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NEUMANN PROBLEM IN A CLASS OF HARMONIC FUNCTIONS IN DOMAINS WITH A PIECEWISE-LYAPUNOV BOUNDARY

Abstract. The Neumann boundary value problem is considered in a finite simply connected domain D with piecewise Lyapunov boundary free from zero interior angles. The solution is sought in the class of harmonic functions u satisfying (1), where Γ_r is the image of the circle $|z| = r$ under the conform mapping of the unit disc onto D.

რეზიუმე. ნულოვანი უკუქცევის არამქონე უბან-უბან ლიაპუნოვის წირით შემოსაზღვრულ სასრულ ცალადბმულ D არეში განხილულია ნეიმანის სასაზღვრო ამოცანა ისეთი ჰარმონიული u ფუნქციების კლასში, რომელთათვისაც შესრულებულია (1) პირობა. სადაც Γ_r არის ერთეულოვანი წრის D არეზე კონფორმულად ასახვისას $|z| = r$ წრეწირის ანასახი. გამოკვლეულია სასღვრის გეომეტრიის გავლენა ამოხსნადობაზე და აგებულია ამონახსნები.

The solvability of boundary value problems for PDE depends on geometrical properties of boundaries of the domains under consideration. (see, e.g. [1]–[2]). In [2], we investigated the Dirichlet problem in a class of harmonic functions which are the real part of an analytic function from the Smirnov class $E_p(D)$, $p > 1$, when the boundary of the domain D is an arbitrary piecewise-Lyapunov curve Γ free from zero interior angles (admitting however cusps with the angle 2π). In the present paper, under the same conditions with respect to Γ we investigate the Neumann problem under the assumption that partial derivatives of an unknown harmonic function have bounded p-mean integrals along certain sequence of curves converging to the boundary.

1^0. Let D be a plane simply connected finite domain whose Jordan oriented boundary Γ consists of a finite number of Lyapunov arcs meeting at the points t_k, $k = \overline{1,n}$, with interior (with respect to D), angles of sizes $\pi\nu_k$, $0 < \nu_k \leq 2$. Let, moreover, $z = z(w)$ be a function mapping conformally the unit circle $U = \{w : |w| < 1\}$ onto the domain D and $w = w(z)$ be the inverse function.

We say that the harmonic in D function $u(x,y) \equiv u(z)$ $(z = x + iy)$ belongs to the class $e'_p(D)$, $p > 0$, if

$$\sup_{r<1} \int_{\Gamma_r} \left(\left|\frac{\partial u}{\partial x}\right|^p + \left|\frac{\partial u}{\partial y}\right|^p \right) |dz| < \infty, \tag{1}$$

where Γ_r is images of the circumference of radius r under the mapping $z = z(w)$.

Let $v(z)$ be a function harmonically conjugate to $u(z)$, and $\phi(z) = u(z) + iv(z)$. Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, while $\phi'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$, from (1) we find that

$$\sup_{r<1} \int_{\Gamma_r} |\phi'(z)|^p |dz| < \infty,$$

that is, $\phi'(z)$ belongs to the Smirnov class $E_p(D)$ (see, e.g., [3], Ch. X). Thus $e'_p(D) = \operatorname{Re} E'_p(D)$, where $E'_p(D) = \{\phi : \phi' \in E_p(D)\}$. This implies that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ have angular boundary values $\left(\frac{\partial u}{\partial x}\right)^+$ and $\left(\frac{\partial u}{\partial y}\right)^+$ which are summable to the p-th power on Γ, i.e., belonging to $L_p(\Gamma)$.

Consider the Neumann problem formulated as follows: find a function u from the conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in D, \quad u \in e'_p(D), \quad p > 1,$$

$$\left(\frac{\partial u}{\partial n}\right)_\Gamma = f(t), \quad f \in L_p(\Gamma), \tag{2}$$

where $\left(\frac{\partial u}{\partial n}\right)_\Gamma = \left(\frac{\partial u}{\partial x}\right)^+ \cos(n,x) + \left(\frac{\partial u}{\partial y}\right)^+ \cos(n,y) \equiv \left(\frac{\partial u}{\partial x}\right)^+ (-\sin\alpha(t)) + \left(\frac{\partial u}{\partial y}\right)^+ \cos\alpha(t)$ and $\alpha(t)$ is the angle between the tangent to Γ at t and the abscissa axis.

The boundary condition from (2) is assumed to be fulfilled almost at all points of Γ.

Let $u = \operatorname{Re}\phi$. Since $\phi' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$, the boundary condition from (2) can be written as

$$\operatorname{Re}[ie^{i\alpha(t)}\phi'(t)] = f(t). \tag{3}$$

Consequently, any solution of the problem (2) generates a solution ϕ of problem (3) for which $\phi' \in E_p(D)$.

2^0. Following [4]–[5], we reduce the problem (3) to the problem of linear conjugation. To this end, we assume that

$$\sqrt[p]{z'(w)}\phi'(z(w)) = \Psi(w), \quad f(z(\tau)) \equiv g_1(\tau), \quad |\tau| = 1$$

and write (3) in the form

$$\operatorname{Re}\left[\frac{i\exp i\alpha(z(\tau))}{\sqrt[p]{z'(\tau)}}\Psi^+(\tau)\right] = g_1(\tau). \tag{4}$$

The function $\Psi(w)$ in the circle U belongs to the Hardy class H_p (see, e.g., [3], Chs. IX–X). Suppose

$$\Omega(w) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi(\frac{1}{\bar{w}})}, & |w| > 1. \end{cases}$$

Then $\Omega \in \widetilde{\mathcal{K}}_p(\gamma)$ ([5], Lemma 4), where $\gamma = \{|\tau| : |\tau| = 1\}$, and

$$\widetilde{\mathcal{K}}_p(\gamma) = \left\{ \Psi : \Psi(w) = \frac{1}{2\pi i} \int\limits_\gamma \frac{\psi(\tau)d\tau}{\tau - w} + \text{const}, \ |w| \neq 1, \ \psi \in L_p(\gamma) \right\}.$$

Assume also $\alpha(\tau) \equiv \alpha(z(\tau))$. It follows from (4) that

$$\exp i\alpha(\tau) \left[\sqrt[p]{z'(\tau)} \right]^{-1} \Omega^+(\tau) + \overline{i \exp i\alpha(\tau)} \left[\overline{\sqrt[p]{z'(\tau)}} \right]^{-1} \Omega^-(\tau) = 2g_1(\tau),$$

whence

$$\Omega^+(\tau) = - \exp(2i\alpha(\tau)) \sqrt[p]{z'(\tau)} \left[\overline{\sqrt[p]{z'(\tau)}} \right]^{-1} \Omega^-(\tau) + g(\tau), \qquad (5)$$
$$\text{where} \quad g(\tau) = -i \exp(-i\alpha(\tau))2f(z(\tau)), \quad g \in L_p(\gamma).$$

The homogeneous problem corresponding to (5) will be

$$\Omega^+(\tau) = \exp(-2i\alpha(\tau)) \sqrt[p]{z'(\tau)} \left[\overline{\sqrt[p]{z'(\tau)}} \right]^{-1} \Omega^-(\tau). \qquad (5_0)$$

Assume $\Omega_*(w) = \Omega\left(\frac{1}{\bar{w}}\right)$, $|w| \neq 1$.

For the restriction on U of the solution $\Omega \in \widetilde{\mathcal{K}}_p(\gamma)$ of the problem (5) to provide a solution of the problem (4) of the class H_p, it is necessary and sufficient for Ω to satisfy

$$\Omega(w) = \Omega_*(w) \qquad (6)$$

(see [4], §§ 40–43, [2]). Thus we can conclude that the problem (2) is equivalent to the problem (5) in the class $\widetilde{\mathcal{K}}_p(\gamma)$ with the additional condition (6).

If Ω satisfies the boundary condition (5), then the same condition is also satisfied by the function $\Omega_*(w)$ ([4], §41). It can be easily proved that if $\Omega \in \widetilde{\mathcal{K}}_p(\gamma)$, then $\Omega_* \in \widetilde{\mathcal{K}}_p(\gamma)$ as well. Therefore the function $\frac{1}{2}(\Omega(w) + \Omega_*(w))$ which already satisfies (6) will be a solution of the problem (5) from the class $\widetilde{\mathcal{K}}_p(\gamma)$.

In particular, if (5) possesses a unique solution Ω, then Ω_* coincides with it, and hence in this case the condition (6) is fulfilled for Ω.

3^0. **The Homogeneous Problem.** Let

$$y(w) = \exp\left(\frac{1}{2\pi i} \int\limits_\Gamma \frac{-2i\alpha(\tau)d\tau}{\tau - w} \right), \quad |w| \neq 1, \qquad (7)$$

where $\alpha(\tau)$ is assumed to be continuous on the arcs (τ_k, τ_{k+1}), $(k = \overline{1, n}, \tau_{n+1} = \tau_1)$ and to have at the points τ_k one-sided limits equal to the angle formed

by the tangent at t_k with the abscissa axis. Thus $\alpha(\tau)$ has at the points τ_k a discontinuity equal to the angle size between the left and the right one-sided tangents at t_k or, which is the same, equal to $\pi(1 - \nu_k)$. Moreover, when the point τ leaves the circumference γ to the left, the function $\alpha(\tau)$ admits an increment equal to 2π. Let us choose on Γ a point τ_0, $\tau_0 \neq t_k$, and assume it to be an initial point on γ. Then $\alpha(\tau)$ at this point has a discontinuity equal to 2π. This implies that for the w-s close to γ, the following relation is valid:

$$y(w) \approx (w - \tau_0)^{-2} \prod_{k=1}^{n} (w - \tau_k)^{-(1-\nu_k)} y_0(w), \tag{8}$$

where y_0 is continuous and different from zero both in $\overline{U}$ and in CU (CU is the supplement of U to the whole plane).

Let

$$\widetilde{X}(w) = \begin{cases} -y(w) \sqrt[p]{z'(w)}, & |w| < 1, \\ y(w) \sqrt[p]{\overline{z'(1/\overline{w})}}, & |w| > 1. \end{cases} \tag{9}$$

Since Γ is a piecewise-Lyapunov curve with interior angles of sizes $\pi\nu_k$, $0 < \nu_k \leq 2$, according to Warschawski's theorem ([6], see also [7], Ch. I), we have

$$z'(w) = \prod_{k=1}^{n} (w - \tau_k)^{\nu_k - 1} z_0(w), \tag{10}$$

where z_0 is a function different from zero and satisfying Hölder's condition on $\overline{U}$.

Due to (8) and (10), we obtain from (9) that

$$\widetilde{X}(w) = (w - \tau_0)^{-2} \prod_{k=1}^{n} (w - \tau_k)^{-\frac{\nu_k - 1}{p'}} X_0(w), \quad p' = p/p - 1, \tag{11}$$

where

$$0 < m \leq |X_0(w)| \leq M. \tag{11'}$$

Assume

$$X(w) = \widetilde{X}(w)(w - \tau_0)^2 \prod_{\{k : \nu_k \geq p'\}} (w - \tau_k), \tag{12}$$

where the multipliers $w - \tau_k$ in the product are taken with respect to such k-s that at the point $t_k (= z(\tau_k))$ the curve Γ has the angle equal to $\pi\nu_k$ with $\nu_k \geq p'$. $X(w)$ satisfies the condition (5_0) and possesses the following properties:

(i) it can be represented by a Cauchy type integral with density from $L_p(\gamma)$ and with a polynomial principal part of order $\varkappa + 2$ at infinity, where $\varkappa$ is the number of angular points for which $\nu_k \geq p'$;

(ii) in the neighborhood of the points τ_k for which $\nu_k = p'$ (denote them in the sequel by τ_k'), by (12) and (11') we have

$$X(w) = O((w - \tau_k')^{1/p'});\qquad(13)$$

(iii) $\frac{1}{X} \in \widetilde{\mathcal{K}}_{p'-\varepsilon}(\gamma)$ and has a zero of order $\varkappa + 2$ at the point $z = \infty$;

(iv) the functions $(X^{\pm})^{-1}$ are integrable to the p'-th power on every part of γ which is obtained from γ by eliminating small neighborhoods of the points τ_k'.

If Ω_0 is a solution of the problem (5_0), then $F(w) = \Omega_0(w)X^{-1}\ (w)$ satisfies the condition

$$F^+(\tau) = F^-(\tau),\quad \tau \in \gamma.\qquad(14)$$

Proceeding from properties (i)–(iv) of the function X, we can justify, just as in [2], that in the neighborhood of every point γ different from τ_k, the function $F(w)$ is analytic and may have perhaps only poles. Thus we conclude that

$$\Omega_0(w) = X(w)\sum_k \frac{A_k}{(w - \tau_k')^{n_k}},$$

where n_k are nonnegative integers.

But $\Omega_0 \in \widetilde{\mathcal{K}}_p(\gamma)$ and therefore $A_k = 0$, i.e., $\Omega_0 \equiv 0$. By the same argument, only the constant functions are solutions of the problem (2) for $f = 0$.

4^0. **Inhomogeneous Problem.** If there are no points τ_k' and also cusps with $\nu_k = 2$ for $p \geq 2$ (denote them by τ_k''), then $g/X^+ \in L_{1+\delta}(\gamma)$ and therefore

$$\int_\gamma \frac{g}{X^+}\frac{d\tau}{\tau - w} \in \widetilde{\mathcal{K}}_{1+\delta}(\gamma),\quad \delta > 0.$$

Taking this as a basis, we can suppose that as it is characteristic for the linear problem of conjugation (see [8]), the function

$$\widetilde{\Omega}(w) = \frac{X(w)}{2\pi i}\int_\gamma \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - w}\qquad(15)$$

will be a solution of the problem (5).

From (11) and (11') as well as from the properties of the Cauchy type integral in the unit circle, it easily follows that $\widetilde{\Omega} \in H_\eta,\ \eta > 0$, in the circle U. Let us show that $\widetilde{\Omega} \in H_p$. By virtue of Smirnov's theorem, it suffices to show that $\Omega^+ \in L_p(\gamma)$.

We have

$$\widetilde{\Omega}^+(\tau) = \frac{1}{2}g(\tau) + \frac{X^+(\tau)}{2\pi i}\int_\gamma \frac{g(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta - \tau},\qquad(16)$$

where

$$X^+ = O(\rho(\tau)), \quad \rho(\tau) = \prod_{k=1}^{n} (\tau - \tau_k)^{\frac{1-\nu_k}{p'}}.$$

Under the assumptions with respect to ν_k, we have $\frac{1-\nu_k}{p'} \in (-\frac{1}{p}, \frac{1}{p'})$, and since the Cauchy type singular operator is bounded in $L_p(\gamma, r)$, where $r(\tau) = \prod_{k=1}^{n} (\tau - \tau_k)^{\alpha_k}$ with $\alpha_k \in (-\frac{1}{p}, \frac{1}{p'})$ (see, e.g., [8]), we have $\widetilde{\Omega} \in L_p(\gamma)$. This means that $\widetilde{\Omega} \in H_p$ in U. From this in its turn it follows that $\widetilde{\Omega}$ belongs to $\widetilde{\mathcal{K}}(\gamma)$ if it has the limits when $z \to \infty$. But as far as $X(w)$ has at infinity the limit of order $\varkappa + 2$, it is necessary and sufficient that

$$\int_\gamma \frac{g(\tau)}{X^+(\tau)} \tau^k \, d\tau = 0, \quad k = 0, 1, \dots, |\varkappa|, \tag{17'}$$

or, which is the same,

$$\int_G f(t) \exp\left(\frac{1}{\pi} \int_\gamma \frac{\alpha(\zeta)d\zeta}{\zeta - w(t)}\right) \prod_{\nu_k \geq p'} \frac{w(t) - w(t_0)}{w(t) - w(t_k)} w'(t) w^k(t) dt = 0, \tag{17}$$
$$k = 0, 1, \dots |\varkappa|.$$

If the points t'_k (that is, t_k for which $\nu_k = p'$), do exist then one can construct a function $\widetilde{f}_0 \in L^p(\gamma)$ for which the conditions (17) are fulfilled, but in this case the problem (4), and hence the problem (2), is unsolvable. Such a construction is possible by the same way as for the Dirichlet problem in the class $\operatorname{Re} E_p(D)$ (see [2]). The similar conclusion is valid if $p \geq 2$ and there exist the points t_k on Γ for which $\nu_k = 2$. Consequently, for the problem (5) to be solvable, one has to strengthen the assumptions relative the boundary function f. We assume that

$$f(t) \prod_{\nu_k = p', \nu_k \neq 2} \ln |t - t'_k| \prod_{\nu_k = 2} \ln^{\mu(p)} |t - t''_k| \in L_p(\Gamma), \tag{18}$$

where $\mu(p) = 1$ for $p \geq 2$ and $\mu(p) = 0$ for $p < 2$.

Since for the curves under consideration $\arg(z(\tau) - z(\tau_k))$ are bounded functions and $z(\tau) - z(\tau_k) = (\tau - \tau_k)^{\nu_k} z_1(\tau)$, where $z_1(\tau)$ is a continuous and different from zero function ([6], [7]), it follows from (18) that

$$g(\tau) \prod_{\nu_k = p', \nu_k \neq 2} \ln |\tau - \tau_k| \prod_{\nu_k = 2} \ln^{\mu(p)} |\tau - \tau'_k| \in L_p(\gamma). \tag{18'}$$

Now in the neighborhood of the singular points $\widetilde{\tau}_k$ $\left(= w(t'_k) \text{ or } w(t''_k) \right)$, we have $\frac{g}{X^+} = 0\left((\tau - \widetilde{\tau}_k)^{-\frac{1}{p}} \ln^{-1}(\tau - \widetilde{\tau}_k)\right)$ which implies $\frac{g}{X^+} \in L_1(\gamma)$. Then the integral in (15) makes sense, and the function $\widetilde{\Omega}$ defined by (15) belongs to H_δ, $\delta > 0$. Again, for this function to belong to $\widetilde{\mathcal{K}}_p(\gamma)$, it is sufficient

that the conditions $\Omega^+ \in L_p(\gamma)$ and (17′) be fulfilled. This time, in the integral from the equality (16) we have

$$X^+(\tau) = O\left(r(\tau) \prod_{\nu_k=2} (\tau - \tau_k'') \prod_{\nu_k=p'} (\tau - \tau_k')^{-\frac{\nu_k-1}{p'}+1} \right).$$

Represent the function g as

$$g(\tau) = \frac{g(\tau) \prod_k \ln(\tau - \widetilde{\tau}_k)}{\prod_k \ln(\tau - \widetilde{\tau}_k)} = \frac{g^*(\tau)}{\prod_k \ln(\tau - \widetilde{\tau}_k)}.$$

Owing to (18′), we have $g^* \in L_p(\gamma)$. We write the equality (15) in the form

$$\widetilde{\Omega}^+(\tau) = \frac{1}{2}g(\tau) + \frac{\lambda(\tau)\rho(\tau) \prod(\tau - \widetilde{\tau}_k)^{1/p'}}{2\pi i} \times$$

$$\times \int_\gamma \frac{g^*(\xi)}{\lambda(\xi)\rho(\xi)} \prod [(\xi - \widetilde{\tau}_k)^{1/p'} \ln(\xi - \widetilde{\tau}_k)]^{-1} \frac{d\xi}{\xi - \tau}, \qquad (19)$$

where $0 < m \le |\lambda(\xi)| \le M$. Since the points $\widetilde{\tau}_k$ are separated, using Theorem 1 from [2] we conclude that $\widetilde{\Omega} \in L_p(\gamma)$. If we assume that conditions (17) are fulfilled, then $\widetilde{\Omega} \in \widetilde{\mathcal{K}}_p(\gamma)$ and is a solution of the problem (5). According to 3^0, (5) has no other solutions. Thus the condition (6) is fulfilled for $\widetilde{\Omega}$, and therefore the function $\phi'(z(w)) = \Psi(w) = \frac{\widetilde{\Omega}(w)}{\sqrt[p]{z'(w)}}$, $|w| < 1$, is a solution of the problem (4).

Proceeding from the above, we can easily find a solution of the problem (2) as well, namely

$$u(z) = \mathrm{Re} \left[\int_{w(0)}^{w(z)} \frac{\widetilde{\Omega}(\xi)d\xi}{\sqrt[p]{z'(\xi)}} \right] + C, \qquad (20)$$

where C is an arbitrary real constant and the integration is performed over any rectifiable path connecting the points o and w.

Remark. If Γ is a smooth curve, then the points t_k are absent, so $\varkappa = 0$. Moreover, one can prove that $\exp\left(\frac{1}{\pi} \int_\gamma \frac{\alpha(\xi)d\xi}{\xi - w(t)} \right) = \frac{e^{-i\alpha(t)}}{w'(t)}$. (To this end, we choose $\tau_0 = 1$ and take into account that in the case where a domain is bounded by a Lyapunov curve, the equality $i \ln z'(w) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{Re}[i \ln z']}{\tau} \frac{\tau+w}{\tau-w} d\tau$ is valid and that $\int_0^{2\pi} \frac{\theta d\xi}{\xi - \tau} = \ln(1 - \tau) - i\theta - \pi i$). Then the condition (17) takes the form

$$\int_\gamma f(t)e^{-i\alpha(t)}dt = \int_\gamma f(t)ds = 0,$$

which coincides with the condition of solvability of the Neumann problem when considered in the class of functions for which partial derivatives are continuous up to the boundary.

5^0. Let us summarize the above-stated results in the form of

Theorem. *Let D be a simply connected domain whose Jordan boundary Γ consists of a finite number of Lyapunov arcs meeting at the points t_k with the angles (measured from the interior of D) of size $\pi\nu_k$, $0 < \nu_k \leq 2$ and e'_p, $p > 1$, is the class of harmonic functions satisfying the condition (1). Denote by $\varkappa$ a number the of angular points at which $\nu_k \geq p'$, $p' = p/p - 1$, and by $X(w)$ the function defined by the formulas (12) and (9).*

If $f \in L_p(\gamma)$ and there are no points t'_k for which $\nu_k = p'$ and, in the case $p \geq 2$ also the points t''_k for which $\nu_k = 0$, then for the Neumann problem to be solvable in the class e'_p, it is necessary and sufficient that conditions (17) be fulfilled.

If there exist the points t'_k and t''_k possessing the above-mentioned properties, then the problem (2) is, generally speaking, unsolvable for any $f \in L_p(\Gamma)$ under the condition (17). In the case where the conditions (18) and (17) are fulfilled, the problem is solvable.

In all the cases, the solution is given by the formula (20) in which $\widetilde{\Omega}$ is the solution of the problem (5) given by the equality (15) and C is an arbitrary constant.

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