t_2$, then with regard for the case considered above, we arrive at

$$\int\limits_{t_2}^{t_1} f_1(\overrightarrow{\tau}, d\tau) * f_2(\overrightarrow{\tau}, d\tau)) = \Big(\int\limits_{t_2}^{t_1} f_i(\overrightarrow{\tau}, d\tau)\Big) * \Big(\int\limits_{t_2}^{t_1} f_{3-i}(\overrightarrow{\tau}, d\tau)\Big), \quad \forall i \in \{1, 2\}.$$

Equating the elements inverse to the left and right-hand sides, we complete the proof of the case (a). ■

2. One-Parameter Integral and Formulas of Partial Integration

2.1. One-parameter Integral. Let $f : [0, \eta] \to M$, $\eta > 0$. For every interval $[a, b]$ we may assume f to be the mapping with respect to $t : (t, s) \to f(s)$ maps the rectangle $[a, b] \times [0, \eta]$ into M. Hence for $\sigma = \{a = s_0 \le \xi_1 \le s_1 \le \dots \le \xi_n \le s_n = b\}$ such that $|\sigma| < \eta$, $\overrightarrow{\sum}_\sigma f = f(\Delta s_1) * f(\Delta s_2) * \dots * f(\Delta s_n)$, and $\overleftarrow{\sum}_\sigma f = f(\Delta s_n) * \dots * f(\Delta s_2) * f(\Delta s_1)$ are defined correctly, and we may speak on the integrability of f in terms of Definition 2.

It appears that there may exist simultaneously the left and the right integrals of such (incomplete) subintegral functions; and if they do, then they are equal (notation, of course, reflects this fact). Indeed, given $a \le b$, let us define $u : \Sigma(a, b) \to \Sigma(a, b)$: if $\sigma = \{a = s_0 \le \xi_1 \le s_1 \le \dots \le$

$\xi_n \le s_n = b\}$, then $u(\sigma) = \{a = u_0 \le \eta_1 \le u_1 \le \eta_n \le u_n = b\}$, where $u_k = a + (b - s_{n-k})$, $k \in \{0, 1, \dots, n\}$, $\eta_k = a + (b - \xi_{n-k+1})$, $k \in \{1, \dots, n\}$. Clearly, $u(u(\sigma)) = \sigma$ (i.e., u is one-to-one), $|u(\sigma)| = |\sigma|$, and $\sum_{u(\sigma)}^{\rightarrow} f = \sum_{\sigma}^{\leftarrow} f$, $\sum_{\sigma}^{\rightarrow} f = \sum_{u(\sigma)}^{\leftarrow} f$ for sufficiently fine σ. Thus each of the following directednesses

$$\Big\{ \sum_{\sigma}^{\rightarrow} f \Big\}_{\sigma_0 \le \sigma \in \Sigma(a,b)} \quad \Big\{ \sum_{\sigma}^{\leftarrow} f \Big\}_{\sigma_0 \le \sigma \in \Sigma(a,b)}$$

is a subdirectedness of the other.

The case $a > b$ can be easily reduced to that considered above.

Proposition 5. Let for some $\eta > 0$ $f : [0, \eta] \to M$, $a, b \in \mathbb{R}$, and let there exist $\int\limits_a^b f(d\tau)$. Then there exists $\int\limits_{a+t}^{b+t} f(d\tau)$, $\forall t \in \mathbb{R}$, and

$$\int\limits_a^b f(d\tau) = \int\limits_{a+t}^{b+t} f(d\tau), \quad \forall t \in \mathbb{R}. \tag{6}$$

Proof. Evidently, it suffices to consider the case $a \le b$. Denote

$$\sum^1 = \Sigma(a,b), \quad \sum^2 = \Sigma(a+t, b+t)$$

and define $u : \Sigma(a,b) \to \Sigma(a+t, b+t)$ as follows: to every $\sigma = \{a = s_0 \le \xi_1 \le s_1 \le \dots \le \xi_n \le s_n = b\}$ there corresponds

$$u(\sigma) = \{a + t = s_0 + t \le \xi_1 + t \le s_1 + t \le \dots \le \xi_n + t \le s_n + t = b + t\}.$$

It is clear that u is one-to-one and $|u(\sigma)| = |\sigma|$, $\forall \sigma \in \sum^1$. Moreover, for sufficiently fine $\sigma \in \sum^2$, we have $\sum_{u^{-1}(\sigma)} f = \sum_{\sigma} f$. Therefore $\Big\{ \sum_{\sigma} f \Big\}_{\sigma \in \sum^2 |\sigma| < \eta}$ is the subdirectedness of the directedness $\Big\{ \sum_{\sigma} f \Big\}_{\sigma \in \sum^1 |\sigma| < \eta}$. Consequently, (6) holds. ■

Proposition 6. Let for some $\eta > 0$ $f : [0, \eta] \to M$ and let for every $t \ge 0$ there exist $\int\limits_0^t f(d\tau)$. Then $\Big\{ \int\limits_0^t f(d\tau) \Big\}_{t \ge 0}$ is a one-parameter semigroup in M, i.e.,

$$\int\limits_0^{t_1+t_2} f(d\tau) = \Big(\int\limits_0^{t_1} f(d\tau) \Big) * \Big(\int\limits_0^{t_2} f(d\tau) \Big), \quad \forall t_1, t_2 \ge 0.$$

Proof. Combining the results of Propositions 3 and 5, we can prove the above proposition:

$$\int_0^{t_1+t_2} f(d\tau) = \Big(\int_0^{t_1} f(d\tau)\Big) * \Big(\int_{t_1}^{t_1+t_2} f(d\tau)\Big) =$$
$$= \Big(\int_0^{t_1} f(d\tau)\Big) * \Big(\int_0^{t_2} f(d\tau)\Big). \quad \blacksquare$$

Corollary 1. *If in the conditions of Proposition 6 there is also $f(0) = e$, then $\left\{\int_0^t f(d\tau)\right\}_{t\geq 0}$ is a one-parameter submonoid in M.*

Denote by $\widehat{\Sigma}(a,b)$ the set of all partitions of the interval $[a,b]$ of the form $\sigma = \{a = t_0 < \cdots < t_n = b\}$ (we imply that $a < b$). As usual,

$$\Delta t_i = t_i - t_{i-1}, \quad |\sigma| = \max\{\Delta t_i | i = 1,\dots,n\}.$$

$\widehat{\Sigma}(a,b)$ is ordered as follows: $\sigma_1 \leq \sigma_2$ if $|\sigma_1| \geq |\sigma_2|$.

Lemma 1. *Let for some $\eta > 0$ $f : [0,\eta] \to M$, $f(0) = e$ and $t \geq 0$. Then each of the following two directednesses*

$$\Big\{\sum_\sigma f\Big\}_{\sigma_0 \leq \sigma \in \Sigma(0,t)} \quad \textit{and}$$
$$\{f(\Delta\tau_1) * f(\Delta\tau_2) * \cdots * f(\Delta\tau_n)\}_{\widehat{\sigma} \leq \{0=\tau_0<\cdots<\tau_n=t\} \in \widehat{\Sigma}(0,t)}$$

is the subdirectedness of the other.

Proof. Denote

$$\alpha_\sigma = \sum_\sigma f, \quad \sigma_0 \leq \sigma \in \Sigma(0,t),$$
$$\beta_\nu = f(\Delta\tau_1) * f(\Delta\tau_2) * \cdots * f(\Delta\tau_n),$$
$$\nu_0 \leq \nu = \{0 = \tau_0 < \cdots < \tau_n = t\} \in \widehat{\Sigma}(0,t),$$

and construct the mappings $u : \Sigma(0,t) \to \widehat{\Sigma}(0,t)$ and $v : \widehat{\Sigma}(0,t) \to \Sigma(0,t)$ as follows.

Let $\sigma = \{0 = \tau_0 \leq \xi_1 \leq \tau_1 \leq \cdots \leq \xi_n \leq \tau_n = t\} \in \Sigma(0,t)$. Denote $s_0 = \tau_0$, $s_1 = \big\{\tau_j \in \{\tau_0,\dots,\tau_n\} | \tau_{j-1} = s_0, \tau_j > s_0\big\}$ and so on. If the constructed in such a way set $\{s_0,\dots,s_k\}$ does not involve $\{\tau_0,\dots,\tau_n\}$, then $s_{k+1} = \big\{\tau_j \in \{\tau_0,\dots,\tau_n\} | \tau_{j-1} = s_k, \tau_j > s_k\big\}$.

Not more than in n steps we obtain $\{s_0,\dots,s_p\}$ such that

$$\{s_0,\dots,s_p\} = \{\tau_0,\dots,\tau_n\}.$$

Now we can determine $u : u(\sigma) = \{0 = s_0 < \cdots < s_p = t\} \in \widehat{\Sigma}(0,t)$.

The mapping v can be defined in a more simple manner: to every $\nu = \{0 = \tau_0 < \cdots < \tau_n = t\}$ there corresponds

$$v(\nu) = \{0 = \tau_0 \le \xi_1 = \tau_1 \le \cdots \le \xi_n = \tau_n = t\} \in \Sigma(0,t).$$

Taking into account the properties of u and v, the following identities complete the proof:

$$\alpha_{v(\nu)} = \beta_\nu, \quad \nu_0 \le \nu = \{0 = \tau_0 < \cdots < \tau_n = t\} \in \widehat{\Sigma}(0,t),$$
$$\beta_{u(\sigma)} = \alpha_\sigma, \quad \sigma_0 \le \sigma \in \Sigma(0,t). \quad \blacksquare$$

Corollary 2. *Under the conditions of the lemma, $g \in M$ is the integral of the function f on $[0,t]$ (from 0 to t) if and only if the directedness*

$$\{f(\Delta\tau_1) * f(\Delta\tau_2) * \cdots * f(\Delta\tau_n)\}_{\nu_0 \le \{0=\tau_0<\cdots<\tau_n=t\} \in \widehat{\Sigma}(0,t)}$$

converges to g.

Proposition 7. Let for some $\eta > 0$ $f : [0,\eta] \to M$, $f(0) = e$ and $t \ge 0$. Then the existence of each of the following two integrals

$$\int_0^t f(d\tau) \quad \text{and} \quad \int_0^1 f(t \cdot d\tau)$$

implies the existence of the other one and their equality.

Proof. Consider the nontrivial case $t > 0$ and introduce the notation:

$$\alpha_\nu = f(\Delta s_1) * f(\Delta s_2) * \cdots * f(\Delta s_m),$$
$$\nu_0 \le \nu = \{0 = s_0 < \cdots < s_m = t\} \in \widehat{\Sigma}(0,t),$$
$$\beta_\sigma = f(t * \Delta\tau_1) * f(t * \Delta\tau_2) * \cdot * f(t * \Delta\tau_n),$$
$$\sigma_0 \le \sigma = \{\{0 = \tau_0 < \cdots < \tau_n = 1\} \in \widehat{\Sigma}(0,1)\},$$

where σ_0 and ν_0 are fixed sufficiently fine partitions.

Determine $u : \widehat{\Sigma}(0,1) \to \widehat{\Sigma}(0,t)$ as follows: to every $\sigma = \{0 = \tau_0 < \cdots < \tau_n = 1\} \in \widehat{\Sigma}(0,1)$ there corresponds $u(\sigma) = \{0 = t\tau_0 < \cdots < t\tau_n = t\} \in \widehat{\Sigma}(0,t)$. Obviously, u is one-to-one and

$$|u(\sigma)| = t|\sigma|, \quad \forall\sigma \in \widehat{\Sigma}(0,1), \quad |u^{-1}(\sigma)| = t^{-1}|\sigma|, \quad \forall\sigma \in \widehat{\Sigma}(0,t).$$

The identities

$$\alpha_{u(\sigma)} = \beta_\sigma, \quad \forall\sigma \in \widehat{\Sigma}(0,1), \quad \sigma \ge \sigma_0,$$
$$\beta_{u^{-1}(\nu)} = \alpha_\nu, \quad \forall\nu \in \widehat{\Sigma}(0,t), \quad \nu \ge \nu_0,$$

with regard for the properties of the mapping u prove that

$$\{\alpha_\nu\}_{\nu_0 \le \nu \in \widehat{\Sigma}(0,t)} \quad \text{and} \quad \{\beta_\sigma\}_{\sigma_0 \le \sigma \in \widehat{\Sigma}(0,1)}$$

are subdirectednesses of each other, which by virtue of Lemma 1 proves out proposition. ■

2.2. Formulas of "Partial Integration". For every invertible $g \in M$, let us determine an automorphism of the monoid M:

$$Ad_g f = g * f * \overline{g}, \quad \forall f \in M.$$

It is easily seen that $Ad_g e = e$, $Ad_g(f_1 * f_2) = Ad_g f_1 * Ad_g f_2$, $Ad_g(\overline{f}) = \overline{Ad_g f}$ when f is invertible in M and if $\{p(t)\}_{t\geq 0}$ is a one-parameter semi-group in M, then $\{Ad_g p(t)\}_{t\geq 0}$ is also a one-parameter semi-group.

Proposition 8. Let $g : [0, \eta] \to M$, $\eta > 0$, $\{p(t)\}_{t\geq 0}$ be a one-parameter subgroup in M, and for some $a, b \in \mathbb{R}$ let there exist the integral $\int_a^b Ad_{p(\overrightarrow{\tau})} g(d\tau)$ $\left(\int_a^b Ad_{p(\overleftarrow{\tau})} g(d\tau)\right)$. Then $\exists \int_{a+t}^{b+t} Ad_{p(\overrightarrow{\tau})} g(d\tau)$ $\left(\int_{a+t}^{b+t} Ad_{p(\overleftarrow{\tau})} g(d\tau)\right)$, and $\forall t \in \mathbb{R}$ we have.

$$\int_{a+t}^{b+t} Ad_{p(\overrightarrow{\tau})} g(d\tau) = Ad_{p(t)} \left[\int_a^b Ad_{p(\overrightarrow{\tau})} g(d\tau) \right] \tag{7}$$

$$\left(\int_{a+t}^{b+t} Ad_{p(\overleftarrow{\tau})} g(d\tau) = Ad_{p(t)} \left[\int_a^b Ad_{p(\overleftarrow{\tau})} g(d\tau) \right] \right).$$

Proof. Let $a \leq b$ and let there exist $\int_a^b Ad_{p(\overrightarrow{\tau})} g(d\tau)$. Take some $t \in \mathbb{R}$ and determine $u : \Sigma(a+t, b+t) \to \Sigma(a, b)$ as follows: to every $\sigma = \{a+t = s_0 \leq \xi_1 \leq s_1 \leq \cdots \leq \xi_n \leq s_n = b+t\}$ there corresponds $u(\sigma) = \{a = s_0 - t \leq \xi_1 - t \leq s_1 - t \leq \cdots \leq \xi_n - t \leq s_n - t = b\}$. Clearly, $|u(\sigma)| = |\sigma|$ and u is one-to-one. For the sake of simplicity we also denote $f(\tau, s) = Ad_{p(\tau)} g(s)$. The evident equality $\overrightarrow{\sum_\sigma} f = Ad_{p(t)} \overrightarrow{\sum_{u(\sigma)}} f$, with regard for the properties of u, proves that

$$\left\{ \overrightarrow{\sum_\sigma} f \right\}_{\sigma \in \Sigma(a+t, b+t), |\sigma| < \eta} \tag{8}$$

is a subdirectedness of the directedness $\left\{ Ad_{p(t)} \overrightarrow{\sum_\sigma} f \right\}_{\sigma \in \Sigma(a,b), |\sigma| < \eta}$ which converges to the right-hand side of (7). Consequently, (8) also converges to the right-hand side of (7). By the definition of the integral, this means that (8) converges to the left-hand side (7) as well. Thus, by virtue of the uniqueness of the limit, (7) holds.

Let now $a > b$, and let there exist $\int\limits_a^b Ad_{p(\vec{\tau})} g(d\tau)$. Hence there also exists its inverse $\int\limits_a^b Ad_{p(\vec{\tau})} g(d\tau)$. Since $b < a$, for every t we have

$$\int\limits_{b+t}^{a+t} Ad_{p(\vec{\tau})} g(d\tau) = Ad_{p(t)} \left[\int\limits_b^a Ad_{p(\vec{\tau})} g(d\tau) \right],$$

and both sides are invertible. Equating their inverse elements, we obtain (7).

The version of Proposition 8 given in square brackets is proved in a similar way. ■

Proposition 9. Let $\{p(t)\}_{t\geq 0}$ and $\{q(t)\}_{t\geq 0}$ be one-parameter subgroups in M, and for some $a, b \in \mathbb{R}$ let there exist the integral $\int\limits_a^b Ad_{q(\vec{\tau})} p(d\tau)$. Then there exists $\int\limits_a^b Ad_{p(\vec{\tau})} q(d\tau)$, and the equality

$$\int\limits_a^b Ad_{p(\vec{\tau})} q(d\tau) = p(a) * q(-a) * \left[\int\limits_a^b Ad_{q(\vec{\tau})} p(d\tau) \right] * q(b) * p(-b) \quad (9)$$

takes place.

Proof. Let $a \leq b$ and there exist $\int\limits_a^b Ad_{q(\vec{\tau})} p(d\tau)$. We construct $u : \Sigma(a,b) \to \Sigma(a,b)$ as follows: to every $\sigma = \{a = s_0 \leq \xi_1 \leq s_1 \leq \cdot \leq \xi_n \leq s_n = b\}$ there corresponds $u(\sigma) = \{a = \xi_0 = s_0 \leq \xi_1 \leq s_1 \leq \cdots \leq \xi_n \leq s_n = \xi_{n+1} = b\}$. By the construction, $|u(\sigma)| \leq 2|\sigma|$.

$$\overrightarrow{\sum_{\sigma}} \left(Ad_{p(\tau)} q(d\tau) \right) =$$
$$= p(\xi_1) * q(-s_0) * q(s_1) * p(-\xi_1) * \cdots * p(\xi_n) * q(-s_{n-1}) * q(s_n) * p(-\xi_n) =$$
$$= p(\xi_0) * q(-s_0) * [q(s_0) * p(-\xi_0) * p(\xi_1) * q(-s_0)] *$$
$$* [q(s_1) * p(-\xi_1) * p(\xi_2) * q(-s_1)] *$$
$$* \cdots * [q(s_{n-1}) * p(-\xi_{n-1}) * p(\xi_n) * q(-s_{n-1})] *$$
$$* [q(s_n) * p(-\xi_n) * p(\xi_{n+1}) * q(-s_n)] * q(s_n) * p(-\xi_{n+1}) =$$
$$= p(a) * q(-a) * \left[\overrightarrow{\sum_{u(\sigma)}} (Ad_{q(\tau)} p(d\tau)) \right] * q(b) * p(-b).$$

Consequently,

$$\left\{ \sum_{\sigma}^{\rightarrow} (Ad_{p(\tau)} q(d\tau)) \right\}_{\sigma \in \Sigma(a,b)} \tag{10}$$

is a subdirectedness of the directedness

$$\left\{ p(a) * q(-a) * \left[\sum_{\nu}^{\rightarrow} (Ad_{q(\tau)} p(d\tau)) \right] * q(b) * p(-b) \right\}_{\nu \in \Sigma(a,b)},$$

which, by the conditions of the proposition and due to the continuity of the binary operation in M, converges to the right-hand side of (9). Hence, (10) converges to the right-hand side of (9) which, by the definition of the integral, proves (9).

The case where $a > b$ easily follows from the above proven. ■

Proposition 10. Let $\{p(t)\}_{t \geq 0}$ and $\{q(t)\}_{t \geq 0}$ be one-parameter subgroups in M and for some $a, b \in \mathbb{R}$ let there exist the integral $\int_a^b Ad_{q(-\overleftarrow{\tau})} p(-d\tau)$. Then there exists $\int_a^b Ad_{p(\overleftarrow{\tau})} q(d\tau)$, and

$$\int_a^b Ad_{p(\overleftarrow{\tau})} q(d\tau) = p(b) * q(b) * \left[\int_a^b Ad_{q(-\overleftarrow{\tau})} p(-d\tau) \right] * q(-a) * p(-a).$$

Proof. Just as in the case of Proposition 9, the proof is actually a simple checking. ■

The results proved in Propositions 9 and 10 are naturally associated with the formulas of partial integration. As is seen, the formula for the left integral is of more familiar form.

3. Integral representation of c_0-subgroups of operators

Let $A : D(A) \to X$ be a linear operator acting in the Banach space X. One of the basic results of the theory of subgroups of operators, the Hille-Iosida-Phillips theorem ([5], Ch. VIII. p. 1), states that the linear (possibly unbounded) operator A in the Banach space X generates a strongly continuous semi-group of operators $\{U(t)\}_{t \geq 0}$ (i.e., a c_0-semi-group) if and only if A is densely defined, closed and has a resolvent satisfying

$$|(\lambda - A)^{-n}|_{B(x)} \leq \alpha(\lambda - \beta)^{-n}, \quad n = 1, 2, \ldots, \lambda > \beta, \tag{11}$$

for some constants $\alpha \geq 1$ and $\beta \geq 0$.

In this case, $U(t)$ is a strong limit of operators of the type $\left(I - \frac{t}{n} A\right)^{-n}$ as $n \to \infty$ ([6], Ch. IX), where I is the identical mapping of the space X onto itself. This fact is contained in the definition of the exponent of the unbounded operator A and also in the notation $U(t) = \exp(tA)$ (some authors write $U(t) = \exp(-tA)$).

Despite the fact that such a definition of the exponent for an unbounded generating operator is accepted and widespread, it gives rise to a dissatisfaction: in the general case, for calculation of $\exp(tA)$ it is impossible to use the series $\sum_{i=0}^{n} \frac{t^i A^i}{i!}$ or the strong limit of operators of the type $\left(I + \frac{t}{n}A\right)^n$ (as $n \to \infty$), since the domain of definition of A^n contracts with the growth of n. Thus the definition of the exponent of the unbounded operator contains certain conditionality. In our opinion, application of the Riemann chronological integral allows us to achieve as much clearness as possible.

Theorem 1. *Let a linear operator A in the Banach space X generate the strongly continuous semi-group $\{U(s)\}_{s\geq 0}$. Then*

$$U(s) = \circledcirc \int_0^s (I - dt \cdot A)^{-1}, \quad \forall s \geq 0. \tag{12}$$

The integral is taken in the monoid $B(X)$ (which is considered to have the unity I, operation of composition and the strong convergence).

Proof. Let us consider the case nontrivial $s > 0$ and divide the proof into several parts.

(a) For any sufficiently fine partition $\sigma_0 \in \widehat{\Sigma}(0, s)$, the following directedness is defined correctly:

$$\{S(\sigma)\}_{\sigma_0 \leq \sigma \in \widehat{\Sigma}(0,s)}, \tag{13}$$

where

$$S(\sigma) = (I - \Delta\tau_1 \cdot A)^{-1} \cdots (I - \Delta\tau_m \cdot A)^{-1}, \quad \sigma = \{0 = \tau_0 < \cdots < \tau_m = s\}.$$

A generates a c_0-semi-group. Therefore, by the Hille-Iosida-Phillips theorem, A is closed, densely defined, and for some $\alpha \geq 1$ and $\beta \geq 0$ there takes place (11).

$(I - tA) = t(t^{-1} - A)$. Therefore for every $t \in (0, \beta^{-1})$, $\exists (I - tA)^{-1}$ and

$$(I - tA)^{-1} = \frac{1}{t}\Big(\frac{1}{t} - A\Big)^{-1}. \tag{14}$$

Thus, if $\sigma_0 \in \widehat{\Sigma}(0, s)$ such that $|\sigma_0| \leq \frac{1}{\beta+1}$ (in the sequel, this will be assumed to be the case), then every term of the directedness (13) is defined correctly.

(b) Directedness (13) is bounded in $B(X)$.

(11) is equivalent to the following condition ([3], p. 244, Prop. 3.3):

$$\Big| \prod_{j=1}^{k} (\lambda_j - A)^{-1} \Big|_{B(X)} \leq \alpha \prod_{j=1}^{k} (\lambda_j - \beta)^{-1}, \quad \lambda_j > \beta, \quad k = 1, 2, \ldots$$

which along with (14) for every $\sigma = \{0 = t_0 < \dots < t_m = s\} \geq \sigma_0$ yields

$$|S(\sigma)|_{B(X)} = |(I - \Delta t_1 \cdot A)^{-1} \cdots (I - \Delta t_m \cdot A^{-1})|_{B(X)} \leq$$
$$\leq \alpha(1 - \Delta t_1 \cdot \beta)^{-1} \cdots (1 - \Delta t_m \cdot \beta)^{-1}.$$

For an arbitrary $j \in \{1, \dots, m\}$, we have

$$\frac{1}{1 - \Delta t_j \cdot \beta} = 1 + \frac{\Delta t_j \cdot \beta}{1 - \Delta t_j \cdot \beta} \leq 1 + \frac{\Delta t_j \cdot \beta}{1 - \frac{\beta}{\beta+1}} =$$
$$= 1 + \Delta t_j \cdot \beta(\beta + 1) \leq \exp(\Delta t_j(\beta + \beta^2)).$$

Consequently,

$$|(I - \Delta t_1 \cdot A)^{-1} \cdots (I - \Delta t_m \cdot A)^{-1}|_{B(X)} \leq$$
$$\leq \alpha \cdot \exp((\Delta t_1 + \dots + \Delta t_m)(\beta + \beta^2)), \tag{15}$$
$$|S(\sigma)|_{B(X)} \leq \alpha \cdot \exp(s(\beta + \beta^2)). \tag{16}$$

(c) $D(A^2)$ is dense in X since for every $\lambda > \beta$

$$D(A^2) = (\lambda - A)^{-1} D(A), \quad D(A) = (\lambda - A)^{-1} X,$$

where $D(A)$ is dense and $(\lambda - A)^{-1}$ is the bounded operator.
(d) For every $x \in D(A^2)$,

$$\{S(\sigma)x\}_{\sigma_0 \leq \widehat{\Sigma}(0,s)} \tag{17}$$

is a converging directedness.

Let $x \in D(a^2)$, $\sigma = \{0 = \tau_0 < \dots < \tau_m = s\}$, and let σ_1 be a refinement of σ:

$$\sigma_1 = \{0 = \tau_0 < \tau_{1\,0} < \dots < \tau_{1\,p(1)} = \tau_1 = \tau_{2\,0} < \dots < \tau_{m\,p(m)} = \tau_m = s\},$$

and σ, $\sigma_1 \geq \sigma_0$.

$$|S(\sigma)x - S(\sigma_1)x| = \big|(I - \Delta\tau_1 \cdot A)^{-1} \cdots (I - \Delta\tau_m \cdot A)^{-1} x -$$
$$-(I - \Delta\tau_{1\,1} \cdot A)^{-1} (I - \Delta\tau_{1\,2} \cdot A)^{-1} \cdots (I - \Delta\tau_{m\,p(m)} \cdot A)^{-1} x\big| \leq$$
$$\leq \big|((I - \Delta\tau_1 \cdot A)^{-1} - (I - \Delta\tau_{1\,1} \cdot A)^{-1}) \cdots (I - \Delta\tau_{1\,p(1) \cdot A})^{-1}) \cdot$$
$$\cdot (I - \Delta\tau_2 \cdot A)^{-1} \cdots (I - \Delta\tau_m \cdot A)^{-1} x\big| +$$
$$+ |(I - \Delta\tau_{1\,1} \cdot A)^{-1} \cdots (I - \Delta\tau_{1\,p(1)} \cdot A)^{-1} \cdot$$
$$\cdot ((I - \Delta\tau_2 \cdot A)^{-1} - (I - \Delta\tau_{2\,1} \cdot A)^{-1} \cdot (I - \Delta\tau_{2\,p(2)} \cdot A)^{-1}) \cdot$$
$$\cdot (I - \Delta\tau_3 \cdot A)^{-1} \cdots (I - \Delta\tau_m \cdot A)^{-1} x| +$$
$$+ \dots + |(I - \Delta\tau_{1\,1} \cdot A)^{-1} \cdots (I - \Delta\tau_{(m-1)p(m-1)} \cdot A)^{-1}$$
$$\big((I - \Delta\tau_m \cdot A)^{-1} - (I - \Delta\tau_{m\,1} \cdot A)^{-1} \cdots (I - \Delta\tau_{m\,p(m)} \cdot A)^{-1}\big) x|. \tag{18}$$

For an arbitrary $\lambda > \beta$,

$$A(\lambda - A)^{-1} x = (\lambda - A)^{-1} A x, \quad \forall x \in D(A). \tag{19}$$

Therefore for every $t_1, t_2 \in (0, \beta^{-1}/2)$, on $D(A^2)$ there takes place the following operator equality:

$$
\begin{gathered}
(I-(t_1+t_2)\cdot A)^{-1}-(I-t_1\cdot A)^{-1}(I-t_2\cdot A)^{-1}= \\
=(I-(t_1+t_2)\cdot A)^{-1}-(I-(t_1+t_2)\cdot A)^{-1}(I-(t_1+t_2)\cdot A)\cdot \\
\cdot(I-t_1\cdot A)^{-1}(I-t_2\cdot A)^{-1}=(I-(t_1+t_2)\cdot A)^{-1}- \\
-(I-(t_1+t_2)\cdot A)^{-1}\big\{\big[(I-(t_1+t_2)\cdot A)-(I-t_2\cdot A)(I-t_1\cdot A)\big]+ \\
+(I-t_2\cdot A)(I-t_1\cdot A)\big\}(I-t_1\cdot A)^{-1}(I-t_2\cdot A)^{-1}= \\
=t_1t_2(I-(t_1+t_2)\cdot A)^{-1}(I-t_1\cdot A)^{-1}(I-t_2\cdot A)^{-1}A^2. \qquad (20)
\end{gathered}
$$

Taking into account (19) and (20), for every $j \in \{1, \ldots, m\}$ on $D(A^2)$ we have

$$
\begin{gathered}
(I-\Delta\tau_i\cdot A)^{-1}-(I-\Delta\tau_{i\,1}\cdot A)^{-1}\cdots(I-\Delta\tau_{i\,p(i)}\cdot A)^{-1}= \\
=\big\{(I-\Delta\tau_i\cdot A)^{-1}-(I-\Delta\tau_{i\,1}\cdot A)^{-1}(I-(\Delta\tau_{i\,2}+\cdots+\Delta\tau_{i\,p(i)})A)^{-1}\big\}+ \\
+\big\{(I-\Delta\tau_{i\,1}\cdot A)^{-1}\big(I-(\Delta\tau_{i\,2}+\cdots+\Delta\tau_{i\,p(i)})A\big)^{-1}- \\
-(I-\Delta\tau_{i\,1}\cdot A)^{1}(I-\Delta\tau_{i\,2}\cdot A)^{-1}\cdot \\
\cdot\big(I-(\Delta\tau_{i\,3}+\cdots+\Delta\tau_{i\,p(i)})A\big)^{-1}\big\}+\cdots+ \\
+\big\{(I-\Delta\tau_{i\,1}\cdot A)^{-1}\cdots(I-\Delta\tau_{i(p(i)-2)}\cdot A)^{-1}\cdot \\
\cdot\big(I-(\Delta\tau_i(p(i)-1)+\Delta\tau_{i\,p(i)})A\big)^{-1}- \\
-(I-\Delta\tau_{i\,1}\cdot A)^{-1}\cdots(I-\Delta\tau_{i\,p(i)}\cdot A)^{-1}\big\}= \\
=\Delta\tau_{i\,1}(\Delta\tau_{i\,2}+\cdots+\Delta\tau_{i\,p(i)})\cdot(I-\Delta\tau_i\cdot A)^{-1}(I-\Delta\tau_{i\,1}\cdot A)^{-1}\cdot \\
\cdot\big(I-(\Delta\tau_{i\,2}+\cdots+\Delta\tau_{i\,p(i)})A\big)^{-1}A^2+ \\
+\Delta\tau_{i\,2}(\Delta\tau_{i\,3}+\cdots\Delta\tau_{i\,p(i)})\cdot \\
\cdot(I-\Delta\tau_{i\,1}\cdot A)^{-1}\big(I-(\Delta\tau_{i\,2}+\cdots+\Delta\tau_{i\,p(i)})A\big)^{-1}\cdot \\
\cdot(I-\Delta\tau_{i\,2}\cdot A)^{-1}\big(I-(\Delta\tau_{i\,3}+\cdots+\Delta\tau_{i\,p(i)})A\big)^{-1}A^2+ \\
+\cdots+\Delta\tau_{i(p(i)-1)}\Delta\tau_{i\,p(i)}\cdot \\
\cdot(I-\Delta\tau_{i\,1}\cdot A)^{-1}\cdots(I-\Delta\tau_{i(p(i)-2)}\cdot A)^{-1}\cdot \\
\cdot\big(I-(\Delta\tau_{i(p(i)-1)}+\Delta\tau_{i\,p(i)})A\big)^{-1}\cdot \\
\cdot(I-\Delta\tau_{i(p(i)-1)}A)^{-1}(I-\Delta\tau_{i\,p(i)}\cdot A)^{-1}A^2.
\end{gathered}
$$

By virtue of this fact and because of (15), the inequality (18) results in

$$
|S(\sigma)x-S(\sigma_1)x|\le s\alpha^2\cdot\exp(3s(\beta+\beta^2))|\sigma||A^2x|. \qquad (21)
$$

Given an arbitrary $\varepsilon>0$, let $A^2x\neq 0$. We take $\widehat{\sigma}\in\widehat{\Sigma}(0,s)$ such that

$$
|\widehat{\sigma}|\le\min\big\{|\sigma_0|,\ \big(2s\alpha^2\cdot\exp(3s(\beta+\beta^2))|A^2x|\big)^{-1}\big\}.
$$

Then for any $\sigma_1 \geq \widehat{\sigma}$ and $\sigma_2 \geq \widehat{\sigma}$ we have

$$|S(\sigma_1)x - S(\sigma_2)x| \leq |S(\sigma_1)x - S(\sigma_3)x| + |S(\sigma_2)x - S(\sigma_3)x| \leq \varepsilon,$$

where the partition σ_3 is inscribed in σ_1 and σ_2 simultaneously. Due to the arbitrariness of ε, the directedness (17) is fundamental. If $A^2x = 0$, then the fundamentality (17) is obvious.

Thus X is a complate space, and therefore the directedness (17) converges for every $x \in D(A^2)$. Denote its limit by Sx.

(e) In the monoid $B(X)$ with the operation of composition and with a strong convergence there exists $\circledcirc \int\limits_0^s (I - dt \cdot A)$.

Indeed, the directedness (13) is bounded in $B(X)$, the directedness (17) converges for every $x \in D(A^2)$ and $D(A^2)$ is dense in X. Therefore, according to the well-known corollary of the principle of uniform boundedness ([5], Ch. II, p. 1), there exists the limit $Sx = \lim_\sigma S(\sigma)x$ for every $x \in X$ and S is a continuous operator. Hence (13) converges strongly to some operator S from $B(X)$. By the definition of the integral, this implies that there exists $\circledcirc \int\limits_0^s (I - dt \cdot A)^{-1}$ which is equal to S.

(f) Prove finally that the equality (12) holds.

$\{S_n\}_{n\in N}$ is a subdirectedness of the directedness (13), where $S_n = (I - \frac{s}{n}A)^{-1} \cdots (I - \frac{s}{n}A)^{-1} = (I - \frac{s}{n}A)^{-n}$ (this is obvious if we consider the mapping $n \to \{0 < \frac{s}{n} < \frac{2s}{n} < \cdots \frac{ns}{n} = s\} : \mathbb{N} \to \widehat{\Sigma}(0,s))$. Consequently, in the monoid $B(X)$

$$\circledcirc \int\limits_0^s (I - dt \cdot A)^{-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (I - \frac{s}{n}A)^{-n}. \tag{22}$$

On the other hand, the semi-group $U(\cdot)$ generated by the operator A is constructed with the help of the strong limit $U(t) = \lim_{n\to\infty}(I - \frac{t}{n}A)^{-n}$ ([6], Ch. IX, p. 594), which together with (22) provides us with (12). ■

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