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BOUNDARY VALUE PROBLEMS FOR ANALYTIC AND HARMONIC FUNCTIONS IN DOMAINS WITH NONSMOOTH BOUNDARIES. APPLICATIONS TO CONFORMAL MAPPINGS

Abstract. In the present monograph, on the basis of the Cauchy type integral theory discontinuous boundary value problems for analytic functions with oscillating conjugacy coefficients and boundaries are studied. For analytic functions from Smirnov classes, the complete solution of the Riemann-Hilbert problem in domains with arbitrary piecewise smooth boundaries is presented. On the basis of the investigation of the linear conjugation problem, the boundary properties of derivatives of functions conformally mapping the unit circle onto a domain admitting a boundary with tangential oscillation less than π, are studied. From new representations derived for the above-mentioned functions, some well-known results of Lindelöf, Kellog and Warschawski as well as their generalizations are obtained; the Dirichlet and Neumann problems for harmonic functions from Smirnov classes are investigated; the picture of solvability is described completely; the non-Fredholm case is exposed; an influence of geometric properties of boundaries on the solvability is revealed; in all cases of solvability explicit formulas for the solutions in terms of Cauchy type integrals and conformally mapping functions are given.

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რეზიუმე. წინამდებარე მონოგრაფიაში კოშის ტიპის ინტეგრალის თეორიის საფუძველზე ანალიზური ფუნქციებისთვის შესწავლილია წყვეტილი სასაზღვრო ამოცანები რხევადი შეუღლების კოეფიციენტებით და საზღვრებით. სმირნოვის კლასის ანალიზური ფუნქციებისათვის ამოხსნილია რიმან-ჰილბერტის ამოცანა ნებისმიერი უბან-უბან გლუვი წირებით შემოსაზღვრული არეების შემთხვევაში. წრფივი შეუღლების ამოცანის გამოკვლევის საფუძველზე შესწავლილია იმ ასახვების წარმოებულების სასაზღვრო თვისებები, რომლებიც ერთეულოვან წრეს ისეთ არეზე გადასახავენ, რომლის საზღვრის მხების რხევა π-ზე ნაკლებია. ზემოხსენებული ფუნქციების ახალი წარმოდგენებიდან მიღებულია ლინდელოფის, კელოგის და ვარშავსკის ზოგიერთი ცნობილი შედეგი და მათი განზოგადებები. გამოკვლეულია დირიხლესა და ნეიმანის ამოცანები სმირნოვის კლასის ჰარმონიული ფუნქციებისათვის. მთლიანადაა აღწერილი ამოხსნადობის სურათი. განხილულია არაფრედჰოლმური შემთხვევა. შესწავლილია საზღვრების გეომეტრიული თვისებების გავლენა ამოხსნადობაზე. მოცემულია ამონახსნების ცხადი ფორმულები კოშის ტიპის ინტეგრალებისა და კონფორმული ასახვების ტერმინებში.

Introduction

The idea of writing this monograph was stimulated by the authors' latest investigations on two ranges of problems. The first includes boundary value problems for analytic and harmonic functions in domains with piecewise smooth boundaries, while the second is related with finding new properties of conformal mappings of the unit circle onto simply connected domains with non-smooth boundaries using solutions of the above-mentioned problems.

Below we describe our investigations against the background of the theory of boundary value problems for analytic functions. These problems are first encountered in B. Riemann [136]. Important results in this direction have been obtained by Yu.V. Sokhotskiĭ, D. Hilbert, I. Plemelj and T. Carleman.

Let Γ denote either a curve or a finite family of nonintersecting curves. The main objects of our study are the following problems:

(a) The Riemann problem: find a function ϕ from a given class A of the functions, analytic on the plane cut along Γ, whose boundary values satisfy the conjugacy condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t),$$

where G and g are functions prescribed on Γ, and ϕ^+ and ϕ^- are the boundary values of ϕ on Γ .

(b) The Riemann-Hilbert problem: in the domain bounded by Γ, find an analytic function $\phi(z)$ such that its boundary values $\phi^+(t)$ satisfy the condition

$$\mathrm{Re}[G(t)\phi^+(t)] = f(t), \quad t \in \Gamma,$$

where G and f are functions given on Γ.

Depending on the assumptions imposed on the unknown functions, the boundary value problems are conditionally divided into three groups: (i) continuous problems with a continuous (up to the boundary) solution; (ii) piecewise continuous problems, when the continuity is violated only at a finite number of boundary points; (iii) all other problems of discontinuous type.

Fundamental results which stimulated intensive research of these problems were obtained by F.D. Gakhov [37] and N.I. Muskhelishvili [103]. In their first works, F.D. Gakhov considered the problem for the continuous case, whereas N.I. Muskhelishvili investigated it in a more general, piecewise continuous form. Subsequently, I.N. Vekua [160–161] suggested a new approach for investigating a general, continuous, linear boundary value problem of Hilbert type. Using Plemelj's results, N.I. Muskhelishvili and N.P. Vekua studied the continuous boundary value problem for several unknown functions.

The discontinuous boundary value problem has mainly been treated in classes of analytic functions representable by Cauchy type integrals with densities from Lebesgue spaces. It was I.I. Privalov who first considered this problem in a particular case.

Systematic investigations of these problems are connected with the name of B.V. Khvedelidze. In his works, the conjugacy coefficients were assumed to be continuous or piecewise continuous and the boundary to belong to the Lyapunov class. Later on, discontinuous boundary value problems with piecewise continuous coefficients and related singular integral equations were studied by many authors.

The most general results related to the spectral theory of singular integral operators with piecewise continuous coefficients on regular curves in Lebesgue weighted spaces are presented by A. Böttcher and Ju.I. Karlovich in the monograph [7] (therein one can also find an extensive bibliography on the above-mentioned problems).

The investigation of discontinuous boundary value problems with oscillating conjugacy coefficients has always been regarded as one of challenging tasks. The problems on complete characterization of the coefficients G admitting a factorization and on construction of solutions of discontinuous boundary value problems in classes of Cauchy type integrals remains still unsolved.

An essential progress in this direction has been achieved by I.B. Simonenko [141] and I.I. Danilyuk [18]. In their works, the coefficients of boundary value problems were assumed to be functions having an infinite set of discontinuity points, at which, generally speaking, at least one of the one-sided limits does not exist. At the same time, as boundary curves I.I. Danilyuk investigated the curves with bounded rotation.

All the above-mentioned investigations were accompanied by the improvement of the techniques of Cauchy type singular integrals and operators, the derivation of new weighted inequalities for these operators, and the development of some methods of Functional Analysis.

Of the tools applied to discontinuous boundary value problems for analytic functions, the basic are the methods of factorization of functions given on the boundary, as well as the theory of Cauchy type integrals with densities from Lebesgue spaces.

As for the methods and results of investigations achieved in this area, the monographs by B.V. Khvedelidze [68] and I.I. Danilyuk [21] are worth mentioning.

Another important method of treating the boundary value problems is that of singular integral equations. The point is that these problems for various classes of analytic functions are equivalently reduced to singular integral equations. We mean the well-known Carleman-Vekua method applied to continuous and piecewise continuous problems, as well as I.B. Simonenko's theorem on the equivalence of factorization of the coefficient G in the class of Cauchy type integrals with density from $L^p(\Gamma)$ and noetherianness of a linear singular integral equation in this space. In a number of cases, we employ a combination of these two methods.

On the basis of investigations connected with the boundedness of a singular operator in a weighted space and with the belonging to a Smirnov class of the exponential function of a Cauchy type integral with a bounded density, we have managed (in the case of general boundaries) to widen the class of admissible coefficients in discontinuous boundary value problems and the class of domains in which these problems are posed (see [78–79]).

Recently it became clear ([65, 117, 120]) that using the above-mentioned results, one can determine properties of the derivative and the argument of the derivative of the function which maps conformally the unit circle onto a domain with a non-smooth boundary. Representations of conformal mappings and their derivatives are given, providing us with information about their global properties and the behaviour near angular points of the boundary. Results generalizing the well-known Warszawski theorem are also presented. We obtained a new, lucid proof even in the case studied by the above-mentioned author.

Using the solution of the problem of linear conjugation, we managed to describe boundary properties of the derivative and the argument of the derivative of the function which maps conformally the unit disk onto a domain whose boundary admits tangential oscillations not greater than π. The obtained result involves, as a particular case, the E. Lindelöf theorem on the continuity in the closed circle of the argument of the derivative of the conformally mapping function, when the boundary of the domain is smooth.

One of the central places in the present monograph is given to the Riemann-Hilbert boundary value problem in domains with piecewise smooth boundaries. First we consider in these domains the Dirichlet and Neumann boundary value problems in classes of harmonic functions which are nothing but real parts of analytic functions from Smirnov classes.

A great interest to boundary value problems for ellyptic equations in domains with non-smooth boundaries is motivated by the fact that such domains are the most natural ones in many physical processes described by these equations.

The study of the above-mentioned problems in domains with Lyapunov boundaries goes back to G. Giraud and S. Mikhlin.

The period starting from 1977 is marked by intensive investigations carried out in this direction. The Dirichlet problem with boundary conditions from the class L^p $(1 < p < \infty)$ was solved by B.E. Dahlberg for C^1-domains [16], whereas for domains with Lipschitz boundaries (when the boundary function belongs to the class L^p, $2 - \varepsilon \leq p < \infty$, with ε depending on the domain), E.B. Fabes, M. Jodeit Jr. and N. Riviere [33], using the well-known A.P. Calderon theorem on the boundedness of a singular operator over C^1-curves, transferred successfully the classical potential methods to the C^1-domains. As E.B. Fabes, M. Jodeit Jr. and T.E. Lewis [34] have shown, the picture becomes more complicated in the case of non-smooth boundaries. The point is that the integral equations, to which elliptic equations are reduced, are found unsolvable in the above-mentioned domains for some p depending on the extent of non-smoothness of the boundary.

The most significant results referring to boundary value problems for elliptic equations in domains with non-smooth boundaries are contained in the works of V.G. Maz'ya and A.A. Soloviev [93–95], V.G. Maz'ya, S.A. Nazarov and B.A. Plamenevskiĭ [96], V.A. Kondrat'ev and O.A. Oleĭnik [85], M. Dauge [22], etc.

A tight connection between harmonic (in a plane domain) and analytic functions made it possible to extend the methods and results stated for boundary value problems for analytic functions to those for harmonic functions. A method developed in this direction by N.I. Muskhelishvili has been successfully applied by us to basic boundary value problems for harmonic functions in domains with non-smooth boundaries.

We obtained a complete picture of solvability for the above-mentioned problems and found that it depends mainly on the geometry of the boundary. The presence of angular points is occasionally the reason of their unsolvability or many-valued solvability. The most effective tool in studying the problem is provided by the above-mentioned properties of derivatives of functions mapping conformally the unit circle onto a domain with a piecewise smooth boundary, as well as by two-weight inequalities for singular integrals ([30], [80]). The latter allow one to point out for the boundary functions the Lebesgue spaces with logarithmic weight, for which the problem becomes solvable.

In all the cases of solvability, the solutions of the Dirichlet and the Neumann problems are constructed in quadratures by means of the Cauchy type integrals and conformally mapping functions.

Next, in Smirnov classes we investigated the more general Riemann-Hilbert problem for functions, analytic in domains with a piecewise smooth boundary. A picture illustrating the solvability of the problem is presented; the influence of the coefficient and the boundary geometry on the character of solvability is shown; in all the cases of solvability, the solutions are given in quadratures.

Revert now to two-sided discontinuous boundary value problems for analytic functions. When investigating the problem of linear conjugation, as unknowns we usually consider Cauchy type integrals with densities from the Lebesgue class L^p for $1 < p < \infty$. To the same class must belong the function g in the boundary condition of the Riemann problem. If the latter is only summable, then the boundary value problem has, as is seen, no solution even in the simplest cases. This is connected with the fact that the boundary function of the Cauchy type integral is not always summable, and hence a Cauchy type integral with a summable density fails to be representable by the Cauchy integral.

Here naturally arises the problem of extending the notion of the Lebesgue integral in such a way that the functions mentioned above would be integrable in a new sense. Such an extension takes its origin in the work of A.N. Kolmogorov [84] in which he proves that the function conjugate to a 2π-periodic summable function is B-integrable, and hence the conjugate trigonometric series is a Fourier B-series. An analogous result for the A-

integral has been proved by E.S. Titchmarsh [154]. Further, P.L. Ul'yanov [156–158] has shown that the boundary values of the Cauchy type integral on Lyapunov contours are always A-integrable, and the Cauchy type integral itself is representable by the Cauchy A-integral. An application of the A-integral to the theory of Cauchy type integrals and to the solution of a boundary value problem and the associated singular integral equation is given in [59]. On the other hand, it is known that the A-integral has a number of significant shortcomings preventing it to be a convenient tool.

In [60–61], it has been shown with the aid of the A- and B-integrals that most of the results obtained for conjugate functions and for Cauchy-Lebesgue type integrals do not depend on specific properties of these integrals. They hold true for any generalization of the Lebesgue integral in whose sense the conjugate function is integrable and its integral equals zero. This is the way how the notion of the so-called $\widetilde{L}$-integral originated.

In [58] it is stated that if the density of the Cauchy type integral is summable, then its angular boundary values are $\widetilde{L}$-integrable, and the Cauchy type integral is representable in the domain by the Cauchy $\widetilde{L}$-integral. On the basis of the $\widetilde{L}$-integration, we established many new properties of the Cauchy type integral. Namely, we generalized the well-known formulas of inversion of the singular Cauchy integral which allowed us to investigate the discontinuous problem of linear conjugation in the case where the function g is Lebesgue summable, and the conjugacy coefficient G is Hölder continuous, differing from zero. Solutions of the problem are sought in the class of Cauchy type $\widetilde{L}$-integrals. Moreover, all the solutions of the problem are constructed explicitly. The new point in our work is that the boundary curve is regular (Carleson curve).

All these results are reflected in the present monograph. We endevoured to reproduce as complete a picture as possible of authors' investigations connected with discontinuous boundary value problems and with the theory of conformal mappings.

Chapter I is devoted to the investigation of boundary value problems on the basis of the function theory. Results from nonlinear harmonic analysis are presented herein, including continuous operators generated by singular integrals on general contours, one- and two-weight inequalities for these operators and a theorem on belonging a Smirnov class of a Cauchy type integral. In terms of the so-called p-mean singular integrals for an individual function, criteria for the representability by the Cauchy type integral are found.

The notion of generalized $\widetilde{L}$-integral is introduced, and a number of new properties of Cauchy type $\widetilde{L}$-integrals are established.

Chapter II proposes results obtained for discontinuous Riemann-Privalov boundary value problems in the case of more general, oscillating coefficients in the boundary conditions. Simultaneously the class of boundary curves is expanded.

Discontinuous boundary value problems in classes of analytic functions representable by Cauchy type integrals with densities from weighted Lebesgue spaces are treated by reducing them to analogous non-weighted problems. Boundary value problems on infinite lines are considered as well.

In the same chapter, the Riemann-Hilbert problem in a class of Cauchy type integrals with densities from weighted Lebesgue spaces is reduced to the problem of linear conjugation for a circle, with coefficients reflecting all singularities of the weight, boundary curve and initial coefficients. The latter allows one (varying sets of unknown functions, coefficients and boundary curves) to investigate the Riemann-Hilbert problem in various statements, using the results of §§2–4 of Chapter I and the properties of conformally mapping functions. One of such possibilities is realized in Chapter IV where the problem is considered in a domain with an arbitrary piecewise smooth boundary.

At the end of Chapter II, we study boundary value problems of linear conjugation in a class of functions representable by Cauchy $\widetilde{L}$-integrals, when the boundary curve satisfis Carleson condition and the non-vanishing coefficient is of the Hölder class.

Results of Chapter II are found to be very efficient for revealing new properties of functions which map conformally the unit circle onto a simply connected domain with a non-smooth boundary. Just to these questions is devoted Chapter III. The main point here is to represent a conformally mapping function, its derivative and the argument of the derivative, and to illustrate their behaviour in the neighbourhood of angular points (including cusps). Some properties of these functions are described for curves admitting tangential oscillations.

Chapter IV gives the complete solution of the Riemann-Hilbert problem in domains with an arbitrary piecewise smooth boundary; a comprehensive treatment of the Dirichlet and the Neumann problems for real parts of analytic functions from Smirnov classes, is carried out. Non-Fredholm cases are considered; a picture of the solvability is described completely; the solutions are given in quadratures.

It should be noted that in the present work we do not touch upon such important areas as: boundary value problems and singular integral equations with matrix coefficients and shifts; the cases of the infinite index; the problems in classes of generalized analytic functions, etc.

Some of the results presented here are scattered in authors' earlier papers, mainly in the announced form. A substantial part of the monograph involves latest results, some of them still not published. As we see it, the exposition of all these results in a unified form clarifies their influence on each other.

Chapters in the present work are divided into sections which in their turn are subdivided into subsections. Formulas in each section are supplied with a pair of numbers, the first one indicating the section number and the second one formula's ordinal number. References within each chapter are given as the section number and the formula number, while references to results from other chapters have additionally the chapter number.

References contain papers and monographs which are connected directly with the problems under consideration. The list is by no means complete.

We would be very pleased if this monograph will give the reader an impulse to further interest in this area. At the same time, we would be very grateful if the reader will point out imperfections or mistakes of any kind, which inevitably occur.

Note finally that the authors' interest to the subject matter has in many respects been stimulated by a many-year collaboration (within the framework of one department) with Academician B.V. Khvedelidze, a well-known specialist of discontinuous boundary value problems and singular integral equations.

Basic Ingredients

In this section we present definitions, notation and propositions which will be used throughout the paper. Definitions within separate chapters or section will be given in appropriate places.

0.1. $\mathbb{C}$ is the extended complex plane. $\mathbb{R} = (-\infty, +\infty)$.

0.2. $U = \{z : |z| < 1\}$, $\gamma = \{z : |z| = 1\}$.

0.3. $H(\alpha)$, $0 < \alpha \leq 1$ is the class of functions satisfying Hölder condition with the exponent α; $H = \cup_{\alpha \in [0,1]} H(\alpha)$ is the Hölder class; $H(1)$ is the Lipschitz class.

0.4. $C(E)$ is the Banach space of all continuous on E functions with usual norm.

0.5. Under a curve is meant an oriented, rectifiable, Jordan curve on which as a parameter the arc length is chosen starting from any fixed point. The equation of the curve in this case is $t = t(s)$, $0 \leq s \leq l$, where l is its length. The parameter s is called the arc abscissa. The function $t = t(s)$ is assumed to be periodically extended on $\mathbb{R}$.

Every curve on the plane $\mathbb{C}$ generates the set of Γ-images of the segment $[0, l]$ for the mapping $t = t(s)$. The curve will often be identified with this set. In particular, by saying that a closed Jordan curve divides the plane into two connected sets we mean that the set Γ possesses this property. An analogous remark refers to the expressions such as "a curve bounds a domain D", "a function is given on a curve", etc.

0.6. We will say that the function $f = f(t)$ defined on Γ is almost everywhere finite, integrable, and so on, if the function $f_*(s) = f(t(s))$ possesses the corresponding property on $[0, l]$. If f is integrable, we assume

$$\int_\Gamma f(t)dt = \int_0^l f(t(s))t'(s)ds. \tag{0.1}$$

0.7. $L^p(\Gamma)$, $p \geq 1$ is the space of summable in the p-th degree functions f with the norm

$$\|f\|_{p,\Gamma} = \|f\|_p = \left(\int_\Gamma |f|^p ds\right)^{1/p}. \tag{0.2}$$

If w is a measurable, almost everywhere finite and different from zero function, then $L^p(\Gamma, w)$ is the space of functions with the norm

$$\|f\|_{p,w} = \|fw\|_p, \quad L^\infty(\Gamma) = \{f : \|f\|_\infty = \operatorname*{ess\,sup}_{t\in\Gamma} |f(t)| < \infty\}.$$

Suppose $L(\Gamma) = L^1(\Gamma)$, $L(\Gamma; w) = L^1(\Gamma; w)$.

The spaces $L^p(\Gamma; w)$ and $L^{p'}(\Gamma; w^{-1})$, $p > 1$, $p' = \frac{p}{p-1}$ are conjugate ones.

0.8. Let $f \in L(\Gamma)$ and $t_0 = t(s_0) \in \Gamma$. Denote by $\Gamma_\varepsilon(t_0)$, $\varepsilon < \frac{l}{4}$ the portion of the curve Γ left after removing a small arc with the ends $t(s_0 - \varepsilon)$ and $t(s_0 + \varepsilon)$. If

$$\lim_{\varepsilon \to 0} \frac{1}{\pi i} \int\limits_{\Gamma_\varepsilon(t_0)} \frac{f(t)dt}{t - t_0} = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int\limits_{s_0+\varepsilon}^{l-s_0-\varepsilon} \frac{f(t(s))t'(s)ds}{t(s) - t(s_0)} = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{f(t)dt}{t - t_0},$$

then we say that there exists the singular Cauchy integral (or the singular integral) of the function f at the point t_0.

In the sequel we put

$$(S_\Gamma f)(t_0) = \frac{1}{\pi i} \int\limits_{\Gamma} \frac{f(t)dt}{t - t_0}, \quad t_0 \in \Gamma. \tag{0.3}$$

0.9. $D(S_\Gamma)$ is the set of those functions $f \in L(\Gamma)$ for which $(S_\Gamma f)(t_0)$ exists for almost all $t_0 \in \Gamma$. As is known (see also, Ch.I, §3), for rectifiable Γ

$$D(S_\Gamma) = L(\Gamma). \tag{0.4}$$

0.10. The operator

$$S_\Gamma : f \to S_\Gamma f \tag{0.1}$$

is called the Cauchy operator. Sometimes instead of S_Γ we will write S. A norm of the operator S_Γ, when it acts boundedly (continuously) from $L^p(\Gamma)$ to $L^s(\Gamma)$, $p \geq s > 0$, will be denoted by $\|S_\Gamma\|_{p,s}$.

If Γ is a straight line, then $S_\Gamma f$ is usually called the Hilbert transform. When $\Gamma = \gamma$, assuming $\tau = \exp i\sigma$, $t = \exp is$, we have

$$\frac{d\tau}{\tau - t} = \Big(\frac{1}{2} \operatorname{ctg} \frac{\sigma - s}{2} + \frac{i}{2}\Big) d\sigma,$$

and therefore

$$(S_\gamma f)(e^{is}) = (-i\widetilde{f})(s) + \frac{1}{2} f(s), \tag{0.6}$$

where

$$\widetilde{f}(s) = -\frac{1}{2\pi} \int\limits_0^{2\pi} f(\sigma) \operatorname{ctg} \frac{\sigma - s}{2} d\sigma. \tag{0.7}$$

The function $\widetilde{f}$ is called the function conjugate to f.

0.11. Classes of Curves. Let $t = t(s)$, $0 \leq s \leq l$, be the equation of the curve Γ. It will be called: (a) a smooth curve if t' is continuous (and in the case of its closedness, $t'(0) = t'(l)$); (b) Lyapunov curve, if $t' \in H$; (c) a piecewise smooth curve, if t' is piecewise continuous, and a piecewise Lyapunov curve, if t' is a piecewise Hölder function; (d) a curve with bounded rotation (Radon's curve) if t' is of finite variation.

K is the set of those curves Γ for which

$$\inf \frac{|t_1 - t_2|}{s(t_1, t_2)} = k > 0, \tag{0.8}$$

where s is the length of the arc Γ (the smaller one if Γ is closed) connecting the points t_1 and t_2.

A closed Jordan curve Γ is called Smirnov curve, if the finite domain D bounded by this curve is a Smirnov domain, i.e., the function $w = \ln |z'(w)|$ is representable by a Poisson integral, where $z = z(w)$ is a conformal mapping of U onto D.

Any curve of the class K is a Smirnov curve [131].

$R_{p,s}$, $p \geq 1$, $s \leq p$, is the set of all Jordan curves for which the operator S_Γ is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$.

$$R_p = R_{p,p}. \tag{0.9}$$

$$R = \bigcap_{p>1} R_p. \tag{0.10}$$

0.12. The operator S is of a strong type (p, s), if it is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$, and of a weak type (p,p), if $\forall f \in L^p(\Gamma)$ and $\forall \lambda > 0$,

$$m(t : |(Sf)(t)| > \lambda) < \frac{c}{\lambda^p} \int_\Gamma |f|^p ds, \tag{0.11}$$

where c does not depend on λ and f, and m is the Lebesgue measure.

If for any converging in $L^p(\Gamma)$, $p \geq 1$, sequence f_n the sequence $S_\Gamma f_n$ converges in measure, then we will say that S_Γ is continuous in measure.

0.13. $W_p(\Gamma)$, $p > 1$, is the set of weighted (i.e., measurable, almost everywhere finite and different from zero) functions w such that $\forall f \in L^p(\Gamma; w)$,

$$\int_\Gamma |(S_\Gamma f)w|^p ds \leq M_p \int_\Gamma |fw|^p ds, \tag{0.12}$$

where M_p does not depend on f.

A_p is the Muckenhoupt class, the set of given on γ weighted positive functions w such that

$$\sup_I \Big(\frac{1}{|I|} \int_I w ds\Big)\Big(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} ds\Big)^{p-1} < \infty, \tag{0.13}$$

where $I \subset \gamma$ is an arbitrary arc of length less than 2π, $|I|$ is its length.

0.14. Let Γ be a closed Jordan curve bounding a finite domain D^+, and let D^- be a complement to $\mathbb{C}$ of the set $D^+ \cup \Gamma$. A curve lying in D^+ (D^-) and ending at the point t is said to be a non-tangential path, if in the neighbourhood of t it lies in some angle of size less than π, with the vertex at the point t and the bisectrix coinciding with the normal to Γ. If the function $\phi = \phi(z)$ defined in $D^+(D^-)$ tends along any non-tangential path to the limit $\phi^+(t)$ $(\phi^-(t))$ as $z \in D^+(D^-)$, $z \to t$, then this limit is called the angular boundary value from the left (from the right) of the function ϕ at the point t. If $\phi^+(t)$ ($\phi^-(t)$) exists almost everywhere on Γ, then the function $\phi^+ = \phi^+(t)$ will be called an angular boundary value of ϕ on Γ from the left (from the right).

0.15. Let Γ be a closed, rectifiable Jordan curve, and let $f \in L(\Gamma)$. The function

$$(\mathcal{K}_\Gamma f)(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}, \quad z\overline{\in}\Gamma,$$

is termed a Cauchy-type integral with density f.

If $\phi(z) = (\mathcal{K}_\Gamma f)(z)$ and $\phi^+(t) = f(t)$ almost everywhere on Γ, that is,

$$\phi(z) = (\mathcal{K}_\Gamma \phi^+)(z), \quad z \in D^+,$$

then we will say that ϕ is represented in the domain D^+ by the Cauchy integral. The Cauchy integral in D^- is defined analogously.

The Cauchy type integral has angular boundary values ϕ^+ and ϕ^- which are defined by the Sokhotskiĭ-Plemelj formulas

$$\phi^{\pm}(t) = \pm\frac{1}{2}f(t) + \frac{1}{2\pi i}\int_\Gamma \frac{f(\tau)d\tau}{\tau - t}, \quad t \in \Gamma. \tag{0.14}$$

This statement is valid by Privalov's lemma ([133], p. 190) and the equality (0.4).

In order for a Cauchy type integral $\phi = \mathcal{K}_\Gamma f$ to be representable in D^+ (D^-) by a Cauchy integral, it is necessary and sufficient that ϕ^+ belong to $L(\Gamma)$ and the equality $S_\Gamma\phi^+ = \phi^+$ ($S_\Gamma\phi^- = -\phi^-$) take place ([68], p. 100).

0.16. $$\mathcal{K}^p(\Gamma; w) = \left\{\phi : \phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(t)dt}{t-z}, \quad z\overline{\in}\Gamma, \quad \varphi \in L^p(\Gamma; w)\right\}.$$

$$\widetilde{\mathcal{K}}^p(\Gamma; w) = \{\phi : \phi = \phi_0 + \text{const}, \quad \phi_0 \in \mathcal{K}^p(\Gamma; w)\}.$$

$$\widetilde{\mathcal{K}}^p_n(\Gamma; w) = \left\{\phi : \phi = \phi_0 + q_n, \quad \phi_0 \in \mathcal{K}^p(\Gamma; w), \quad q_n = \sum_{k=0}^n a_k z^k\right\}.$$

When $w = 1$, these classes will be denoted, respectively, by $\mathcal{K}^p(\Gamma)$, $\widetilde{\mathcal{K}}^p(\Gamma)$ and $\widetilde{\mathcal{K}}^p_n(\Gamma)$.

0.17. H^p, $p > 0$, is the Hardy class, the set of analytic in U functions ϕ for which

$$\sup_{r\in(0,1)}\int_0^{2\pi} |\phi(re^{i\theta})|^p d\theta < \infty. \tag{0.15}$$

0.18. Let D be a simply connected domain bounded by a rectifiable curve Γ. Then $E^p(D)$ or E^p, $p > 0$, is the Smirnov class of analytic in D functions

ϕ for which there exists a sequence of closed curves $\Gamma_n \subset D$ converging to Γ such that

$$\sup_n \int_{\Gamma_n} |\phi(z)|^p |dz| < \infty. \tag{0.16}$$

If D is an infinite domain bounded by Γ, then $\widetilde{E}^p(D)$ is a set of analytic in D functions ϕ for which $\Psi(w) = \phi\left(\frac{1}{w} + z_0\right) \in E^p(\widetilde{D})$, where $\widetilde{D}$ is the finite domain into which the function $w = \frac{1}{z - z_0}$, $z_0 \overline{\in} D$, maps D (this class obviously does not depend on the choice of z_0).

0.19. Basic properties of the functions from E^p:

(1) $\phi \in E^p(D)$ if and only if

$$\sup_r \int_{\Gamma_r} |\phi(z)|^p |dz| < \infty, \tag{0.17}$$

where Γ_r is the image of the circumference $|z| = r$ under a conformal mapping of U onto D (see, e.g., [133]).

(2) The function $\phi \in E^p(D)$ possesses angular boundary values on Γ, and the boundary function belongs to $L^p(\Gamma)$ ([133], p. 205).

(3) The class $E^1(D)$ coincides with the class of functions representable in D by the Cauchy integral ([133], p. 205–206).

(4) Smirnov's theorem. If D is a Smirnov domain, $\phi \in E^p(D)$ and $\phi^+ \in L^q(D)$, $q > p$, then $\phi \in E^q(D)$.

(5) Let D be a simply connected domain with the boundary Γ, $e \subset \Gamma$, $\operatorname{mes} e > 0$ and let ϕ_n be a sequence of analytic in D functions. If $\phi_n \in E^p(D)$, $\sup \|\phi_n^+\|_p < \infty$ and ϕ_n^+ converges in measure on e to the function f, then ϕ_n converges in D to the function $\phi \in E^p(D)$, and on e we have $\phi^+ = f$ (see, e.g., [133], pp. 268–9).

0.20. Stein's interpolation theorem ([150]). If M is a linear operator acting from one space of measurable functions into the other,

$$1 \le r_1, r_2, s_1, s_2 \le \infty, \quad r^{-1} = (1-t)r_1^{-1} + tr_2^{-1},$$
$$s^{-1} = (1-t)s_1^{-1} + ts_2^{-1}, \quad 0 < t < 1,$$

and

$$\|(Mf)k_i\|_{s_i} \le c_i \|fu_i\|_{r_i}, \quad i = 1, 2,$$

then

$$\|(Mf)k\|_s \le c\|fu\|_r, \tag{0.19}$$

where $k = k_1^{1-t}k_2^t$, $u = u_1^{1-t}u_2^t$, $c = c_1^{1-t}c_2^t$,

0.21. Let a linear operator A map the Banach space X into the Banach space Y. We will say that A is a Noetherian operator if: (i) the equation $Ax = y$ is solvable for any right-hand side of y which satisfies the condition $f(y) = 0$, where f is an arbitrary solution of the conjugate homogeneous equation $A^*f = 0$; (ii) zero subspaces $N(A)$ and $N(A^*)$ of the operators A and A^* are finite-dimensional. If A is a Noetherian operator, and λ and μ are the dimensions of the subspaces $N(A)$ and $N(A^*)$, then the difference $\varkappa = \lambda - \mu$ is called the index of the operator A ($\operatorname{ind} A$).

If A is a Noetherian operator, and V is a compact operator acting from X to Y, then the operator $A + V$ is likewise Noetherian, and $\operatorname{ind}(A+V) = \operatorname{ind} A$ ([3]).

CHAPTER I
ON SINGULAR AND CAUCHY TYPE INTEGRALS

§ 1. On the Definition of Cauchy Singular Integral

In the literature we meet with two seemingly different notions of the principal value of the Cauchy integral of the function $f(t)(t-t_0)^{-1}$, when the integral is taken along the curve Γ and $t_0 \in \Gamma$. According to the first approach, the arc $\gamma_\varepsilon(t_0)$ which is the part of Γ contained in the circle of radius $\varepsilon > 0$ with center at t_0 is cut off the curve, then the integral is taken over the remaining part and its limit as $\varepsilon \to 0$ is considered. When using the second approach, the arc $\gamma_\varepsilon(t_0)$ in the above definition is replaced by the symmetric with respect to t_0 arc of the length 2ε. In subsection 1.2 we will show that these integrals exist in almost all points simultaneously and have equal values.

In determining the curvilinear integral, under the notion of a curve we usually mean a family of equivalent in a certain sense paths. Below it will be shown how one can obtain the definition of the same character for a singular integral.

Recall first some conventional definitions.

We say that the paths $\mu_1 : \sigma \to t_1(\sigma) = x_1(\sigma) + iy_1(\sigma)$, $\sigma \in [\alpha_1, \beta_1]$ and $\mu_2 : \tau \to t_2(\tau) = x_2(\tau) + iy_2(\tau)$, $\tau \in [\alpha_2, \beta_2]$ are equivalent (or μ_2 is obtained from μ_1 by a change of parameter) if there exists a strongly increasing absolutely continuous function $\sigma = \sigma(\tau)$ which maps $[\alpha_2, \beta_2]$ onto $[\alpha_1, \beta_1]$ and satisfies $t_2(\tau) = t_1(\sigma(\tau))$.

A class $\{\mu\} = \Gamma$ of equivalent paths will be called a curve Γ. General image on $\mathbb{C}$ of segments $[\alpha_\mu, \beta_\mu]$ for the mapping μ will also be denoted by Γ. If Γ is a rectifiable curve, $t = t(\sigma)$, $\sigma \in [\alpha, \beta]$, is the equation of an arbitrary path of μ from Γ and

$$s(\sigma) = \int\limits_{\alpha}^{\sigma} |t'(u)| du,$$

then the path $\mu_s : s \to t(\sigma(s))$, $s \in [o, l]$, $l = s(\beta)$, where $\sigma = \sigma(s)$ is the function inverse to $s = s(\sigma)$, does not depend on the choice of μ. This path will be called the arc path. The arc path equation $\zeta = \zeta(s)$ is called the equation of Γ with respect to the arc abscissa. Almost everywhere on $[0, l]$ we have $|\zeta'(s)| = 1$.

We say that the set $e \subset \Gamma$ (Γ-image of the curve on $\mathbb{C}$) is measurable if the set $\zeta^{-1}(e) \subset \mathbb{R}$ is Lebesgue measurable (ζ^{-1} is the function inverse to ζ); the measure of e is assumed to be equal to that of the set $\zeta^{-1}(e)$.

Let $\mu : \sigma \to t(\sigma)$, $\sigma \in [\alpha, \beta]$ be a path in which $t = t(\sigma)$ is an absolutely continuous function, and $t_0 = t(\sigma_0)$, $0 < \varepsilon < (\beta - \alpha)/2$. Denote by $\mu_\varepsilon(t_0)$ the image of the set $(\sigma_0 - \varepsilon, \sigma_0 + \varepsilon)$ and let $\Gamma_{\mu,\varepsilon}(t_0) = \Gamma \backslash \mu_\varepsilon(t_0)$. If there exists

$$\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int\limits_{\Gamma_{\mu,\varepsilon}(t_0)}\frac{f(t)dt}{t-t_0}=$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int\limits_{\sigma_0+\varepsilon<\sigma<\sigma_0+\beta-\alpha-\varepsilon}\frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma)-t(\sigma_0)}=\frac{1}{\pi i}\int\limits_{\Gamma(\mu)}\frac{f(t)dt}{t-t_0}, \tag{1.1}$$

then we will say that at the point t_0 there exists a singular integral along the path μ, and will denote it by

$$S(\mu,f)(t_0)=\frac{1}{\pi i}\int\limits_{\Gamma(\mu)}\frac{f(t)dt}{t-t_0}. \tag{1.2}$$

Along with the above-said, consider also the following definition of a singular integral. Let $t=t(s)$, $0\le s\le l$, be the equation of the curve Γ with respect to the arc abscissa, $0<s_0<l$, $\varepsilon<\frac{l}{4}$. Denote by $\gamma_{0,\varepsilon}(t_0)$ the least connected arc Γ passing through t_0 with the ends on the circumference $|t-t_0|=\varepsilon$, and let $\Gamma_{\mu_0,\varepsilon}(t_0)=\Gamma\backslash\mu_{0,\varepsilon}(t_0)$. Denote

$$(S_\Gamma^{(0)}f)(t_0)=\lim_{\varepsilon\to 0}\frac{1}{\pi i}\int\limits_{\Gamma_{\mu_0,\varepsilon}(t)}\frac{f(t)dt}{t-t_0}=\frac{1}{\pi i}\int\limits_{\Gamma(\mu_0)}\frac{f(t)dt}{t-t_0}. \tag{1.3}$$

1.1. Independence of a singular integral on the curve parametrization.

Theorem 1.1. *Let $\Gamma=\{\mu\}$ be a rectifiable Jordan curve and $f\in L(\Gamma)$. If $\mu_s:s\longrightarrow\zeta(s)$, $0\le s\le l$ is an arc path and $\mu:\sigma\longrightarrow t(\sigma)$, $\alpha\le\sigma\le\beta$ is an arbitrary, absolutely continuous path from $\{\mu\}$, then from the existence almost everywhere of either of the two singular integrals $S(\mu_s,f)(\zeta(s_0))$ or $S(\mu,f)(t(\sigma_0))$, it follows the existence of the other and their equality.*

Proof. Let $t_0=t(\sigma_0)=\zeta(s_0)$ and let the relations

$$\Big(\int\limits_{\alpha}^{u}|t'(u)|du\Big)'=|t'(\sigma_0)|\ne 0, \tag{1.4}$$

$$\lim_{\delta\to 0}\frac{1}{\delta}\int\limits_{s_0}^{s_0+\delta}|f(\zeta(s))|ds=|f(\zeta(s_0))| \tag{1.5}$$

be fulfilled. Let us show that for such points t_0 the existence of a singular integral along either of the paths μ_s or μ results in the existence along the other and their equality.

Let $\alpha < \sigma_0 < \beta$ and let $\varepsilon > 0$ be an arbitrary number, provided $\alpha < \sigma_0 - \varepsilon < \sigma_0 + \varepsilon < \beta$. Denote

$$\eta = \eta(\varepsilon) = \int\limits_{\sigma_0-\varepsilon}^{\sigma_0} |t'(u)|du. \tag{1.6}$$

The function $\eta = \eta(s)$ is decreasing. Let $\varepsilon = \varepsilon(\eta)$ be an inverse to it function and

$$\nu(\eta) = \int\limits_{\sigma_0}^{\sigma_0+\varepsilon(\eta)} |t'(u)|du. \tag{1.7}$$

The functions ζ and t will be assumed to be extended periodically on $\mathbb{R}$ with periods l and $\beta - \alpha$, respectively.

Consider the difference

$$I(\varepsilon) = \int\limits_{s_0+\eta}^{s_0+l-\eta} \frac{f(\zeta(s))\zeta'(s)ds}{\zeta(s)-\zeta(s_0)} - \int\limits_{\sigma_0+\varepsilon}^{\sigma_0+\beta-\alpha-\varepsilon} \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma)-t(\sigma_0)} \tag{1.8}$$

and prove that $I(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since μ_s and μ are paths from an equivalence class, there exists a function $\sigma = \sigma(s)$, $0 \le s \le l$, such that $t(\sigma(s)) = \zeta(s)$. Change the variable σ in the second integral of (1.8) by the equality $\sigma = \sigma(s)$. Then

$$|I(\varepsilon)| = \left| \int\limits_{I} \frac{f(\zeta(s))\zeta'(s)ds}{\zeta(s)-\zeta(s_0)} \right|, \tag{1.9}$$

where I is the interval with the ends $s_0 + \eta$ and $s_0 + \nu(\eta)$. By virtue of (1.4), it can be easily verified that

$$\lim_{\sigma\to\sigma_0} \frac{t(\sigma)-t(\sigma_0)}{s(\sigma)-s(\sigma_0)} = \lim_{\sigma\to\sigma_0} \frac{t(\sigma)-t(\sigma_0)}{\sigma-\sigma_0}\frac{\sigma-\sigma_0}{s(\sigma)-s(\sigma_0)} = \frac{t'(\sigma_0)}{|t'(\sigma_0)|}. \tag{1.10}$$

This implies the existence of a positive number $k = k(\sigma_0)$ such that if σ lies in the small neighbourhood of the point σ_0, then

$$k < |t(\sigma) - t(\sigma_0)||s(\sigma) - s(\sigma_0)|^{-1}. \tag{1.11}$$

Denote $\rho(\eta) = \min(\eta, \nu(\eta))$. Then, taking into account (1.11) and (1.9), we get

$$|I(\varepsilon)| \le \frac{1}{k\rho(\eta)} \left| \int\limits_{I} |f(\zeta(s))|ds \right| = \frac{1}{k\rho(\eta)} \left| \int\limits_{s_0+\eta}^{s_0+\nu(\eta)} |f(\zeta(s))|ds \right| =$$

$$= \frac{1}{k\rho(\eta)} \left| \eta \frac{1}{\eta} \int\limits_{s_0}^{s_0+\eta} |f(\zeta(s))| ds - \nu(\eta) \frac{1}{\nu(\eta)} \int\limits_{s_0}^{s_0+\nu(\eta)} |f(\zeta(s))| ds \right|. \quad (1.12)$$

According to the assumption (1.5), we have

$$\frac{1}{\eta} \int\limits_{s_0}^{s_0+\eta} |f(\zeta(s))| ds = |f(\zeta(s_0))| + \alpha(\eta),$$

$$\frac{1}{\nu(\eta)} \int\limits_{s_0}^{s_0+\nu(\eta)} |f(\zeta(s))| ds = |f(\zeta(s_0))| + \beta(\eta),$$

where $\alpha(\eta)$ and $\beta(\eta)$ are infinitely small together with η. This and (1.12) yield

$$|I(\varepsilon)| \leq \frac{|\eta - \nu(\eta)|}{\rho(\eta)} \frac{|f(\zeta(s_0))|}{k} + \frac{\eta}{k\rho(\eta)} \alpha(\eta) + \frac{\nu(\eta)}{k\rho(\eta)} \beta(\eta). \quad (1.13)$$

From (1.6) and (1.7), by virtue of (1.4) we have $\nu(\eta)\eta^{-1} \to 1$. Therefore, if $\varepsilon \to 0$, then $|\eta - \nu(\eta)|\rho^{-1}(\eta) \to 0$, and $\eta\rho^{-1}(\eta)$ and $\nu(\eta)\rho^{-1}(\eta)$ are bounded. But then from (1.13) we conclude that $\lim I(\varepsilon) = 0$. This and (1.8) imply that the integrals $S(\mu, f)(t(\sigma_0))$ and $S(\mu_s, f)(\zeta(s_0))$ in conditions (1.4)–(1.5) exist only simultaneously. Since the set of those points $t_0 \in \Gamma$ for which equalities (1.4) and (1.5) are fulfilled simultaneously has a complete measure, we can conclude that the theorem is valid. ■

Remark. The above proven theorem can be regarded as justification of the formula of change of variables in the singular integral. Namely, because of (0.4) and by Theorem 1.1, the following assertion is valid:

Proposition 1.1. *If $s = s(\sigma)$ is an absolutely continuous, increasing function mapping $[\alpha, \beta]$ onto $[0, l]$ and $\zeta(s(\sigma)) = t(\sigma)$, then*

$$\frac{1}{\pi i} \int\limits_0^l \frac{f(\zeta(s))\zeta'(s) ds}{\zeta(s) - \zeta(s_0)} = \frac{1}{\pi i} \int\limits_\alpha^\beta \frac{f(t(\sigma))t'(\sigma) d\sigma}{t(\sigma) - t(\sigma_0)} \quad (1.14)$$

for almost all $s_0 \in [0, l]$.

It follows from Theorem 1.1 that if the curve Γ is a class of equivalent, absolutely continuous paths and for any path μ_0 from this class the singular integral $S(\mu_0, f)(t_0)$ exists on a set of complete measure, then for any other path μ from Γ the set of those points at which the integral $S(\mu, f)(t_0)$ exists, also possesses a complete measure. The integrals along these paths have the same values. Hence, for all the paths we introduce the unique notation $S(\Gamma; f)$ or $S_\Gamma f$, and for every separate point $\zeta_0 = \zeta(s_0)$ we assume that

$$(S_\Gamma f)(\zeta_0) = \frac{1}{\pi i} \int\limits_0^l \frac{f(\zeta(s))\zeta'(s)ds}{\zeta(s) - \zeta(s_0)} \,.$$

1.2. Integrals $S_\Gamma^{(0)} f$ and $S_\Gamma f = S(\mu_s, f)$ coincide almost everywhere.

Theorem 1.2. *If $f \in L(\Gamma)$ and conditions (1.4) and (1.5) are fulfilled, then for almost all $t_0 \in \Gamma$ the equality*

$$S(\mu_s, f)(t_0) = (S_\Gamma^{(0)} f)(t_0)$$

holds.

Proof. Let $\varepsilon > 0$ and let $\mu_{0,\varepsilon}(t_0)$ be a least connected arc of the path μ_s passing through t_0 with the ends on the circumference $|t - t_0| = \varepsilon$. Denote these ends by t' and t''. Then $\mu_{0,\varepsilon}(t_0) = t't''$. Let $t' = t(s')$, $t'' = t(s'')$. Suppose $\eta = \min(|s'-s_0|, |s''-s_0|)$, $\lambda = \max(|s'-s_0|, |s''-s_0|)$ and consider the difference

$$I^0(\varepsilon) = \int\limits_{\Gamma_\varepsilon(t_0)} \frac{f(t)dt}{t - t_0} - \int\limits_{\Gamma\setminus\mu_{0,\varepsilon}(t_0)} \frac{f(t)dt}{t - t_0} = \int\limits_{I^0} \frac{f(t(s))t'(s)ds}{t(s) - t(s_0)},$$

where I^0 is the interval with the ends $s_0 + \varepsilon$, $s_0 + \lambda$. We have to prove that $\lim_{\varepsilon\to 0} I^0(\varepsilon) = 0$. Just as in proving Theorem 1.1, we obtain the estimate

$$|I^0(\varepsilon)| < \frac{|\eta - \lambda|}{k\eta} + \frac{1}{k}|\alpha(\eta)| + \frac{\lambda}{\eta k}|\beta(\eta)|. \tag{1.15}$$

But for a rectifiable curve having a tangent at almost every points, the quantities λ and η as $\varepsilon \to 0$ are equivalent to the chords $|t' - t_0|$ and $|t'' - t_0|$. By virtue of the above-said $|\eta - \lambda|\eta^{-1} \to 0$ and from (1.15) we conclude that $I^0(\varepsilon)$ vanishes. This completes the proof of the theorem. ■

§ 2. Measure Continuity of the Operator S_Γ and Its Consequences

When considering the operator S_Γ in different spaces of summable functions, it is impossible to say something about integral properties of functions $S_\Gamma f$ without some additional assumptions imposed on Γ. For instance, we know examples of rectifiable curves Γ_0 and Γ_1 and function $f_0 \in C(\Gamma_0)$ such that $S_{\Gamma_0} f_0 \overline{\in} L(\Gamma)$ and $S_{\Gamma_1}(1) \overline{\in} \bigcup\limits_{p>0} L^p(\Gamma_1)$ (see [51], [62] and also subsection 3.4).

However one can state that in the case of arbitrary rectifiable curves Γ the operator S_Γ is continuous in measure (see 0.12).

Let $M(\Gamma)$ be a metric space of measurable almost everywhere finite on Γ functions with the metric

$$\rho(\varphi, f) = \int_0^l \frac{|\varphi(t(s)) - f(t(s))|}{1 + |\varphi(t(s)) - f(t(s))|}\, ds.$$

As is known, the convergence of the sequence f_n to f in $M(\Gamma)$ is equivalent to the convergence of f_n to f in measure.

Theorem 2.1. *If Γ is a rectifiable Jordan's curve, $p \geq 1$ and $\frac{1}{w} \in L^{p'}(\Gamma)$, $p' = \frac{p}{p-1}$, then the operator S_Γ is continuous from $L^p(\Gamma; w)$ to $M(\Gamma)$.*

Proof. It follows from the condition $\frac{1}{w} \in L^{p'}(\Gamma)$ that $L^p(\Gamma; w) \subset L(\Gamma)$. For any natural n consider the operator

$$I_n : f \to I_n f, \quad (I_n f)(t) = \int_{\Gamma_{\frac{1}{n}}(t)} \frac{f(\tau)d\tau}{\tau - t}, \quad t \in \Gamma,$$

and show that

$$m_n = \inf_{t \in \Gamma, \tau \in \Gamma_{\frac{1}{n}}(t)} |\tau - t| > 0. \tag{2.1}$$

Indeed, if we assume that $m_n = 0$, then there exists convergent sequences $\sigma^{(k)}$ and $s^{(k)}$ such that

$$|\tau(\sigma^{(k)}) - \tau(s^{(k)})| \to 0, \quad \frac{1}{n} < |\sigma^{(k)} - s^{(k)}| < l - \frac{1}{n}.$$

If the limits of these sequences are equal to σ^* and s^*, then from the latter inequalities we respectively obtain $t(\sigma^*) = t(s^*)$ and $\sigma^* \neq s^*$, which is impossible because Γ is the Jordan curve.

Thus $m_n > 0$. Therefore the operators I_n are continuous from $L^p(\Gamma; w)$ to $M(\Gamma)$. Since for rectifiable curves $D(S_\Gamma) = L(\Gamma)$ (see 0.4), the sequence $I_n f$ for every $f \in L^p(\Gamma; w) \subset L(\Gamma)$ converges to $S_\Gamma f$ almost everywhere, and hence in measure too. This means that the sequence $I_n f$ converges to $S_\Gamma f$ in $M(\Gamma)$. The spaces $L^p(\Gamma; w)$ and $M(\Gamma)$ are of the type F and therefore, owing to the well-known principle (see, e.g., [17]), we conclude that S_Γ is continuous from $L^p(\Gamma; w)$ to $M(\Gamma)$. ■

Theorem 2.2. *If the operator S_Γ maps the space $L^p(\Gamma; w_1)$ into $L^s(\Gamma; w_2)$, $p \geq s \geq 1$, $w_1^{-1} \in L^{p'}(\Gamma)$ then the S_Γ is continuous from $L^p(\Gamma; w_1)$ to $L^s(\Gamma; w_2)$.*

Proof. The continuity of the operator S_Γ from $L^p(\Gamma; w_1)$ to $L^s(\Gamma; w_2)$ is equivalent to that of the operator

$$T : f \to Tf, \quad (Tf)(t) = \frac{w_2(t)}{\pi i} \int_\Gamma \frac{f(\tau)}{w_1(\tau)} \frac{d\tau}{\tau - t}$$

from $L^p(\Gamma)$ to $L^s(\Gamma)$.

Let us show that the operator T is closed from $L^p(\Gamma)$ to $L^s(\Gamma)$, i.e., the assumptions $\|f_n - f\|_p \to 0$, $\|Tf_n - \psi\|_s \to 0$ imply $Tf = \psi$. Indeed, the sequence $f_n w_1^{-1}$ converges in $L(\Gamma)$ to $f w_1^{-1}$. Then by Theorem 2.1, $S_\Gamma(f_n w_1^{-1})$ converges in measure to $S_\Gamma(f w_1^{-1})$. Hence, there exists a sequence n_k such that $w_2 S_\Gamma(f_{n_k} w_1^{-1})$ converges in measure to $w_2 S_\Gamma(f w_1^{-1})$, i.e., to Tf. Consequently, $\psi = Tf$ and hence the operator T is closed from $L^p(\Gamma)$ to $L^s(\Gamma)$. From the closed graph theorem it now follows that the operator T is continuous from $L^p(\Gamma)$ to $L^s(\Gamma)$. ■

§ 3. On the Continuity of the Operator S_Γ in Lebesgue Spaces

There is a vast literature devoted to the questions of continuity of the Cauchy operator in Lebesgue spaces. N.Luzin proved that S_γ is continuous in $L^2(\gamma)$. M. Riesz showed that S_γ is continuous in $L^p(\gamma)$ for every $p > 1$ (M. Riesz's theorem), i.e., $\gamma \in R$. Later it was established that curves of continuous curvature, Lyapunov and piecewise-Lyapunov curves, curves with bounded rotation and so on, belong to the class R ([98], [66], [45], [20]). Substantial progress in this direction has been achieved in a work by A. Calderon, who proved that smooth curves belong to R [9].

This result implies that [28]

$$D(S_\Gamma) = L(\Gamma). \tag{3.1}$$

As is shown in [51]:

If Γ is a closed, rectifiable curve from $R_{p,s}$, $s \geq 1$, then Γ is a Smirnov curve.

Let $\rho > 0$, Γ be a rectifiable curve and $\zeta \in \Gamma$. Denote by $l_\zeta(\rho)$ the linear measure of that part of Γ which falls into the circle with center ζ and radius ρ.

Theorem 3.1 (David [23]). *In order for the curve Γ to belong to the class R, it is necessary and sufficient that condition*

$$\sup_{\zeta \in \Gamma, \rho > 0} l_\zeta(\rho)\rho^{-1} < \infty \tag{3.2}$$

be fulfilled. If it is satisfied, then the operator S_Γ is of weak type $(1,1)$.

The curves satisfying the condition (3.2) are called as regular.

It is easy to see that if S_Γ is of weak type (1,1), then the inequality

$$\left(\int_\gamma |S_\gamma f|^\delta ds \right)^{\frac{1}{\delta}} \leq M_\delta \int_\gamma |f| ds \tag{3.3}$$

holds for any $\delta < 1$. Consequently,

$$\text{if} \quad \Gamma \in R \quad \text{then} \quad \Gamma \in \bigcap_{\delta < 1} R_{1,\delta}. \tag{3.4}$$

Theorem 3.1 provides us with a complete characteristic of curves of the class R. Despite this fact, it is useful to have conditions of belonging of curves to the class R proceeding from their other characteristics. Theorems facilitating the treatment of S_Γ from this point of view are apparently of interest. The more so that in a number of cases they give additional information on this operator. In the remaining part of §3, we will present some of results obtained in this direction. They will be used in Chapter II when studying boundary value problems.

3.1. Classes of curves J and J^*. We will consider the following classes of curves.

Class J_0. A rectifiable Jordan curve Γ with the equation $t = t(s)$, $0 \le s \le l$ belongs to the class J_0 if there exists a smooth Jordan curve μ with the equation $\mu = \mu(s)$, $0 \le s \le l$, such that

$$\operatorname*{ess\,sup}_{0\le\sigma\le l} \int_0^l \left| \frac{t'(\sigma)}{t(\sigma)-t(s)} - \frac{\mu'(\sigma)}{\mu(\sigma)-\mu(s)} \right| d\sigma < \infty. \tag{3.5}$$

Class J.

$$J = J_0 \cap K. \tag{3.6}$$

Class J^*. A Jordan curve $\Gamma \in K$ belongs to the class J^* if it can be divided into a finite number of arcs from the class J with tangents at the ends.

Smooth curves obviously belong to the class J. To the same class belong the curves with bounded rotation not involving cusps (details for the conditions from the definition of J with $\mu(s) = s$ to be fulfilled for such curves see in [21], p. 146–147). By definition, any piecewise smooth curve with no cusps belong to the class J^* , as well as curves of the class K composed of unclosed smooth curves and of curves with bounded variation.

Show that $J_0 \subset R$. Moreover, the following theorem is valid.

Theorem 3.2. *Let $\Gamma \subset J_0$, and*

$$(S^*_\Gamma f)(s) = \sup_\varepsilon \left| \int\limits_{\varepsilon<|s-\sigma|<\frac{l}{2}} \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma)-t(s)} \right|.$$

Then the operator $S^*_\Gamma : f \to S^*_\Gamma f$ is continuous in $L^p(\Gamma)$, $p > 1$.

Since $|(S_{\Gamma_\varepsilon(t(s))} f)(s)| \le |(S^*_\Gamma f)(s)|$ for any $\varepsilon > 0$, and $S_{\Gamma_\varepsilon} f$ converges almost everywhere to $S_\Gamma f$ as $\varepsilon \to 0$ (since $f \in D(S_\Gamma)$), from the inequality $\|S_{\Gamma_\varepsilon} f\|_p \le \|S^*_\Gamma f\|$ and from the assertion of the theorem we obtain that $\forall p > 1$ $\Gamma \in R_p$, and hence $\Gamma \in R$.

Proof of Theorem 3.2. First of all we note that A. P. Calderon has proved in [9] that the operator S^*_μ in the case of smooth curves μ is of strong type

(p,p), $p>1$ and of weak type $(1,1)$. On this basis, we show first that S^*_Γ is of weak type $(1,1)$. We have

$$t'(s)\frac{t'(\sigma)}{t(\sigma)-t(s)}=$$
$$=-\Big(\frac{t'(s)}{t(s)-t(\sigma)}-\frac{\mu'(s)}{\mu(s)-\mu(\sigma)}\Big)t'(\sigma)+\mu'(s)\frac{t'(\sigma)}{\mu'(\sigma)}\frac{\mu'(\sigma)}{\mu(\sigma)-\mu(s)}. \quad (3.7)$$

Since $|t'(s)|=1$, $|\mu'(\sigma)|=1$ almost everywhere on $[0,l]$, from (3.7) it follows that

$$(S^*_\Gamma f)(s)\le\int\limits_0^l\Big|\frac{t'(s)}{t(s)-t(\sigma)}-\frac{\mu'(s)}{\mu(s)-\mu(\sigma)}\Big||f(\sigma)|d\sigma+$$
$$+\sup_{\varepsilon>0}\Big|\int\limits_{\varepsilon<|s-\sigma|<\frac{l}{2}}f(\sigma)\frac{t'(\sigma)}{\mu'(\sigma)}\frac{\mu'(\sigma)}{\mu(\sigma)-\mu(s)}d\sigma\Big|. \quad (3.8)$$

Denote

$$(Nf)(s)=\int\limits_0^l\Big|\frac{t'(s)}{t(s)-t(\sigma)}-\frac{\mu'(s)}{\mu(s)-\mu(\sigma)}\Big||f(\sigma)|d\sigma.$$

The second summand in (3.8) is equal to $S^*_\mu(f\frac{t'}{\mu'})$. Furthermore, we have

$$m\{s:(Nf)(s)>\lambda\}\le\frac{1}{\lambda}\int\limits_0^l\Big(\int\limits_0^l\Big|\frac{t'(s)}{t(s)-t(\sigma)}-\frac{\mu'(s)}{\mu(s)-\mu(\sigma)}\Big||f(\sigma)|d\sigma\Big)ds.$$

Inverting the order of integration on the right-hand side, we get

$$m\{s:(Nf)(s)>\lambda\}\le\frac{1}{\lambda}\int\limits_0^l\Big(\int\limits_0^l\Big|\frac{t'(s)}{t(s)-t(\sigma)}-\frac{\mu'(s)}{\mu(s)-\mu(\sigma)}\Big|ds\Big)|f(\sigma)|d\sigma.$$

The condition (3.5) implies that

$$m\{s:(Nf)(s)>\lambda\}\le\frac{c}{\lambda}\int\limits_0^l|f(\sigma)|d\sigma. \quad (3.9)$$

The latter means that the operator $N:f\to Nf$ is of weak type (1,1). By the above-mentioned Calderon's theorem, we have

$$m\{s:S^*_\mu(f\frac{t'}{\mu'})(s)>\lambda\}\le\frac{c_1}{\lambda}\int\limits_0^l|f(\sigma)|d\sigma. \quad (3.10)$$

On the basis of (3.8)–(3.10), we conclude that S^*_Γ is of weak type (1.1).

Consider the operator $N_1 : f \longrightarrow N_1 f$.

$$(N_1 f)(s) = \sup_{\varepsilon} \Big| \int\limits_{\varepsilon < |s-\sigma| < \frac{l}{2}} \Big(\frac{t'(\sigma)}{t(\sigma) - t(s)} - \frac{\mu'(\sigma)}{\mu(\sigma) - \mu(s)} \Big) f(\sigma) d\sigma \Big|.$$

Obviously,

$$(N_1 f)(s) \leq (S_\Gamma^* f)(s) + (S_\mu^* f)(s).$$

By virtue of the above-proved result, N_1 is also of weak type (1,1). On the other hand, the operator N_1 by (3.5) is of strong type (∞, ∞). Therefore Stein's interpolation theorem (see 0.20) allows one to conclude that N_1 is continuous in $L^p(\Gamma)$ for any $p > 1$. Next, owing to the representation

$$\frac{t'(\sigma)}{t(\sigma) - t(s)} = \Big(\frac{t'(\sigma)}{t(\sigma) - t(s)} - \frac{\mu'(\sigma)}{\mu(\sigma) - \mu(s)} \Big) + \frac{\mu'(\sigma)}{\mu(\sigma) - \mu(s)},$$

we have

$$(S_\Gamma^* f)(s) \leq (N_1 f)(s) + (S_\mu^* f)(s).$$

Since S_μ^* is continuous in the spaces $L^p(\mu)$, it follows that S_Γ^* is continuous in $L^p(\Gamma)$ for arbitrary p, $1 < p < \infty$. ■

Consider now some properties of the curves from the classes J and J^*.

Proposition 3.1. *Let $\Gamma \in J^*$. Then for every point $c \in \Gamma$ there exists an arc $\Gamma_c \subset \Gamma$ such that $c \in \Gamma_c$ and $\Gamma_c \in J$.*

Proof. Let $\Gamma = \cup_{j=1}^n \Gamma_j$, $\Gamma_j = \Gamma_{c_j c_{j+1}} \subset J$. It suffices to verify the validity of the assertion for the points $c = c_j$. Denote by μ_j and μ_{j+1} those smooth curves which satisfy the condition (3.5) for the curves Γ_j and Γ_{j+1}. Since the condition (3.5) along with $\mu(s)$ is also fulfilled for the function $A\mu + B$, by the choice of the variables A and B we can find that μ_j and μ_{j+1} have tangents at the point c_j coinciding with one-sided tangents of Γ_j and Γ_{j+1}. The curve $\mu = \mu_j \cup \mu_{j+1}$ will be piecewise smooth, the smoothness being violated at the point c_j only. It follows from the condition $\Gamma \in K$ that c_j is an angular point (different from the cusp). Moreover, μ may be self-intersecting. Choose the arcs $\mu_j' \subset \mu_j$ and $\mu_{j+1}' \subset \mu_{j+1}$ with the ends at c_j so that they would lie in non-intersecting small angles with the vertex at c_j and one-sided tangents as bisectrices at c_j. The piecewise smooth arc $\mu' = \mu_j' \cup \mu_{j+1}'$ will be a Jordan arc.

Let $\mu = \mu_j(s)$ be the equation of the arc μ_j. Replace the arc μ_j' by δ_j' in whose equation $\delta_j' = a\mu_j + b$ and the constants a and b are chosen in such a way that $\delta_j' \cup \mu_{j+1}'$ is a smooth arc passing through c_j. Let now Γ_j' and Γ_{j+1}' be the arcs respectively on Γ_j and Γ_{j+1} such that $\Gamma_j' \cap \Gamma_{j+1}' = \{c_j\}$ and their length is equal to that of the arcs δ_j' and μ_{j+1}'. The arc $\Gamma_j' \cup \Gamma_{j+1}' = \Gamma_{c_j}$ satisfies the conditions of our assertion.

Moreover, from the proof we can derive the existence of Γ_c with tangents at the ends. ■

Proposition 3.2. *Let the open curves Γ_{ab} and Γ_{bc} of the class J have no common points except b, and let $\Gamma_{ac} \subset K$. Then $\Gamma_{ac} \subset J$.*

Proof. Let $t = t_i(s)$, $0 \leq s \leq l_i$, $i = 1, 2$, be equations with respect to the arc coordinate of the curves Γ_{ab} and Γ_{bc}, respectively. As far as they belong to J, there exist smooth Jordan curves μ_i, $i = 1, 2$, with the equations $\mu = \mu_i(s)$, $0 \leq s \leq l_i$, for which

$$M_i = \operatorname*{ess\,sup}_{0 \leq \sigma \leq l_i} \int_0^{l_i} \left| \frac{t_i'(s)}{t_i(s) - t_i(\sigma)} - \frac{\mu_i'(s)}{\mu_i(s) - \mu_i(\sigma)} \right| ds < \infty. \tag{3.11}$$

Let $\mu = \mu_1 \cup \mu_2$. Since the condition (3.11) is also satisfied by $A\mu_i + B$, without restriction of generality one can assume that μ passes through b and has a tangent at this point, i.e., μ is a smooth curve.

Let

$$t(s) = \begin{cases} t_1(s), & 0 \leq s \leq l_1, \\ t_2(s - l_1), & l_1 < s \leq l_1 + l_2 = l. \end{cases}$$

$$\mu(s) = \begin{cases} \mu_1(s), & 0 \leq s \leq l_1, \\ \mu_2(s - l_1), & l_1 < s \leq l_1 + l_2 = l. \end{cases}$$

Show that the inequality (3.5) is fulfilled for t and μ. As the curve μ is smooth, it follows that $\Gamma_{ac} \in J$.

For the validity of (3.5) it suffices to show that

$$\operatorname*{ess\,sup}_{0 \leq \sigma \leq l} \int_0^{l_1} \left| \frac{t'(s)}{t(s) - t(\sigma)} - \frac{\mu'(s)}{\mu(s) - \mu(\sigma)} \right| ds < \infty, \tag{3.12}$$

and

$$\operatorname*{ess\,sup}_{0 \leq \sigma \leq l} \int_{l_1}^{l_1 + l_2} \left| \frac{t'(s)}{t(s) - t(\sigma)} - \frac{\mu'(s)}{\mu(s) - \mu(\sigma)} \right| ds < \infty. \tag{3.13}$$

Prove the validity of the inequality (3.12). If $0 \leq \sigma \leq l_1$, then the inequality

$$\operatorname*{ess\,sup}_{0 \leq \sigma \leq l_1} \int_0^{l_1} \left| \frac{t'(s)}{t(s) - t(\sigma)} - \frac{\mu'(s)}{\mu(s) - \mu(\sigma)} \right| ds < \infty$$

follows from (3.11). Let now $\sigma > l_1$. We have

$$I = \int_0^{l_1} \left| \frac{t'(s)}{t(s) - t(\sigma)} - \frac{\mu'(s)}{\mu(s) - \mu(\sigma)} \right| ds =$$

$$= \int_0^{l_1} \Big| \frac{t'(s)\int_s^\sigma \mu'(u)du - \mu'(s)\int_s^\sigma t'(u)du}{(t(s)-t(\sigma))(\mu(s)-\mu(\sigma))} \Big| ds.$$

Select a number s_1, $0 < s_1 < l_1$, such that $l_1 - s_1 < \sigma - l_1$. Then

$$I = \int_0^{l_1} \Big| \frac{t'(s)\int_s^{s_1} \mu'(u)du - \mu'(s)\int_s^{s_1} t'(u)du}{(t(s)-t(\sigma))(\mu(s)-\mu(\sigma))} \Big| ds +$$

$$+ \int_0^{l_1} \Big| \frac{t'(s)\int_{s_1}^{l_1} \mu'(u)du - \mu'(s)\int_{s_1}^{l_1} t'(u)du}{(t(s)-t(\sigma))(\mu(s)-\mu(\sigma))} \Big| ds +$$

$$+ \int_0^{l_1} \Big| \frac{t'(u)\int_{l_1}^{\sigma} \mu'(u)du - \mu'(s)\int_{l_1}^{\sigma} t'(u)du}{(t(s)-t(\sigma))(\mu(s)-\mu(\sigma))} \Big| ds = I_1 + I_2 + I_3. \quad (3.14)$$

Since $\Gamma_{ac} \in K$ and the curve μ is smooth, there exists $m > 0$ such that $|t(s) - t(\sigma)| \geq m|s-\sigma|$, $|\mu(s) - \mu(\sigma)| \geq m|s-\sigma|$, $s, \sigma \in (0, l)$. Then

$$I_1 = \int_0^{l_1} \Big| \frac{t'(s)\int_s^{s_1} \mu'(u)du - \mu'(s)\int_s^{s_1} t'(u)du}{(t(s)-t(s_1))(\mu(s)-\mu(s_1))} \frac{t(s)-t(s_1)}{t(s)-t(\sigma)} \frac{\mu(s)-\mu(s_1)}{\mu(s)-\mu(\sigma)} \Big| ds \leq$$

$$\leq \int_0^{l_1} \Big| \frac{t'(s)\int_s^{s_1} \mu'(u)du - \mu'(s)\int_s^{s_1} t'(u)du}{(t(s)-t(s_1))(\mu(s)-\mu(s_1))} \Big| \Big| \frac{s-s_1}{s-\sigma} \Big|^2 \frac{1}{m^2} ds. \quad (3.15)$$

By virtue of our assumption that $l_1 - s_1 < \sigma - l_1$, we have $\sup\limits_{0 \leq s \leq l_1} \left| \frac{s-s_1}{s-\sigma} \right| < 1$, and therefore

$$I_1 \leq M_1 m^{-2}. \quad (3.16)$$

Further,

$$I_2 \leq \int_0^{l_1} \frac{2|s_1 - l_1| ds}{m^2 (s-\sigma)^2} \leq \frac{2|s_1 - l_1|}{m^2} \frac{l_1}{\sigma - l_1} \leq \frac{2 l_1}{m^2}, \quad (3.17)$$

$$I_3 \leq \int_0^{l_1} \frac{2|\sigma - l_1|}{m^2 (s-\sigma)^2} \leq \frac{2}{m^2}. \quad (3.18)$$

The inequality (3.12) follows immediately from (3.14) and (3.16)-(3.18). The inequality (3.13) can be proved analogously. This completes the proof of the proposition. ■

3.2. Continuity of the operator S_Γ in the Lebesgue spaces is equivalent to belonging to Smirnov classes of the Cauchy type integral. First we present some assertions which are concerned with a continuous extension of the operator S_Γ from the set to its closure.

Lemma 3.1. *Let Γ be a closed, rectifiable Jordan curve bounding the finite domain D and let $B(\Gamma)$ be a linear set from $L^s(\Gamma)$, $s \geq 1$, such that for any $\varphi \in B(\Gamma)$ the function $K_\Gamma\varphi$ belongs to $E^p(D)$ and there exists a number $M_{p,s}$ such that*

$$\|S_\Gamma\varphi\|_p \leq M_{p,s}\|\varphi\|_s. \tag{3.19}$$

Then for every φ from the closure of the set $B(\Gamma)$ in the space $L^s(\Gamma)$ (i.e., for $\forall\varphi \in \overline{B_s(\Gamma)}$), we have: (i) $K_\Gamma\varphi \in E^p(D)$; (ii) the inequality (3.19) is valid.

Proof. By (3.19), the operator S_Γ admits a continuous extension up to $\overline{B_s(\Gamma)}$. Denote it by $\widetilde{S_\Gamma}$. Then for every $\varphi \in \overline{B_s(\Gamma)}$ we have

$$\|\widetilde{S}_\Gamma\varphi\| \leq M_{p,s}\|\varphi\|_s. \tag{3.20}$$

Let $\|\varphi_n - \varphi\|_s \to 0$, $\varphi_n \in B(\Gamma)$, $\varphi \in \overline{B_s(\Gamma)}$. Suppose $\phi_n(z) = (K_\Gamma\varphi_n)(z)$. By the assumption of the lemma, $\phi_n \in E^p(D)$, while $\phi_n^+ = \frac{1}{2}\varphi_n + \frac{1}{2}S_\Gamma\varphi_n$, by the Sokhotskiĭ–Plemelj formula. This implies that ϕ_n^+ converges in $L^p(\Gamma)$ to the function $\frac{1}{2}(\varphi + \widetilde{S}_\Gamma\varphi)$. Now, all the assumptions of Theorem (5) from 0.19 in which $e = \Gamma$, $f = \frac{1}{2}(\varphi + \widetilde{S}_\Gamma\varphi)$, may be regarded to be fulfilled for the sequence ϕ_n. According to this theorem, the sequence ϕ_n converges in D to some function $\phi \in E^p(D)$, for which

$$\phi^+ = \frac{1}{2}(\varphi + \widetilde{S}_\Gamma\varphi). \tag{3.21}$$

But the sequence ϕ_n converges in D to the function $\phi(z) = (K_\Gamma\varphi)(z)$ which thus turns out to be a function of the class $E^p(D)$. Hence the assertion (i) of the lemma is valid. Since $(K_\Gamma\varphi)^+ = \frac{1}{2}(\varphi + S_\Gamma\varphi)$, from (3.21) we obtain the equality $S_\Gamma\varphi = \widetilde{S}_\Gamma\varphi$, $\varphi \in \overline{B_s(\Gamma)}$ which proves the assertion (ii) as well. ■

Lemma 3.2. *If a closed curve $\Gamma \in R_p$, $p > 1$, then Γ belongs to $R_{p'}$.*

Proof. Let Q_{z_0} be the set of rational functions of the type

$$\varphi(z) = \sum_{k=0}^{n} a_k(z - z_0)^k + \sum_{k=-1}^{n} a_k(z - z_0)^k = p(z) + q(z), \quad z \in D^+, \tag{3.22}$$

where D^+ is a finite domain bounded by the curve Γ. Then $S_\Gamma\varphi = p - q$, and the equality

$$\int_\Gamma \varphi S_\Gamma \psi dt = -\int_\Gamma \psi S_\Gamma \varphi dt, \quad \varphi, \psi \in Q_{z_0} \tag{3.23}$$

can be easily verified.

On the basis of the equality

$$\|\varphi\|_{p'} = \sup_{\|\psi\|_p \le 1} \left| \int_\Gamma \varphi\psi dt \right|,$$

using (3.23), we obtain

$$\|S_\Gamma\varphi\|_{p'} \le \|S\|_p \|\varphi\|_{p'}, \quad \varphi \in Q_{z_0}.$$

Applying now Lemma 3.1 to the set $B(\Gamma) = Q_{z_0}$, since $\overline{(Q_{z_0})_{p'}} = L^{p'}(\Gamma)$, we conclude that S_Γ is continuous in $L^{p'}(\Gamma)$, i.e. $\Gamma \in R_{p'}$. ■

Theorem 3.3. *If the curve Γ, bounding the finite domain D, belongs to $R_{s,p}$, $s \ge 1$, then for any $\varphi \in L^s(\Gamma)$ the Cauchy type integral $K_\Gamma\varphi$ belongs to $E^p(D)$.*

Indeed, if we take again $B(\Gamma) = Q_{z_0}$, then (3.19) is considered to be fulfilled by the condition $\Gamma \in R_{s,p}$, and since $\overline{(Q_{z_0})_s} = L^s(\Gamma)$, the assertion of the theorem is the consequence of Lemma 3.1.

Corollary. *If $\Gamma \in R$, $\varphi \in L^p(\Gamma)$ then $K_\Gamma\varphi \in E^p(D)$. In particular, if $\varphi \in L^\infty(\Gamma)$, then $K_\Gamma\varphi \in \cap_{p>1} E^p(D^+)$. If $\Gamma \in \cap_{s<1} R_{1,s}$ (in particular, if $\Gamma \in R$ (see (3.4)), then for any $\varphi \in L(\Gamma)$ the Cauchy type integral $K_\Gamma\varphi \in \cap_{s<1} E^s(D)$.*

Let us prove now an analogue of Lemma 3.1 for an arbitrary curve.

Lemma 3.3. *Let Γ be a simple, rectifiable curve and let $B(\Gamma)$ be a linear set from $L^s(\Gamma)$. If*

$$\|S_\Gamma\varphi\|_p \le M_{p,s}\|\varphi\|_s, \quad \varphi \in B(\Gamma), \quad s \ge 1, \tag{3.24}$$

then this inequality is also valid on $\overline{B_s(\Gamma)}$. In particular, if $B(\Gamma)$ is dense everywhere in $L^s(\Gamma)$, then $\Gamma \in R_{s,p}$.

Proof. By (3.24), S_Γ extends on $\overline{B_s(\Gamma)}$ and the extended operator $\widetilde{S}_\Gamma$ is continuous from $\overline{B_s(\Gamma)}$ to $L^p(\Gamma)$. On the other hand, S_Γ is continuous in measure by Theorem 2.1. Let now $\varphi \in \overline{B_s(\Gamma)}$ and $\|\varphi_n - \varphi\|_s \to 0$. Then $S_\Gamma\varphi_n$ converges in $L^p(\Gamma)$ to $\widetilde{S}_\Gamma\varphi$ and to $S_\Gamma\varphi$ in measure. Hence $\widetilde{S}_\Gamma\varphi = S_\Gamma\varphi$, which implies that the assertion of the lemma is valid. ■

Along with Theorem 3.3, we present here one more assertion showing a tight connection between the continuity of the operator S_Γ in the Lebesgue spaces and the belonging to Smirnov classes of the Cauchy type integral.

Theorem 3.4. *Let Γ be a simple, closed curve bounding the domains D^+ and D^-. For the function $\phi = K_\Gamma\varphi$ to belong to the class $E^p(D^+)$ for any $\varphi \in L^s(\Gamma)$, it is necessary and sufficient that the operator S_Γ be continuous from $L^s(\Gamma)$ to $L^p(\Gamma)$. In the case this condition is fulfilled and $p \geq 1$, then $K_\Gamma\varphi$ belongs to $E^p(D^-)$ as well.*

Proof. As for the sufficiency, this theorem is a consequence of Theorem 3.3. The necessity follows from Theorem 2.2, since from the belonging to the class $E^p(D^+)$ of $K_\Gamma\varphi$ for $\varphi \in L^s(\Gamma)$ it follows that S_Γ is defined on the entire $L^s(\Gamma)$ and maps it onto $L^p(\Gamma)$.

Show that if $\Gamma \in R_{s,p}$ and $\phi = K_\Gamma\varphi$, $\varphi \in L^s(\Gamma)$, then ϕ belongs to $E^p(D^-)$. We have

$$S_\Gamma\phi^- = -\frac{1}{2}\,S_\Gamma\varphi + \frac{1}{2}\,S_\Gamma^2\varphi.$$

But from the condition $\phi \in E^p(D^+)$, $p \geq 1$ there follows the equality $S_\Gamma\phi^+ = \phi^+$ [68] which implies that $S^2\varphi = \varphi$. Consequently, $S_\Gamma\phi^- = -\frac{1}{2}(S_\Gamma\varphi - \varphi) = -\phi^-$. Therefore ϕ can be represented by the Cauchy integral in the domain D^-, [68]. Hence $\phi \in E^1(D^-)$. Moreover, $\phi^- L^p(\Gamma)$. From the above-said, according to Smirnov theorem (see 0.19), we can conclude that $\phi \in E^p(D^-)$, since Γ from $R_{s,p}$ is a Smirnov curve. ■

3.3. Connection between the classes R_p and R. For the curves subject to condition the (3.1), the operator S_Γ is bounded in all the spaces $L^p(\Gamma)$, $p > 1$. In the general case we often encounter diverse pictures of continuity violation, among them the curves belonging to $\cap_{q<p} R_{p,q}$ but not to $R_{p,p}(= R_p)$ (see [62] and subsection 3.4). Moreover, there are no curves which belong to R_{p_1} but not to R_{p_2}, $p_1 \neq p_2$, $p_1 > 1$, $p_2 > 1$. Thus the following theorem is valid.

Theorem 3.5. *For any $p > 1$,*

$$R_p = R. \tag{3.25}$$

To prove the theorem, we will need the following

Lemma 3.4. *If an open curve $\Gamma = \Gamma_{ab}$ belongs to $R_{p,s}$ and has tangents at its ends, then there exists a broken line $\delta = \delta_{ba}$ such that $\mu = \Gamma \cup \delta$ is a closed curve of the class $R_{p,s}$.*

Proof. For the normals drawn at the points a and b there exist points a' and b' such that the segments $\overline{aa'}$ and $\overline{bb'}$ do not intersect the curve Γ. Connect the points a' and b' by a broken line δ' which lies wholly in $\mathbb{C}\backslash\Gamma$. Let $\delta = \delta' \cup \overline{aa'} \cup \overline{bb'}$. Then $\mu = \Gamma \cup \delta$ is a closed Jordan curve.

Prove now that there are a neighbourhood of the point a and a constant $k > 0$ such that if we take $t \in \Gamma$ and $t_0 \in \delta$ from this neighbourhood, then

$$|t - t_0| \geq ks(t, t_0). \tag{3.26}$$

Since there exists a tangent at the point a, one can for a given $\varepsilon > 0$ indicate numbers $\rho_1 > 0$, $m > 0$ such that for $|t - a| < \rho_1$ we will have

$$|t - a| \geq ms(t, a), \quad |\arg(t - a) - \arg t'(s_a)| < \varepsilon. \tag{3.27}$$

Suppose $\varepsilon < \alpha < \frac{\pi}{4}$ and draw straight lines μ_1 and μ_2 passing through the point a and forming with the tangent an angle α. One of the straight lines, say μ_1, passes through the angle formed by the tangent and the normal. Obviously, for $|t - a| < \rho_1$ the points of the curve Γ lie in the angle formed by μ_1 and μ_2. Denote by $d_{\delta,\Gamma}$ the least of the diameters of δ and Γ and assume $\rho < \min(|\overline{aa'}|, \rho_1, \frac{1}{4}d_{\delta,\Gamma})$. Draw the circle of radius ρ with the center at a . Let Γ_1 and δ_1 be the portions of the curves contained within the circle (if these sets are unconnected, we take the components containing a). Show that $\Gamma_1 \cup \delta_1$ is a neighbourhood we are seeking for.

Denote orthogonal projections of the points t and t_0 onto a straight line μ_1 by τ and τ_0, respectively. Then

$$|t - t_0| \geq |t - \tau| + |t_0 - \tau_0|, \tag{3.28}$$

$$|t_0 - \tau_0| = |t_0 - a| \cos\alpha = s(t_0, a) \cos\alpha, \tag{3.29}$$

$$|t - \tau| = |t - a| \sin\alpha_t, \tag{3.30}$$

where α_t is the angle lying between the vector at and μ_1.

By (3.27), for $|t-a| < \rho$ we have $\sin\alpha_t \geq \sin\alpha - \varepsilon_0$ for some $\varepsilon_0 \in (0, \sin\alpha)$, and therefore

$$|t - \tau| \geq s(t, a)m_1, \quad m_1 = m(\sin\alpha - \varepsilon). \tag{3.31}$$

Denoting $k = \min(m_1, \cos\alpha)$ and taking into account (3.29) and (3.31), from (3.28) we obtain (3.26).

Just as above we construct a neighbourhood of the point b (on μ) which is a union of arcs $\Gamma_2 \subset \Gamma$, $\delta_2 \subset \delta$ such that if $t \in \Gamma_2$ and $t_0 \in \delta_2$, the inequality (3.26) is valid.

Now we are able to prove that $\mu \in R_{p,s}$.

Let $f \in L^p(\Gamma)$. We have

$$\int\limits_\Gamma |S_\mu f|^p ds \leq$$

$$\leq A\Big[\int\limits_\Gamma |S_\Gamma f|^s ds + \int\limits_\Gamma |S_\delta f|^s ds + \int\limits_\delta |S_\Gamma f|^s ds + \int\limits_\delta |S_\delta f|^s ds\Big]. \tag{3.32}$$

Since Γ and δ belong to $R_{p,s}$, we have

$$\int\limits_{\Gamma} |S_\Gamma f|^s ds \le M_{p,s}^{(1)} \|f\|_p^s, \quad \int\limits_{\delta} |S_\delta f|^\delta ds \le M_{p,s}^{(2)} \|f\|_p^s. \tag{3.33}$$

Estimate the second summand on the right-hand side of (3.32).

Suppose $\Gamma_3 = \Gamma \backslash (\Gamma_1 \cup \Gamma_2)$, $\delta_3 = \delta \backslash (\delta_1 \cup \delta_2)$. Then $\Gamma = \cup_{j=1}^3 \Gamma_j$, $\delta = \cup_{j=1}^3 \delta_j$. Consider the quantities

$$\int\limits_{\Gamma_k} \left| \int\limits_{\delta_i} \frac{f(t)dt}{t-t_0} \right|^s ds_0.$$

If $k=1$, $i=2,3$, then the distance between the sets Γ_k and δ_i is taken to be positive and therefore in this case $|t-t_0| \ge m_2 > 0$. The same estimate is valid for $k=2$, $i=1,3$, and $k=3$, $i=1,2,3$.

In all these cases,

$$\int\limits_{\Gamma_k} \left| \int\limits_{\delta_i} \frac{f(t)dt}{t-t_0} \right|^s ds_0 \le c_{p,s} \|f\|_p^s. \tag{3.34}$$

It remains to consider the cases $k=1$, $i=1$ and $k=3$, $i=3$. The above-proven inequality (3.26) allows one to apply the well-known method of proving convenient for their estimation (see, for e.g., [45]).

We have

$$\int\limits_{\Gamma_1} \left| \int\limits_{\delta_1} \frac{f(t)dt}{t-t_0} \right|^s ds_0 \le$$

$$\le \frac{1}{k^s} \int\limits_{\Gamma_1} \left| \int\limits_{\delta_1} \frac{|f(t(s))|}{s-s_0} ds \right|^{s_0} ds_0 \le \frac{1}{k^s} \int\limits_{-\infty}^{\infty} \left| \int\limits_{-\infty}^{\infty} \frac{f^*(s)}{s-s_0} \right|^s ds_0,$$

where

$$f^*(s) = \begin{cases} |f(t(s))|, & t \in \delta_1, \\ 0, & t \in \Gamma \backslash \delta_1. \end{cases}$$

Using Riesz theorem for $s>1$ and Kolmogorov's inequality for $s<1$ (see (3.2)), we get

$$\left\{ \int\limits_{\Gamma_1} \left| \int\limits_{\delta_1} \frac{f(t)dt}{t-t_0} \right|^s ds \right\}^{1/s} \le c_{p,s} \left\{ \int\limits_{-\infty}^{\infty} |f^*|^{\max(1,s)} ds \right\}^{1/s} \le c_{p,s} \|f\|_p. \tag{3.35}$$

Analogously, we have

$$\int\limits_{\Gamma_3} \left| S_{\delta_3} f \right|^s ds_0 \le c_{p,s} \|f\|_p^s. \tag{3.36}$$

From (3.32)–(3.36) it follows directly that $||S_\mu f||_s \le M_{p,s}||f||_p$, $\mu \in R_{p,s}$. ■

Proof of Theorem 3.5. Let first Γ be a closed curve. For an arbitrary rational function $\varphi \in Q_{z_0}$, $\varphi = p(z) + q(z)$, $z_0 \in D^+$, where D^+, is a domain bounded by the curve Γ, we have $S_\Gamma\varphi = p - q$. Proceeding from the above, the validity of the equality

$$(S_\Gamma\varphi)^2 = -\varphi^2 + 2S_\Gamma(\varphi S_\Gamma\varphi), \quad \varphi \in Q_{z_0}, \tag{3.37}$$

is easily verified.

Since $\Gamma \in R_p$, assuming that $||S_\Gamma||_p = M_p$, we have

$$||S_\Gamma\varphi||_{2p}^2 \le ||\varphi||_{2p}^2 + 2M_p||\varphi S_\Gamma\varphi||_p \le ||\varphi||_{2p}^2 + M_p||\varphi||_{2p}||S_\Gamma\varphi||_{2p},$$

that is,

$$||S_\Gamma\varphi||_{2p}^2 - M_p||\varphi||_{2p}||S_\Gamma\varphi||_{2p} - ||\varphi||_{2p}^2 \le 0.$$

Consequently,

$$||S_\Gamma\varphi||_{2p} \le (M_p + \sqrt{1 + M_p^2})||\varphi||_{2p}, \quad \varphi \in Q_{z_0}. \tag{3.38}$$

According to Lemma 3.1, we conclude that $\Gamma \in R_{2p}$. Applying the Stein interpolation theorem (see 0.20), we establish that $\Gamma \in \cap_{q\ge p}R_q$. Now by Lemma 3.2 we also have $\Gamma \in \cap_{1<q<p'}R_q$, whence it immediately follows that $\Gamma \in R$.

Let now Γ be an open curve. Since in this case we are unable to obtain (3.37) for some dense in $L^p(\Gamma)$ set by direct calculation, we deduce it be means of Lemma 3.4.

Thus, let first $\Gamma = \Gamma_{ab} \in R_p$ and let it have a tangent at the ends. Let also $\mu = \Gamma \cup \delta$ be the curve constructed in Lemma 3.4. Since $\mu \in R$, we can easily obtain from (3.37) the equality

$$(S_\mu\psi)^2 = -\psi^2 + 2S_\mu(\psi S_\mu\psi), \quad \psi \in L^2(\Gamma). \tag{3.39}$$

Let $\varphi \in L^2(\Gamma)$. Taking in (3.39) the functions ψ coinciding with φ on Γ and equal to zero on δ, we obtain for the curves under consideration the desired equality

$$(S_\Gamma\varphi)^2 = -\varphi^2 + 2S_\Gamma(\varphi S_\Gamma\varphi), \quad \varphi \in L^2(\Gamma). \tag{3.40}$$

Let now $\Gamma = \Gamma_{ab}$ be an arbitrary open curve of the class R_p and $\varphi \in Q_{z_0}$. Consider on Γ sequences of points a_n and b_n at which Γ has tangents and which satisfy $a_n \to a$, $b_n \to b$. For $\Gamma_{a_nb_n}$ the equality (3.40) holds. Write it in the form

$$(S_\Gamma\chi_n\varphi)^2 = -(\chi_n\varphi)^2 + 2S_\Gamma(\chi_n\varphi S_\Gamma\chi_n\varphi),$$

where χ_n is the characteristic function of the arc $\Gamma_{a_nb_n}$. Assuming $\varphi_n = \chi_n\varphi$, we write this equality as

$$(S_\Gamma\varphi_n)^2 = -(\varphi_n)^2 + 2S_\Gamma(\varphi_n S_\Gamma\varphi_n). \tag{3.41}$$

Since the sequence φ_n in the space $L^p(\Gamma)$ converges to φ, while $\varphi_n S_\Gamma \varphi_n$, with regard for the inclusion $\Gamma \in R_p$, converges to $\varphi S_\Gamma \varphi$, from (3.41) we obtain (3.40) for $\varphi \in Q_{z_0}$. Having this equality in hand and using Lemma 3.1, we complete the proof just as it has been done in the case of closed curves. ■

3.4. On singular Cauchy integrals on nonregular curves. According to Theorem 3.1, a class of regular curves describes completely the R class. Beyond this set the behaviour of the operators S_Γ is very diverse. Below we shall cite some examples giving an idea of this matter. The criteria for the continuity of the operators S_Γ from $L^p(\Gamma)$ to $L^q(\Gamma)$, $p > q \geq 1$ will also be indicated, when Γ is a countable union of concentric circumferences. The interest to the latter result is due to the fact that it may point to a way which would greatly facilitate the solution of the problem of complete characterization of the class $R_{p,q}$, $p > q$.

1) Non-Smirnov curves. Let Γ be an arbitrary, simple, closed rectifiable non-Smirnov curve bounding a finite domain G. Then, as is known, there exists a function $\phi \in E^1(G)$ such that $\phi^+ \in L^\infty(\Gamma)$ and $\phi \overline{\in} \bigcup\limits_{p>1} E^p(G)$ ([133], p. 258). But the operator S_Γ fails to be continuous even from $L^\infty(\Gamma)$ to $L^p(\Gamma)$ for some $p > 1$. Indeed, if the curve Γ would belong to the class R_{∞,p_0}, $p_0 > 1$, then the function $\phi(z) = (K_\Gamma \phi^+)(z)$, according to Theorem 3.3, would belong to $E^{p_0}(G)$, but it is not the case. Thus, if Γ is the non-Smirnov curve, then

$$\Gamma \overline{\in} \cup_{p>1} R_{\infty,p}. \tag{3.42}$$

2) Example of the curve Γ for which

$$\sup_{\tau \in \Gamma} \frac{s(\tau,t)}{|\tau - t|} < \infty \tag{3.43}$$

for any $t \in \Gamma$, but $\Gamma \overline{\in} R$.

From condition (3.43) we, in particular, find that $\rho^{-1} \sup\limits_{\rho>0} \rho^{-1} l_t(\rho) < \infty$, $t \in \Gamma$. But condition (3.43) is more rigid than the last one. Further, (3.43) is fulfilled for wide subclasses of curves from the class R (curves with bounded ratio of the arc length to the spanning chord and also piecewise smooth curves with cusps). Despite this fact it appears that condition (3.43) fails to guarantee the belonging of the curve to the class R.

Let $\{a_n\}$, $\{\alpha_n\}$ be decreasing sequences of the points tending to zero, provided $n(a_n - \alpha_n) < a_{n+1} - a_n$. Suppose $x_{k,n} = \alpha_n + kn^{-1}(a_n - \alpha_n)$, $k = \overline{1,n}$. Draw through the point α_n a ray which forms with the Ox axis an angle α, $0 < \alpha < \frac{\pi}{2}$, and draw the arcs of the circumference of radius $x_{k,n} - \alpha_n$ with center at α_n which lie between the ray and the Ox axis. Connecting the ends of these arcs alternately on the ray and on the axis, we obtain a continuous arc $\widetilde{\gamma}_n$. Let $\gamma_n = \widetilde{\gamma}_n \cup [a_{n-1}, a_n]$, $\Gamma = \bigcup\limits_{n=1}^{\infty} \gamma_n$. On the basis of the inequality $n(a_n - \alpha_n) < a_{n+1} - a_n$ we can readily verify that

$|\tau - 0|^{-1} S(0,\tau) < 2$, $\tau \in \Gamma$ i.e., condition (3.43) is fulfilled for the point $t = 0$. Analogous inequality for the other points $t \in \Gamma$ is obvious.

Show that $\Gamma \overline{\in} R$. Indeed, from the construction of the curve Γ it follows that

$$(a_n - \alpha_n)^{-1} l_{\alpha_n}(a_n - \alpha_n) > \alpha \sum_{k=1}^{n} \frac{1}{k},$$

that is, $\sup\limits_{t\in\Gamma,\,\rho>0} \rho^{-1} l_t(\rho) = \infty$, and hence $\Gamma \overline{\in} R$.

3) Example of the curve Γ for which

$$(S_\Gamma 1)(t) \overline{\in} \cup_{p>0} L^p(\Gamma). \tag{3.44}$$

Let $\{\delta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with the condition

$$\delta_n > \delta_{n+1}, \quad n \geq 2, \quad \sum_{n=2}^{\infty} \delta_n < \infty,$$

$$\delta_0 = \sum_{k=1}^{\infty} (-1)^{k-1} \delta_{2k}, \quad \delta_1 = 1 - \sum_{k=1}^{\infty} (-1)^k \delta_{2k+1}. \tag{3.45}$$

Suppose $t_n = (x_n, y_n)$, $n = 0, 1, 2, \ldots$ where

$$x_{2n} = \sum_{k=n}^{\infty} (-1)^{k-1} \delta_{2k+1}, \quad x_{2n+1} = x_{2n},$$

$$y_0 = 0, \quad y_{2n-1} = \sum_{k=n}^{\infty} (-1)^{k-1} \delta_{2k}, \quad n = 1, 2, \ldots, \quad y_{2n} = y_{2n-1}.$$

Let further $\Delta_n = [t_n, t_{n+1}]$, $n = 0, 1, \ldots$, and $\Gamma = (\bigcup\limits_{n=0}^{\infty} \Delta_n) \cup \{0\}$.

Direct calculation gives

$$|(S_\Gamma 1)(t)|^p \geq \left| \operatorname{Im} \int_\Gamma \frac{d\tau}{\tau - t} \right|^p = |\arg(t(s)) - \arg(1 - t(s)) + n_0 \pi|^p, \tag{3.46}$$

where $p > 0$, and the continuous branches of $\arg t(s)$ and $\arg(1 - t(s))$ and the number n_0 are chosen such that $\arg t(0) = 0$.

Let $s_n = \sum_{k=0}^{n} \delta_k$, $k = 0, 1, 2, \ldots$. Since $\arg t(s_0)$ is the angle described by the vector $\overrightarrow{0t(s)}$ upon variation of the parameter s from 0 to s_0, it can be easily verified that for $s \in [s_0, s_{n+1})$ we have $n\frac{\pi}{2} < \arg t(s) < (n+1)\frac{\pi}{2}$. Consequently,

$$\int_\Gamma |\arg t(s)|^p\, ds \geq \sum_{n=0}^{\infty} \int_{s_n}^{s_n+1} |\arg t(s)|^p\, ds \geq \left(\frac{\pi}{2}\right)^p \sum_{n=0}^{\infty} n^p \delta_{n+1}. \tag{3.47}$$

As far as $\arg(1-t(s))$ is the function bounded on Γ, taking $\delta_n = n^{-1}\ln^2 n$, $n = 2, 3, \dots$ from (3.46) and (3.47), we find that $(S_\Gamma 1)(t) \overline{\in} \bigcup_{p>0} L^p(\Gamma)$ for the corresponding curve.

4) Example of the curve Γ with the properties

$$\Gamma \in \bigcap_{0<\varepsilon<p} R_{p,p-\varepsilon} \quad \text{for any} \quad p, \quad 1 < p < \infty \quad \text{and} \quad \Gamma \overline{\in} R. \tag{3.48}$$

Let $\varepsilon_n = \exp(-\sqrt{n})$, $n = 1, 2, \dots$ and let Γ be a broken line consisting of segments $\Delta_n = [-\varepsilon_n, \varepsilon_n + i\varepsilon_n]$, $\widetilde{\Delta}_n = [-\varepsilon_n + i\varepsilon_n, \varepsilon_{n+1}]$, $(n = 0, 1, \dots)$ and $[0, 1]$, i.e., $\Gamma = \bigcup_{n=0}^{\infty} (\Delta_n \cup \widetilde{\Delta}_n) \cup [0, 1]$. This curve possesses the property (3.48) (see [62], Theorem 4).

5) Operator S_Γ for Γ being a countable family of concentric circumferences. Let $\{r_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers satisfying the condition $\sum_{k=0}^{\infty} r_k < \infty$, and let Γ be a family of concentric circumferences $\Gamma_n = \{z : |z| = r_n\}$. Consider the following functions connected with Γ.

Let $t \in \overline{\Gamma}$, $\rho > 0$ and $l_t(\rho)$ be the length of that part of Γ which gets into the circle of radius ρ and center at the point t. Let $V_\rho(t)$ be the variation of the function $\arg(\tau - t)$ on $\Gamma_\rho(t)$, where $\Gamma_\rho(t) = \Gamma \cap \{\tau : |\tau - t| > \rho\}$. Put

$$D(t) = \sup_{\rho>0} l_t(\rho)\rho^{-1}, \quad V(t) = \lim_{\rho\to 0} V_\rho(t).$$

Theorem 3.6. *Let $1 \le q < p \le \infty$ and*

$$\sigma = \sigma(p, q) = \begin{cases} pq(p-q)^{-1}, & p < \infty, \\ q, & p = \infty. \end{cases}$$

Then the following statements are equivalent:

(i) *the operator S_Γ is bounded from $L^p(\Gamma)$ to $L^q(\Gamma)$;*

(ii) $\sum\limits_{n=1}^{\infty} \Big(\sum\limits_{k=n}^{\infty} r_k r_n^{-1} \Big)^{\sigma} r_n < \infty$;

(iii) $\sum\limits_{n=1}^{\infty} n^\sigma r_n < \infty$; (iv) $D \in L^\sigma(\Gamma)$; (v) $V \in L^\sigma(\Gamma)$.

The proof of the equivalence of conditions (i)–(iii) can be found in [64]. The remaining statements of the theorem are proved analogously.

It has been shown [64], [127] that in order for the operator S_Γ to be bounded in $L^p(\Gamma)$, $p > 1$ it is necessary and sufficient that the condition

$$\sum_{k=n}^{\infty} r_k \le C r_n, \quad n = 1, 2, \dots, \tag{3.49}$$

be fulfilled, where C is an absolute constant.

A family of concentric circumferences "simulates" principally rectifiable curves with isolated singularities. Taking into account the above-said, we assume that for an arbitrary rectifiable curve the following statement is valid: $\Gamma \in R_{p,q}$, $1 \le q < p \le \infty$, iff $D(t)$ belongs to $L^\sigma(\Gamma)$ (an analogue of Theorem 3.6).

In favour of such an assumption speaks the fact that condition (3.49) for $p = q$ is analogous to Γ David's condition. Its correctness is partially proved by the following

Proposition 3.3 ([127]). *Let Γ be a simple, closed, rectifiable curve. Then the following statements are valid:*

(i) *if $\Gamma \in R_{p,q}$, $1 \le q < p < 2$ or $2 < q < p < \infty$, then $D \in L^{\sigma-\varepsilon}(\Gamma)$ for arbitrary $\varepsilon \in (0, \sigma)$;*

(ii) *if S_Γ is continuous from $L^\infty(\Gamma)$ to $L^q(\Gamma)$, $q > 0$ then $D \in L^q(\Gamma)$.*

§ 4. On the Continuity of the Operator S_Γ in Weighted Lebesgue Spaces

J. Hardy and Y. E. Littlewood [50] were the first who established the boundedness of the conjugacy operator (Hilbert transform) in the spaces $L^p(\Gamma; w)$, $(1 < p < \infty)$, with a power weight w. Later on, various proofs of the above-mentioned result were proposed by other authors. The theorem below has been proved independently.

Theorem (Khvedelidze [66]). *Operator S_Γ is bounded in the space $L^p(\Gamma, w)$, $1 < p < \infty$, where Γ is the Lyapunov curve, and*

$$w(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k}, \quad t_k \in \Gamma, \quad -\frac{1}{p} < \alpha_k < \frac{1}{p}. \tag{4.1}$$

A full description of weights w ensuring the boundedness of the conjugacy operator in $L^2(\gamma; w)$ has been obtained by Helson and Szegö.

Theorem (Helson, Szegö [52]). *In order that $w \in W_2(\gamma)$, it is necessary and sufficient that it be representable in the form*

$$w(x) = e^{u(x)+\widetilde{v}(x)}, \tag{4.2}$$

where u and v are real bounded functions with $\|v\|_\infty < \frac{\pi}{4}$.

The last condition is equivalent to the condition of sufficiency found earlier by V. F. Gaposhkin [39].

Some subsets of weight functions of the class $W_p(\Gamma)$ with singularities distributed over the entire curve were found by I. B. Simonenko [141], I. I. Danilyuk [19] and I. I. Danilyuk and V. Yu. Shelepov [20]. The first of the above-mentioned authors has obtained his result by solving the boundary value problem of linear conjugation.

A complete solution of a one-weight problem for conjugate in $L^p(\gamma, w)$ $(1 < p < \infty)$ functions is given by the following assertion.

Theorem (Hunt, Muckenhoupt and Wheeden [53]). *A 2π-periodic function $w \in W_p(\gamma)$ $(1 < p < \infty)$ if and only if*

$$\|w\|_{L^p(I)} \|\frac{1}{w}\|_{L^{p'}(I)} \leq c|I|, \tag{4.3}$$

where I is an arbitrary interval of length $|I| < 2\pi$, and the constant c does not depend on I.

An analogue of the Helson and Szegö criterion for arbitrary $L^p(\gamma, w)$ was first obtained in [54].

Theorem (Jones [54]). *A function w belongs to $W_p(\Gamma)$ if and only if*

$$w = e^{u+\widetilde{v}_1 - \frac{1}{p'}\widetilde{v}_2}, \tag{4.4}$$

where

$$u \in L^\infty, \ \ \operatorname{Im} v_j = 0, \ \ \|v_j\| \leq \frac{\pi}{2p} \ \text{ and } \ \widetilde{v}_j \in BLO, \ \ j = 1, 2. \tag{4.5}$$

(for definition of the class BLO see [28], p. 279).

4.1. On the functions from W_p allowing one to construct weights from W_{2p}. The main results of this section are Theorem 4.1 and its Corollaries 3 and 5. Let us start with formulation of two simple lemmas.

Lemma 4.1. *An operator S_Γ is continuous from $L^p(\Gamma, w)$ to $L^p(\Gamma, w)$ if and only if the operator*

$$T : \varphi \to T_\varphi, \quad (T_\varphi)(t) = \frac{w(t)}{\pi i} \int\limits_\Gamma \frac{\varphi(\tau)}{w(\tau)} \frac{d\tau}{\tau - t} \tag{4.6}$$

is continuous from $L^p(\Gamma)$ to $L^p(\Gamma)$.

Lemma 4.2. *If $w \in W_p(\Gamma)$, then $w \in L^p(\Gamma)$, $\frac{1}{w} \in L^{p'}(\Gamma)$ and $\frac{1}{w} \in W_{p'}(\Gamma)$. Moreover, $\Gamma \in R$.*

Proof. Show first that $\Gamma \in R$. The use will be made of Stein's theorem (0.20). Suppose in this theorem $M = S_\Gamma$, $r_1 = s_1 = p$, $r_2 = s_2 = p'$, $t = \frac{1}{2}$, $k_1 = w$, $k_2 = \frac{1}{w}$. Then $s = r = 2$, $k = u = 1$ and hence S_Γ is continuous in $L^2(\Gamma)$. By Theorem 3.5, $\Gamma \in R$. The proof of the first part of the above theorem when Γ is a straight line or a Lyapunov curve, is given in [168] and [141], respectively. Word for word these proofs can be applied to the general case by using the Riesz equality (which is valid, since $\Gamma \in R$) and the fact that if S_Γ is defined on $L^p(\Gamma, w)$, then we must necessarily have $L^p(\Gamma, w) \subset L(\Gamma)$. ■

Theorem 4.1. *Let Γ be a rectifiable Jordan curve. If $w^2 \in W_p(\Gamma)$, $1 < p < \infty$, and $(wS_\Gamma \frac{1}{\omega}) \in L^p(\Gamma)$, then $w \in W_{2p}(\Gamma)$.*

Proof. Note that by Lemma 4.2, $\Gamma \in R$. Since, according to the same lemma, as $w^{-2} \in W_{p'}(\Gamma)$, we have $\frac{1}{w} \in L^{2p'}(\Gamma)$. Let now $\varphi \in Q_{z_0}$, $z_0 \overline{\in} \Gamma$. Then $\frac{\varphi}{w} \in L^{2p'}(\Gamma) \subset L^2(\Gamma)$. Since $\Gamma \in R$, we can apply to this function the formula (3.40). Therefore

$$\left[\frac{w(t)}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)}{w(\tau)}\frac{d\tau}{\tau - t}\right]^2 = w^2(S_\Gamma\frac{\varphi}{w})^2 =$$

$$= w^2\Big\{-\frac{\varphi^2}{w^2} + 2S_\Gamma\big[\frac{\varphi}{w}S_\Gamma(\frac{\varphi}{w})\big]\Big\}. \tag{4.7}$$

If we suppose that $T_\varphi = wS_\Gamma\frac{\varphi}{w}$, then (4.7) yields

$$(T_\varphi)^2 = -\varphi^2 + 2w^2 S_\Gamma(\frac{\varphi}{w^2}T_\varphi). \tag{4.8}$$

Show that $T_\varphi \in L^p(\Gamma)$. Indeed, we have

$$(T\varphi)(t) = \frac{w(t)}{\pi i}\int\limits_{\Gamma}\frac{\varphi(\tau)-\varphi(t)}{w(\tau)(\tau - t)}d\tau + \frac{\varphi(t)w(t)}{\pi i}\int\limits_{\Gamma}\frac{d\tau}{w(\tau)(\tau - t)}. \tag{4.9}$$

Next, the condition $w^2 \in W_p(\Gamma)$ implies that $w^2 \in L^p(\Gamma)$, i.e, $w \in L^{2p}(\Gamma) \subset L^p(\Gamma)$, and since $\varphi \in Q_{z_0}$, the first summand on the right-hand side of (4.9) belongs to $L^p(\Gamma)$, while the second one belongs to $L^p(\Gamma)$, by the assumption $(wS_\Gamma\frac{1}{w}) \in L^p(\Gamma)$.

Thus $T_\varphi \in L^p(\Gamma)$. Consequently, $\varphi T_\varphi \in L^p(\Gamma)$. This, according to the condition $w^2 \in W_p(\Gamma)$, results in the inclusion $[w^2 S_\Gamma(\frac{1}{w^2}T_\varphi)] \in L^p(\Gamma)$. From the equality (4.8) for $\varphi \in Q_{z_0}$ we conclude now that $T_\varphi \in L^{2p}(\Gamma)$.

Moreover, on the basis of (4.8) we have

$$||T_\varphi||_{2p}^2 \leq \Big(\int\limits_{\Gamma}|\varphi|^{2p}d\sigma\Big)^{\frac{1}{p}} + 2\Big(\int\limits_{\Gamma}\Big|w^2 S_\Gamma\Big(\frac{1}{w^2}\varphi T_\varphi\Big)\Big|^p d\sigma\Big)^{\frac{1}{p}}. \tag{4.10}$$

The last summand is calculated with regard for the condition $w^2 \in W_p(\Gamma)$.

$$||T_\varphi||_{2p}^2 \leq ||\varphi||_{2p}^2 + 2A_p\Big(\int\limits_{\Gamma}|\varphi T_\varphi|^p d\sigma\Big)^{\frac{1}{p}} \leq ||\varphi||_{2p}^2 +$$

$$+2A_p\Big(\int\limits_{\Gamma}|\varphi|^{2p}d\sigma\Big)^{\frac{1}{2p}}\Big(\int\limits_{\Gamma}|T_\varphi|^{2p}d\sigma\Big)^{\frac{1}{2p}}, \quad A_p = ||S||_{p,w^2}, \tag{4.11}$$

and hence

$$||T_\varphi||_{2p} \leq \big(A_p + \sqrt{A_p^2 + 1}\big)||\varphi||_{2p} = A_{2p}||\varphi||_{2p}. \tag{4.12}$$

The obtained relation (4.12) is valid for rational functions only.

Show that it is valid for arbitrary $\varphi \in L^{2p}(\Gamma)$. Let $\varphi_0 \in L^{2p}(\Gamma)$ and let φ_n be a sequence of functions from Q_{z_0} such that $\|\varphi_n - \varphi_0\|_{2p} \to 0$, $\|\varphi_n\|_{2p} \le \|\varphi_0\|_{2p}$. By (4.12),

$$\|T_{\varphi_n}\|_{2p} \le A_{2p}\|\varphi_n\|_{2p} \le A_{2p}\|\varphi_0\|_{2p}. \tag{4.13}$$

Since $\frac{1}{w} \in L^{2p'}(\Gamma)$, $\frac{\varphi_n}{w}$ converges in $L(\Gamma)$ to the function $\frac{\varphi_0}{w}$. By Theorem 2.1, the sequence $\left(S_\Gamma \frac{\varphi_n}{w}\right)$ converges in measure to $S_\Gamma \frac{\varphi_0}{w}$. Therefore there exists a sequence n_k such that $S_\Gamma \frac{\varphi_{n_k}}{w}$ converges almost everywhere on Γ to $S_\Gamma \frac{\varphi_0}{w}$. But then $T\varphi_0 = wS_\Gamma \frac{\varphi_0}{w}$. Moreover, the inequality (4.13) is fulfilled for $T\varphi_{n_k}$. By the Fatou theorem,

$$\int\limits_\Gamma |T\varphi_0|^{2p} ds \le \overline{\lim}_{k\to\infty} \int\limits_\Gamma |T\varphi_{n_k}|^{2p} ds \le$$
$$\le A_{2p} \int\limits_\Gamma |\varphi_{n_k}|^{2p} ds \le A_{2p} \int\limits_\Gamma |\varphi_0|^{2p} ds \tag{4.14}$$

which implies that $w \in W_{2p}(\Gamma)$. ■

Remark. As the power function shows, the inclusion $w^2 \in W_{2p}(\Gamma)$ does not, generally speaking, follow from the condition $w \in W_p(\Gamma)$. On the other hand, it follows from the condition $w \in W_{2p}(\Gamma)$ that $(wS_\Gamma \frac{\varphi}{w}) \in L^{2p}(\Gamma)$ if $\varphi \in L^{2p}(\Gamma)$. Therefore, assuming $\varphi \equiv 1$, we obtain $(wS_\Gamma \frac{1}{w}) \in L^{2p}(\Gamma) \subset L^p(\Gamma)$. Thus the assumption of the theorem that the function $wS_\Gamma \frac{1}{w}$ belongs to the class $L^p(\Gamma)$ is a necessary one.

Corollary 1. *If $w^2 \in W_p(\Gamma)$, $(wS_\Gamma \frac{1}{w}) \in L^p(\Gamma)$, $p < r < 2p$ then, $w^{\frac{2p}{r}} \in W_r(\Gamma)$.*

Proof. Bearing in mind Theorem from 0.20, we assume that $M = S_\Gamma$, $r_1 = s_1 = p_1$, $r_2 = s_2 = 2p$, $k_1 = u_1 = w^2$, $k_2 = u_2 = w$. By Lemma 4.2 and from the assumptions of the corollary, this theorem is applicable and for $t = 2 - \frac{2p}{r}$ we have $k = u = w^{2(1-t)}w^t = w^{2-t} = w^{\frac{2p}{r}}$. Thus the inequality

$$\left\| w^{\frac{2p}{r}} S_\Gamma \frac{\varphi}{w^{\frac{2p}{r}}} \right\|_r \le M\|\varphi w^{\frac{2p}{r}}\|_r \tag{4.15}$$

is valid. ■

Corollary 2. *If $w \in W_p(\gamma)$, where $1 < p < 2$, then $w = \exp(u + \widetilde{v})$, where u and v are bounded real functions, and $\|v\|_\infty < \frac{\pi}{2p}$.*

Proof. By Corollary 1, we have $\sqrt{w}^{\frac{2p}{r}} \in W_r(\gamma)$. Since $2 \in [p, 2p]$, we can take $r = 2$. Then $\sqrt{w}^p \in W_2(\gamma)$. According to Helson-Szegö's theorem, $w^{\frac{p}{2}} = \exp(u_1 + \widetilde{v}_1)$, where u_1 and v_1 are bounded real functions, and $\|v_1\|_\infty < \frac{\pi}{4}$. But then $w = \exp(\frac{2u_1}{p} + (\widetilde{\frac{2v_1}{p}})) = \exp(u + \widetilde{v})$, $u = \frac{2u_1}{p}$, $v = \frac{2v_1}{p}$. Clearly, $\|v\|_\infty < \frac{\pi}{2p}$. ■

Corollary 3. *The function of the kind* $w = \exp(u + \widetilde{v})$*, where* u *and* v *are the bounded real functions and* $\|v\|_\infty < \frac{\pi}{2\max(p,p')}$*, belongs to* $W_p(\gamma)$*.*

Proof. (i) Let first $p = 4$, i.e. $w = \exp(u + \widetilde{v})$, $\|v\|_\infty < \frac{\pi}{8}$. Since $w^2 = \exp(2u + 2\widetilde{v})$, $\|2\widetilde{v}\|_\infty < \frac{\pi}{4}$, by Helson-Szegö's theorem, $w^2 \in W_2(\gamma)$. Moreover, as w^2 and $\frac{1}{w^2}$ belong to $L^2(\gamma)$, then $wS_\Gamma w^{-1} \in L(\gamma)$. Therefore, owing to the theorem, we conclude that $w \in W_4(\Gamma)$.

(ii) Let $2 < p < 4$ and $w = \exp(u + \widetilde{v})$, $\|v\|_\infty < \frac{\pi}{2p}$. The function $w^{\frac{p}{4}}$, because of (i), belongs to $W_4(\gamma)$. Assuming $p = 4$, $r = p$ in Corollary 1, we find that $w \in W_p(\gamma)$.

Hence, Corollary 3 is valid for $p \in [2, 4]$. Repeating the above reasoning, by induction we prove that the corollary is valid when $p \in [2, \infty]$.

(iii) If $1 < p < 2$, then since $\|v\| < \frac{\pi}{2\max(p,p')}$, we have that $\frac{1}{w} \in W_{p'}(\gamma)$, and by virtue of Lemma 4.2, $w \in W_p(\Gamma)$. ■

Corollary 4. *If the operator* T *is bounded with respect to the norm of the space* $L^p(\Gamma)$ *for rational functions from* Q_{z_0}*, then it is bounded in* $L^p(\Gamma)$*.*

It suffices to see that part of the proof of Theorem 4.1 which implies (4.12).

Corollary 5. *If* $w \in W_p(\gamma)$*, then* $w = \exp(u + \widetilde{v})$*, where* u *and* v *are the bounded real functions, and* $\|v\|_\infty < \frac{\pi}{2\min(p,p')}$*.*

Proof. If $1 < p < 2$, then using Corollary 2, we obtain $w = \exp(u + \widetilde{v})$, where $\|v\|_\infty < \frac{\pi}{2p}$. If, however, $p \geq 2$, then applying again Corollary 2, but now to the function $\frac{1}{w} \in W_{p'}(\gamma)$, we obtain $w = \exp(u + \widetilde{v})$, where this time $\|v\|_\infty < \frac{\pi}{2p'}$. ■

In subsection 4.5 we will prove that the result of Corollary 5 remains also valid in the case of Lyapunov curves.

4.2. Criteria of boundedness of S_Γ in $L^p(\Gamma, w)$ for regular curves Γ. In this section we assume that Jordan curve Γ is regular that is,

$$\nu\Gamma(z, r) \leq cr, \quad z \in \Gamma,$$

where $\Gamma(z, r) = B(z, r) \cap \Gamma$, $B(z, r)$ is a circle with center in $z \in \Gamma$ and of radius r, ν is an arc length measure on Γ. As it was noted in §3, the class of regular curves completely describes the class R.

Introduce the following notation:

$$M_\Gamma f(t) = \sup_{\substack{\Gamma(z,r) \ni t \\ 0 < r < \operatorname{diam} \Gamma}} \frac{1}{\nu\Gamma(z, r)} \int\limits_{\Gamma(z,r)} |f(\tau)| d\nu,$$

$$M_\Gamma^{(p)}(t) = (M_\Gamma |f|^p)^{\frac{1}{p}}(t),$$

$$f^{\#}(t) = \sup_{\Gamma(z,r) \ni t} \frac{1}{\nu\Gamma(z, r)} \int\limits_{\Gamma(z,r)} |f(\tau) - f(t)| d\nu,$$

$$f_E = \frac{1}{\nu E} \int_E f(t)\, d\nu,$$

$$wE = \int_E w(t) d\nu \quad \text{for any measurable} \quad E \subset \Gamma.$$

Definition 1. A measurable non-negative function $w \in A_\infty(\Gamma)$ if there exists $\delta > 0$ such that

$$\int_E w(t) d\nu \le c \Big(\frac{\nu E}{\nu \Gamma(z,r)} \Big)^\delta \int_{\Gamma(z,r)} w(t) d\nu$$

for every $\Gamma(z,r)$ and measurable $E \subset \Gamma(z,r)$.

Definition 2. A weight function $w \in A_p(\Gamma)$, $1 < p < \infty$, if

$$\sup_{\substack{z \in \Gamma \\ 0 < r < \operatorname{diam} \Gamma}} \frac{1}{\nu \Gamma(z,r)} \int_{\Gamma(z,r)} w(t) d\nu \Big(\frac{1}{\nu \Gamma(z,r)} \int_{\Gamma(z,r)} w^{1-p'}(t) dt \Big)^{p-1} < \infty.$$

In the sequel, we will need two facts: (i) if $w \in A_p(\Gamma)$ for some $p > 1$, then $w \in A_\infty(\Gamma)$; (ii) the class $A_p(\Gamma)$ is open, i.e. there exists some $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}(\Gamma)$ and $w \in A_{p_1}(\Gamma)$ for arbitrary $p_1 > p$.

As far as a regular curve is one of examples of a homogeneous type space, the above-mentioned properties of $A_p(\Gamma)$ as well as the proof of the propositions below can be found in [40], Chapters 1, 5 and 7.

Proposition A. *Let $1 < p < \infty$, $\Gamma \in R$ and $w \in A_p(\Gamma)$. Then*

$$\int_\Gamma (M_\Gamma f(t))^p w(t) d\nu \le c \int_\Gamma |f(t)|^p w(t) d\nu \tag{4.16}$$

where c does not depend on f.

The principle of the proof is well known (cf., e.g., [153], p.3). It is based on the following covering

Lemma 4.3. *Let Γ be a regular curve and let $E \subset \Gamma$ be a bounded set. Assume that every point $t \in E$ is endowed with a positive number $r(t)$. Then there exists not more than a countable set of points $t_j \in E$ such that $\Gamma_j = \Gamma(t_j, r(t_i))$ are mutually disjoint and $E \subset \bigcup_j \Gamma_j$.*

Proof. Assume that $\sup\{r(t), t \in E\} < \infty$. Otherwise, there exists a point $t \in E$ such that $E \subset B(t, r(t)) \cap \Gamma$, which proves the lemma.

Let us take the point $t_1 \in E$ such that $r(t_1) > \sup\{r(t), t \in E\}$ and suppose that the points $t_1, t_2, \dots, t_{n-1}$ are already chosen. Now we select a point $t_n \in E_n = E \setminus \bigcup_{j=1}^{n-1} B(t_j, 3r(t_j))$ with the condition

$$r(t_n) > \frac{1}{2} \sup\{r(t) : t \in E_n\}.$$

It is clear that $\Gamma(t_n, r(t_n)) \cap B(t_j, r(t_j)) = \varnothing$ for $j < n$. Otherwise,

$$|t_j - t_n| \leq 3r(t_j).$$

Indeed, for $j < n$ we have $r(t_j) > \frac{1}{2}\sup\{r(t) : t \in E_j\} \geq \frac{1}{2}r(t_n)$, and if $\tau \in B(t_n, r(t_n)) \cap B(t_j, r(t_j)) \cap \Gamma$, then

$$|t_j - t_n| \leq |t_j - \tau| + |\tau - t_n| \leq r(t_j) + r(t_n) \leq 3r(t_j).$$

The latter contradicts the fact that $t_n \notin B(t_j, 3r(t_j))$, $j < n$. Hence $\Gamma(t_n, r(t_n)) \cap B(t_j, r(t_j)) = \varnothing$ for $j < n$.

Thus there may occur two cases:

(1) If after our choice of a finite number of points t_j, $j = 1, \dots, n$ we find that $E \backslash \bigcup\limits_{j=1}^{n} B(t_j, 3r(t_j)) = \varnothing$, then the set $\bigcup\limits_{j=1}^{n} B(t_j, 3r(t_j))$ covers the set E.

(2) If this process continues infinitely, then we will have $\lim\limits_{n\to\infty} r(t_n) = 0$. Indeed, let for some $\varepsilon > 0$ and for the subsequence $(t_{i_k})_k$ of (t_j) we have $r(ti_k) > \varepsilon$. On the other hand, these points belong to the bounded set E, and therefore they may find themselves in some ball B. Moreover, the sets $\Gamma(t_j, r(t_j))$ are mutually disjoint, and therefore

$$s(t_{i_k}, t_{i_m}) \geq \min_{k,m} s(t_{i_k}, t_{i_m}) > \varepsilon > 0.$$

This means that a portion of Γ, having an infinite length, appears in the ball B. But this contradicts the regularity of Γ.

Assume now that there exists a point $t \in E$ such that $t \in E \backslash \bigcup\limits_{j=1}^{n} B(t_j, 3r(t_j))$. Then there exists n_0 such that $r(t) > 2r(t_{n_0})$. On the other hand, $r(t) \leq \sup\{r(t) : t \in E_{n_0}\} < 2r(t_{n_0})$, and we conclude that $E \backslash \bigcup\limits_{j=1}^{n} B(t_j, 3r(t_j)) = \varnothing$. ■

Along with the above lemma we will also need the Whitney type covering lemma.

Lemma 4.4. *Let Γ be a regular curve, $E \subset \Gamma$, $E \neq \Gamma$ be a bounded, open set in the sense of the topology of Γ and the number $c \geq 1$. Then there exists a set of balls $\{B_j\}_j = \{B(t_j, R_j)\}$, $t_j \in E$ such that the following conditions are fulfilled:*

(a) $E = \bigcup\limits_j \Gamma_j$, $\Gamma_j = B_j \cap \Gamma$;

(b) *there exists a positive number $\eta = \eta(c)$ such that every point $t \in \Gamma$ belongs at least to η balls $\widetilde{B}_j = \widetilde{B}(t_j, cR_j)$;*

(c) $\widetilde{\widetilde{B}}_j \cap (\Gamma \backslash E) \neq \varnothing$ *for every j, where $\widetilde{\widetilde{B}}_j = B(t_j, 3cR_j)$.*

Proof. Let $t \in E$ and $r(t) = \frac{1}{6c} d(t, \Gamma \backslash E) = \frac{1}{6c}\inf\{|t - \tau| : \tau \notin E\} > 0$. By virtue of Lemma 4.3, there exists a set of balls $\{B(t_j, r(t_j))\}$ such that the sets $\Gamma(t_j, r(t_j))$ are mutually disjoint, and $E \subset \bigcup\limits_j B(t_j, 3r(t_j))$.

For $\tau \in \Gamma(t_j, 3cr(t_j))$ we have

$$|t_j - \tau| < 3cr(t_j) = 3c\frac{1}{6c}d(t_j, \Gamma\backslash E) = \frac{1}{2}d(t_j, \Gamma\backslash E) < d(t_j, \Gamma\backslash E).$$

Let $R_j = 3r(t_j)$ and $B_j = B(t_j, R_j)$. Then, obviously,

$$B_j \cap \Gamma \subset B(t_j, cR_j) \cap \Gamma \subset E.$$

This implies that

$$E \subset (\cup_j B_j) \cap \Gamma \subset \cup_j \widetilde{B}_j \cap \Gamma \subset E.$$

Consequently, $E = \underset{j}{\cup} B_j \cap \Gamma$. Thus item (a) is proved.

Further, since $3cR_j = 9cr(t_j) = \frac{3}{2}d(t_j, \Gamma\backslash E) > d(t_j, \Gamma\backslash E)$, we obtain $\widetilde{\widetilde{B}}_j \cap (\Gamma\backslash E) \neq \varnothing$. It remains to prove (b). Notice first that there exists a number $h > 0$ such that every ball $B(t, r)$ cannot contain more than h^n points $\{t_j\}$ for which $|t_i - t_j| > \frac{r}{2^n}$, $i \neq j$.

Let now $t \in \widetilde{B}_j \cap \Gamma$, $\widetilde{B}_j = B(t_j, cR_j)$. Show that $cR_j < d(t, \Gamma\backslash E)$. Indeed,

$$\begin{aligned} 2cR_j &= 2c3r(t_j) = 6cr(t_j) = d(t_j, \Gamma\backslash E) \leq \\ &\leq |t_j - t| + d(t, \Gamma\backslash E) < cR_j + d(t, \Gamma\backslash E), \end{aligned}$$

whence it follows that

$$cR_j < d(t, \Gamma\backslash E).$$

Moreover, $\widetilde{B}_j \cap \Gamma \subset B(t, 2d(t, \Gamma\backslash E))$. Really, let $\tau \in \widetilde{B}_j \cap \Gamma$. Then

$$|t - \tau| \leq |\tau - t_j| + |t_j - t| < cR_j + cR_j < 2cR_j < 2d(t, \Gamma\backslash E),$$

which denotes the required inclusion.

Moreover,

$$d(t, \Gamma\backslash E) \leq |t - t_j| + d(t_j, \Gamma\backslash E) < cR_j + 2cR_j = 3cR_j,$$

which implies that

$$R_j \geq \frac{d(t, \Gamma\backslash E)}{3c}.$$

For $t \in \widetilde{B}_i \cap \widetilde{B}_j \cap \Gamma$, $i \neq j$, we have $\widetilde{B}_i \cap \Gamma \subset B(t, 2d(t, \Gamma\backslash E))$. Indeed, if $\tau \in \widetilde{B}_i \cap \Gamma$, then we have

$$|\tau - t| \leq |\tau - t_j| + |t_j - t| < 2cR_j < 2d(t, \Gamma\backslash E).$$

Thus all the centers t_j of the balls $\widetilde{B}_i$ containing t lie in $B(t, 2d(t, \Gamma\backslash E)) \cap \Gamma$. But on the other hand,

$$\Gamma \cap B(t_i, r(t_i)) \cap B(t_j, r(t_j)) = \varnothing \quad \text{for} \quad i \neq j$$

and

$$|t_i - t_j| \geq \min\{r(t_i), r(t_j)\} = \frac{1}{3}\min\{R_i, R_j\} \geq$$

$$\geq \frac{1}{6c} d(t, \Gamma \backslash E) = \frac{d(t, \Gamma \backslash E)}{2^{\log_2 6c}}.$$

Consequently, there is a h, $h > 0$, such that a number of balls containing the point t does not exceed $h^{\log_2 6c}$.

Note that in the previous inequality we have used the fact that $|t_i - t_j| \geq \min\{r(t_i), r(t_j)\}$. The latter holds because $t_i \notin B(t_j, r(t_j)) \cap \Gamma$ and $t_j \notin B(t_i, r(t_i)) \cap \Gamma$. $\blacksquare$

Lemma 4.5. *Let Γ be a regular curve, $1 < p < \infty$, and $w \in A_p(\Gamma)$. Then there exists $b > 1$ such that for any $r > 0$ and $z \in \Gamma$ we have*

$$w\Gamma(z, 2r) \leq b w\Gamma(z, r)$$

(the doubling condition).

Proof. From the definition of $A_p(\Gamma)$, regularity of Γ and Hölder inequality, we have

$$w\Gamma(z, 2r) \leq (\nu\Gamma(z, 2r))^p \Bigg(\int\limits_{\Gamma(z,2r)} w^{1-p'}(t) d\nu \Bigg)^{1-p} \leq$$

$$\leq b\nu\Gamma(z, r) \Bigg(\int\limits_{\Gamma(z,r)} w^{1-p'}(t) d\nu \Bigg)^{1-p} \leq$$

$$\leq b \Bigg(\int\limits_{\Gamma(z,r)} w(t) d\nu \Bigg) \Bigg(\int\limits_{\Gamma(z,r)} w^{1-p'}(t) d\nu \Bigg)^{p-1} \times$$

$$\times \Bigg(\int\limits_{\Gamma(z,r)} w^{1-p'}(t) d\nu \Bigg)^{1-p} = b w\Gamma(z, r). \quad \blacksquare$$

Lemma 4.6. *Let μ be a nonnegative Borel measure on $\mathbb{C}$, $\mu\Gamma < \infty$, and there exist $b > 0$ such that*

$$\mu\Gamma(t, 2r) \leq b\mu\Gamma(t, r)$$

for any $t \in \Gamma$ and $r > 0$. Then for an arbitrary point $t_0 \in \Gamma$ there exists a number $R > 0$ such that $\Gamma \subset B(t_0, R)$.

Proof. Suppose to the contrary that there exists a number $t_0 \in \Gamma$ such that for an arbitrary $R > 0$ the set $\Gamma \backslash B(t_0, R)$ is empty. Fix some $R > 0$ and let $z \in \Gamma \backslash B(t_0, 2R)$. It is evident that

$$B\Big(t_0, \frac{|t_0 - z|}{2}\Big) \cap B\Big(z, \frac{|t_0 - z|}{2}\Big) = \varnothing.$$

Moreover,

$$B\Big(z, \frac{|t_0 - z|}{2}\Big) \subset B\Big(t_0, \frac{3}{2}|t_0 - z|\Big).$$

In fact, let $\tau \in B\Big(z, \frac{|t_0-z|}{2}\Big)$. Then, obviously,

$$|t_0-\tau| \le |t_0-z| + |z-\tau| \le |t_0-z| + \frac{|t_0-z|}{2} = \frac{3}{2}|t_0-z|.$$

Moreover,

$$B\Big(t_0, \frac{|t_0-z|}{2}\Big) \subset B\Big(z, \frac{3|t_0-z|}{2}\Big).$$

Indeed, for $\tau \in B(t_0, |\frac{t_0-z}{2}|)$ we have

$$|z-\tau| \le |z-t_0| + |t_0-\tau| \le \frac{|t_0-z|}{2} + |t_0-z| = \frac{3}{2}|t_0-z|.$$

Therefore

$$\mu\Gamma\Big(t_0, \frac{|t_0-z|}{2}\Big) \le \mu\Gamma\Big(z, \frac{3|t_0-z|}{2}\Big)\Big) \le$$
$$\le b_1\mu\Gamma\Big(z, \frac{|t_0-z|}{2}\Big),$$

and thus we obtain

$$\mu\Gamma\Big(t_0, \frac{3}{2}|t_0-z|\Big) \ge \mu\Gamma\Big(t_0, \frac{|t_0-z|}{2}\Big) + \mu\Gamma\Big(z, \frac{|t_0-z|}{2}\Big) \ge$$
$$\ge \mu\Gamma\Big(t_0, \frac{|t_0-z|}{2}\Big) + \frac{1}{b_1}\mu\Gamma\Big(t_0, \frac{|t_0-z|}{2}\Big) =$$
$$= (1+\delta)\mu\Gamma\Big(t_0, \frac{|t_0-z|}{2}\Big) \ge (1+\delta)\mu\Gamma(t_0, R).$$

If we assume that $R_1 = \frac{3}{2}|t_0-z|$, then from the latter we can conclude that

$$\mu\Gamma(t_0, R_1) \ge (1+\delta)\mu\Gamma(t_0, R).$$

Continuing this process, we get a sequence $(R_k)_k$ of positive numbers such that

$$\mu\Gamma(t_0, R_k) \ge (1+\delta)^k\mu\Gamma(t_0, R), \quad k = 1, 2, \dots.$$

Passing to the limit as $k \to \infty$, from the above inequality we find that $\mu\Gamma = \infty$, which contradicts our assumption. ■

Corollary. *Let Γ be a regular curve, $w \in A_p(\Gamma)$, $1 \le p < \infty$. If $w(\Gamma) < \infty$, then for an arbitrary $t_0 \in \Gamma$ there exists a number $R > 0$ such that $\Gamma \subset B(t_0, R)$.*

Let Γ be a regular curve and w be a weight function. For every summable function $f : \Gamma \to \mathbb{R}$ we define on $(0, \infty)$ an equimeasurable function

$$f_w^*(x) = \inf\{\lambda > 0 : w\{t : |f(t)| > \lambda\} \le x\}.$$

We can readily show the validity of the following

Lemma 4.7. *For every $x > 0$ we have*

$$w\{t : |f(t)| > f^*(x)\} \leq x.$$

Lemma 4.8. *Let $w \in A_\infty(\Gamma)$, and let Ω be a measurable subset of Γ, $w\Omega > 0$. Then there exists an open (in the sense of the Γ topology) set $G \supset \Omega$ such that*

$$wG \leq cw\Omega,$$

where the constant c does not depend on Ω.

Proof. Let $G = \{t \in \Gamma : M(\chi_\Omega)(t) > \frac{1}{2}\}$, where

$$M(\chi_\Omega(t)) = \sup_{r>0} \frac{\nu(\Gamma(t,r) \cap \Omega)}{\nu\Gamma(t,r)}.$$

We can easily see that G is an open set in the topology of Γ, and $\Omega \subset G$. From the condition $w \in A_p(\Gamma)$ and also from the inequality of the weak type (p,p) we deduce for the operator M that

$$wG = w\{t : M(\chi_\Omega)(t) > \frac{1}{2}\} \leq c2^p w\Omega. \quad \blacksquare$$

Lemma 4.9. *Let Γ be an unbounded regular curve, and let $w \in A_\infty(\Gamma)$. Then $w\Gamma = \infty$.*

Proof. Suppose to the contrary that $w\Gamma < \infty$. Then owing to Lemmas 4.5 and 4.6, the curve Γ may appear in some ball $B(t,r)$, $t \in \Gamma$, $r > 0$. On the strength of the regularity we obtain $\nu\Gamma = \nu\Gamma(t,r) \leq cr < \infty$, which contradicts our assumption on the unboundedness of the curve Γ. $\blacksquare$

Lemma 4.10. *Let Γ be an unbounded regular curve, and let $w \in A_\infty(\Gamma)$. Then there exists a number $c > 0$ such that for every $x > 0$ we have*

$$(M_\Gamma f)^*_w(x) \leq c(f^\#)^*_w(2x) + (M_\Gamma f)^*_w(2x).$$

Proof. Fix $x > 0$ and assume

$$\Omega = \{t : f^\#(t) > (f^\#)^*_w(2x)\} \cup \{t : M_\Gamma f(t) > (M_\Gamma f)^*_w(2x)\}.$$

By Lemma 4.7, we have

$$\begin{gathered} w\Omega \leq w\{t : f^\#(t) > (f^\#)^*_w(2x)\} + \\ + w\{t : M_\Gamma f(t) > (M_\Gamma f)^*_w(2x)\} \leq 4x < \infty. \end{gathered}$$

By virtue of Lemma 4.8, there exists an open (in the sense of the Γ topology) set $G \supset \Omega$ such that

$$wG \leq c_1 w\Omega < \infty.$$

Since Γ is unbounded, using Lemma 4.9 we find that $w\Gamma = \infty$ and, obviously, $\Gamma \backslash \Omega \neq \varnothing$.

Apply now Lemma 4.4 owing to which there exists for $c=1$ a sequence $(B_j)_j$ of balls with centers on Γ such that $G=\bigcup\limits_j(B_j\cap\Gamma)$, $\sum\limits_j\chi_{B_j\cap\Gamma}(t)\le m$ and $\overline{B}_j\cap(\Gamma\backslash G)\ne\varnothing$, $j=1,2,\dots$, where $B_j=B(t_j,r_j)$ and $\overline{B}_j=B(t_j,3r_j)$.

The lemma will be considered to be proved, if we show that there exists $c_1>0$ such that

$$w\{t:M_\Gamma f(t)>c_1(f^{\#})^*_w(2x)+(M_\Gamma f)^*_w(2x)\}<x$$

for an arbitrary $x>0$.

Suppose

$$E_j=\{t\in B_j\cap\Gamma:M_\Gamma f(t)>c_1(f^{\#})^*_w(2x)+(M_\Gamma f)^*_w(2x)\}.$$

Fix j and let $f=g+h$, where $h=(f-f_{\overline{B}_j\cap\Gamma})\chi_{\overline{B}_j\cap\Gamma}$. As is easily seen, for $t\in B_j\cap\Gamma$ we have

$$g(t)=f(t)-(f-f_{\overline{B}_j\cap\Gamma})\chi_{\overline{B}_j\cap\Gamma}(t)=f_{\overline{B}_j\cap\Gamma}\le(M_\Gamma f)^*_w(2x),$$

since $\overline{B}_j\cap(\Gamma\backslash G)\ne\varnothing$.

On the other hand, for $t\in\Gamma\backslash G$ we have

$$g(t)=f(t)\le M_\Gamma f(t)\le(M_\Gamma f)^*_w(2x),$$

and we can conclude that

$$\|g\|_\infty\le(M_\Gamma f)^*_w(2x).$$

Consequently,

$$M_\Gamma f(t)\le M_\Gamma h(t)+M_\Gamma g(t)\le M_\Gamma h(t)+\|g\|_\infty\le M_\Gamma h(t)+(M_\Gamma f)^*_w(2x).$$

Further, by virtue of the weak type inequality and the definition of the set Ω, we obtain the estimates

$$\begin{gathered}\nu E_j\le\nu\{t\in B_j\cap\Gamma:M_\Gamma h(t)>c_1(f^{\#})^*_w(2x)\}\le\\ \le\frac{c_2}{c_1(f^{\#})^*_w(2x)}\int\limits_\Gamma|h(t)|d\nu=\frac{c_2}{c_1(f^{\#})^*_w(2x)}\int\limits_{\overline{B}_j\cap\Gamma}|f(t)-f_{\overline{B}_j\cap\Gamma}|d\nu\le\\ \le\frac{c_2c_3}{c_1}\,\nu(B_j\cap\Gamma),\quad j=1,2,\dots.\end{gathered}$$

In the above estimate we have used the fact that $\overline{B}_j\cap(\Gamma\backslash G)\ne\varnothing$.

Next, by virtue of the condition $w\in A_\infty(\Gamma)$, we have

$$\begin{gathered}wE_j\le c_4\Big(\frac{\nu E_j}{\nu(B_j\cap\Gamma)}\Big)^\delta w(B_j\cap\Gamma)\le\\ \le c_4\Big(\frac{c_2c_3}{c_1}\Big)^\delta w(B_j\cap\Gamma)\Big),\quad j=1,2,\dots.\end{gathered}$$

Thus we obtain

$$w\{t : M_\Gamma f(t) > c_1(f^\#)^*_w(2x) + (M_\Gamma f)^*_w(2x)\} \le$$

$$\le \sum_j \nu E_j \le mb\Big(\frac{c_2c_3}{c_1}\Big)^\delta wG \le bc_4\big(\frac{c_2c_3}{c_1}\big)^\delta w(\Gamma\cap\Omega) \le$$

$$\le mbc_4\Big(\frac{c_2c_3}{c_1}\Big)^\delta w(\Omega) \le 4mbc_4\Big(\frac{c_2c_3}{c_1}\Big)^\delta x.$$

Take now c_1 so large that

$$4bmc_4\Big(\frac{c_2c_3}{c_1}\Big)^\delta < 1.$$

Then from the previous inequality we conclude that

$$w\{t : M_\Gamma f(t) < c_1(f^\#)^*_w(2x) + (M_\Gamma f)^*_w(2x)\} < x$$

for an arbitrary $x > 0$. ■

Lemma 4.11. *Let Γ be an unbounded regular curve, and let $w \in A_\infty(\Gamma)$. Then the inequality*

$$(M_\Gamma f)^*_w(x) \le c_1\int_x^\infty (f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} + \lim_{x\to\infty}(M_\Gamma f)^*_w(x)$$

holds, where c_1 does not depend on x and f.

Proof. Since Γ is unbounded, by Lemma 4.9 we have $w\Gamma = \infty$.

Applying now Lemma 4.10, we obtain the following estimates:

$$(M_\Gamma f)^*_w(x) \le c(f^\#)^*_w(2x) + (M_\Gamma f)^*_w(2x) \le$$

$$\le c(f^\#)^*_w(2x)\frac{1}{x}\int_x^{2x} d\lambda + (M_\Gamma f)^*_w(2x) \le$$

$$\le 2c\int_x^{2x}(f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} + (M_\Gamma f)^*_w(2x) \le$$

$$\le 2c\int_x^{2x}(f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} + (M_\Gamma f)^*_w(4x) + c(f^\#)^*_w(4x) \le$$

$$\le 2c\int_x^{2x}(f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} + 2c\int_{2x}^{4x}(f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} +$$

$$+(M_\Gamma f)^*(4x) \le 2c\int_x^{2x}(f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} + 2c\int_{2x}^{4x}(f^\#)^*_w(\lambda)\frac{d\lambda}{\lambda} +$$

$$+(M_\Gamma f)_w^*(4x) \le \cdots \le 2c \int\limits_x^{2^n x} (f^\#)_w^*(\lambda)\frac{d\lambda}{\lambda} + (M_\Gamma f)_w^*(2^n x)$$

for arbitrary n and with c independent of f, x and n.

Consequently,

$$(M_\Gamma f)_w^*(x) \le 2c \lim_{n\to\infty} \int\limits_x^{2^n x} (f^\#)_w^*(\lambda)\frac{d\lambda}{\lambda} +$$

$$+ \lim_{x\to\infty}(M_\Gamma f)^*(2^n x) = c_1 \int\limits_x^{\infty}(f^\#)_w^*(\lambda)\frac{d\lambda}{\lambda} + \lim_{x\to\infty}(M_\Gamma f)_w^*(x). \quad \blacksquare$$

Proposition B. *Let Γ be an unbounded regular curve, $1 \le p < \infty$, and let $w \in A_\infty(\Gamma)$. Then there exists $c_p > 0$ such that for any f with the condition $\lim_{x\to\infty}(M_\Gamma f)_w^*(x) = 0$ the inequality*

$$\Big(\int\limits_\Gamma (M_\Gamma f)(t))^p w(t)d\nu\Big)^{1/p} \le c_p\Big(\int\limits_\Gamma (f^\#(t))^p w(t)d\nu\Big)^{1/p} \qquad (4.17)$$

holds.

Proof. By Lemma 4.11 and Hardy's inequality, we deduce the estimate

$$\Big(\int\limits_\Gamma ((M_\Gamma f)(t))^p w(t)d\nu\Big)^{1/p} = \Big(\int\limits_0^\infty ((M_\Gamma f)_w^*(x))^p dx\Big)^{1/p} \le$$

$$\le c\Big(\int\limits_0^\infty\Big(\int\limits_x^\infty (f^\#)_w^*(\lambda)\,\frac{d\lambda}{\lambda}\Big)^p dx\Big)^{1/p} \le c_p\Big(\int\limits_0^\infty ((f^\#)_w^*(x))^p dx\Big)^{1/p} =$$

$$= c_p\Big(\int\limits_\Gamma (f^\#(t))^p w(t)d\nu\Big)^{1/p}. \quad \blacksquare$$

Proposition C. *Let Γ be an unbounded regular curve, and let $w \in A_{p_0}(\Gamma)$ and $p \ge p_0$. Then there exists the constant $c_p > 0$ such that*

$$\Big(\int\limits_\Gamma (M_\Gamma f)^p(t)w(t)d\nu\Big)^{1/p} \le c_p\Big(\int\limits_\Gamma (f^\#(t))^p w(t)d\nu\Big)^{1/p}.$$

Proof. Let $f \in L_w^p(\Gamma)$. Since $w \in A_p(\Gamma)$, we have $M_\Gamma f \in L_w^p(\Gamma)$, and hence

$$\lim_{x\to\infty}(M_\Gamma f)_w^*(x) = 0.$$

The remainder follows from Proposition B, since $w \in A_\infty(\Gamma)$.

Proposition C is the analogue of the well-known theorem due to C. Fefferman and E. M. Stein [35].

Proposition D. *Let $1 < p < \infty$, and $\Gamma \in R$. Then the following pointwise estimation is valid:*

$$(S_\Gamma f)^{\#}(t) \le c M_\Gamma^{(p)} f(t) \tag{4.18}$$

almost everywhere on Γ for arbitrary $f \in L(\Gamma)$.

Proof. Let $t_0 \in \Gamma$, $r > 0$ if Γ is unbounded and $t \in B(t_0, r) \cap \Gamma$. Put $f_1(\tau) = f(\tau)\chi_{B(t_0,2r)\cap\Gamma(\tau)}$ and $f_2(\tau) = f(\tau) - f_1(\tau)$. We have

$$\frac{1}{\nu(\Gamma \cap B(t_0, r))} \int\limits_{\Gamma\cap B(t_0,r)} |S_\Gamma f(\tau) - S_\Gamma f(t_0)| d\nu =$$

$$= \frac{1}{\nu(\Gamma \cap B(t_0, r))} \int\limits_{\Gamma\cap B(t_0,r)} |S_\Gamma f_1(\tau) + S_\Gamma f_2(\tau) - S_\Gamma f_2(t_0)| d\nu.$$

Let

$$F(r,t) = S_\Gamma f_1(t) + S_\Gamma f_2(t) - S_\Gamma f_2(t_0) = S_\Gamma f_1(t) +$$

$$+ \int\limits_{\{\tau\in\Gamma: |\tau - t_0| \ge 2r\}} \Big(\frac{1}{\tau - t} - \frac{1}{\tau - t_0}\Big) f(\tau) d\tau = I_1(t) + I_2(t).$$

Since $\Gamma \in R$, we have

$$\|I_1\|_p = \|S_\Gamma f_1\|_p \le c\|f_1\|_p.$$

Hence

$$\|I_1\|_p = c\bigg(\int\limits_{\Gamma\cap B(t_0,r)} |f(\tau)|^p d\nu\bigg)^{\frac{1}{p}} \le c(\nu(\Gamma \cap B(t_0, 2r))^{\frac{1}{p}} \times$$

$$\times \inf_{\tau\in\Gamma\cap B(t_0,2r)} M^{(p)} f(\tau) \le c_1(\nu\Gamma(t_0, r))^{\frac{1}{p}} \inf_{\tau\in\Gamma\cap B(t_0,r)} M^{(p)} f(\tau).$$

In the last estimate we have used the regularity of Γ.
Further,

$$I_2(t) \le \int\limits_{\{\tau\in\Gamma: |\tau - t_0| \ge 2r\}} \Big|\frac{1}{\tau - t} - \frac{1}{\tau - t_0}\Big| |f(\tau)| d\nu =$$

$$= \int\limits_{\{\tau\in\Gamma: |\tau - t_0| \ge 2r\}} \frac{|t - t_0|}{|\tau - t||\tau - t_0|} |f(\tau)| d\nu.$$

For $\tau \in \Gamma$, $|\tau - t_0| \ge 2r$ and $t \in \Gamma \cap B(t_0, r)$ we have $|\tau - t_0| \le |\tau - t| + |t - t_0| \le |\tau - t| + r \le |\tau - t| + \frac{|\tau - t_0|}{2}$. Hence $|\tau - t_0| < 2|\tau - t|$. Therefore

$$I_2(t) \le 2 \int\limits_{\{\tau\in\Gamma: |\tau - t_0| \ge 2r\}} \frac{|t - t_0|}{|\tau - t_0|^2} |f(\tau)| d\nu =$$

$$= 2\sum_{k\geq 1}\frac{|t-t_0|}{|\tau-t_0|^2}\int\limits_{\{\tau\in\Gamma:2^k r\leq|\tau-t_0|<2^{k+1}r\}}|f(\tau)|d\nu \leq$$

$$\leq c\sum_{k\geq 1}\frac{1}{2^k}(2^{k+1}r)^{-1}\nu\Gamma\cap B(t_0,2^{k+1}r)\inf_{\tau\in\Gamma(t_0,2^{k+1}r)}Mf(\tau)\leq$$

$$\leq c\inf_{\tau\in\Gamma(t_0,r)}Mf(\tau)\sum_{k=1}^{\infty}\frac{1}{2^k}\leq c\inf_{\tau\in\Gamma(t_0,r)}Mf(\tau).$$

Thus

$$\int\limits_{\Gamma(t_0,r)}|F(r,t)|d\nu\leq\int\limits_{\Gamma(t_0,r)}|I_1(t)|d\nu+\int\limits_{\Gamma(t_0,r)}|I_2(t)|d\nu\leq$$

$$\leq(\nu\Gamma(t_0,r))^{\frac{1}{p'}}\Bigg(\int\limits_{\Gamma(t_0,r)}|I_1(t)|^p d\nu\Bigg)^{\frac{1}{p}}+c\nu(\Gamma(t_0,r))\inf_{\tau\in\Gamma(t_0,r)}Mf(\tau)=$$

$$= c\nu(\Gamma(t_0,r))(\inf_{\tau\in\Gamma(t_0,r)}M^{(p)}f(\tau)+\inf_{\tau\in\Gamma(t_0,r)}Mf(\tau))\leq$$

$$\leq c\nu\Gamma(t_0,r)\inf_{\tau\in\Gamma(t_0,r)}M^{(p)}f(\tau),$$

whence we obtain the estimate

$$\frac{1}{\nu\Gamma(t_0,r)}\int\limits_{\Gamma(t_0,r)}|F(r,t)|d\nu\leq c\inf_{\tau\in\Gamma(t_0,r)}M^{(p)}f(\tau),$$

where a constant c does not depend on t_0 and r.

Next,

$$\frac{1}{\nu\Gamma(t_0,r)}\int\limits_{\Gamma(t_0,r)}|S_\Gamma f(\tau)-(S_\Gamma f)_{\Gamma(t_0,r)}|d\nu\leq$$

$$\leq\frac{2}{\nu\Gamma(t_0,r)}\int\limits_{\Gamma(t_0,r)}|F(r,t)|\,d\nu\leq cM^{(p)}f(t)$$

for arbitrary $t\in\Gamma\cap B(t_0,r)$.

Finally we conclude that

$$(S_\Gamma f)^{\#}(t)\leq cM^{(p)}f(t)$$

almost everywhere on Γ. ■

Theorem 4.2. *Let $1<p<\infty$, $\Gamma\in R$. Then for the inequality*

$$\int\limits_{\Gamma}|S_\Gamma f(t)|^p w(t)d\nu\leq c\int\limits_{\Gamma}|f(t)|^p w(t)d\nu \tag{4.19}$$

to be valid with a constant c not depending on f, it is necessary and sufficient that $w\in A_p(\Gamma)$.

Proof. At first we assume that Γ is an unbounded curve. Since the class $A_p(\Gamma)$ is open, there exists some $p_1 > 1$ such that $w \in A_{p/p_1}(\Gamma)$. Using now Propositions D, C and A, we obtain the estimates

$$\int_\Gamma |S_\Gamma f(t)|^p w(t) d\nu \leq$$
$$\leq \int_\Gamma (M_\Gamma(S_\Gamma f)(t))^p w(t) d\nu \leq c_1 \int_\Gamma ((S_\Gamma f)^{\#}(t))^p w(t) d\nu \leq$$
$$\leq c_2 \int_\Gamma (M_\Gamma(|f|^{p_1})(t))^{\frac{p}{p_1}} w(t) d\nu \leq c_3 \int_\Gamma |f(t)|^p w(t) d\nu. \qquad (4.20)$$

Thus we have proved the part of Theorem 4.2 in case of unbounded Γ concerning the sufficiency.

We pass now to the case where the curve Γ is bounded. Assume first that $\Gamma = \Gamma_{ab}$ is an open curve with the ends a and b, and U_Γ is a circle containing Γ. Choose on Γ sequences of points $(a_n)_n$ and $(b_n)_n$ with the condition $a_n \to a$, $b_n \to b$ at which the Γ curve possesses the tangents. For sufficiently large N, we can construct a circle U_N with center at the point a_n containing the arc $a_N a$ and excluding the point b. Since the curve has at the points a_n the tangents, there exists a sector with the vertex at the point a_n such that its intersection with U_N does not contain the points of the curve Γ. Therefore there exists the segment $\overline{a_n a_n'}$, $a_n' \in U_N$ such that $\overline{a_n a_n'} \cap \Gamma = \varnothing$. Since $U_\Gamma \backslash \Gamma$ is the domain, there exists a broken line $\cup_n \overline{a_n A_n}$, $A_n \in \delta U_\Gamma$ with a finite number (say m) of links which does not intersect Γ. Let $\delta_{n,a} = \overline{a_n A_n} \cup \overset{\smile}{A_n A_N}$, where $A_n A_N$ are the arcs of the circumference δU_N – boundary of U_n. Similarly we construct the curves $\delta_{n,b}$ connected analogously with the other end b.

If δ_a and δ_b are nonintersecting rays with vertices at the points A_N and B_N, then the curves $\Gamma_n = \Gamma_{a_n b_n} \cup \delta_{n,a} \cup \delta_{n,b} \cup \delta_a \cup \delta_b$, and U_N may turn out regular, and as is easily seen,

$$\sup_{\xi \in \Gamma_n} \sup_{\rho > 0} \frac{l_\xi(\rho)}{\rho} \leq \sup_{\xi \in \Gamma} \sup_{\rho > 0} \frac{l_\xi(\rho)}{\rho} + m + 2.$$

Consequently, on the basis of the above-proven results we find that

$$\|S_{\Gamma_n}\|_{L^p_w \to L^p_w} \leq M$$

for some M, independent of n. From this we conclude that

$$\|S_{\Gamma_{a_n b_n}}\|_{L^p_w \to L^p_w} \leq M.$$

Next, from the last estimate and from the fact that S_Γ is continuous in L^p, in a standard way we deduce that

$$\|S_{\Gamma_{ab}}\|_{L^p_w \to L^p_w} \leq M.$$

Let now Γ be an arbitrary, simple, closed regular curve. Take on Γ three points a, b and c with the condition $s(a) < s(b) < s(c)$. For $f \in L^p_w(\Gamma)$ we have

$$\int_{\Gamma_{ab}} |S_\Gamma f(t)|^p w(t) d\nu \leq$$
$$\leq 2^p \Big(\int_{\Gamma_{ab}} |S_{\Gamma_{ab}} f(t)|^p w(t) d\nu + \int_{\Gamma_{ab}} |S_{\Gamma_{ba}} f(t)|^p w(t) d\nu \Big).$$

But

$$\int_{\Gamma_{ab}} |S_{\Gamma_{ba}} f(t)|^p w(t) d\nu \leq$$
$$\leq 2^p \Big(\int_{\Gamma_{ab}} |S_{\Gamma_{bc}} f(t)|^p w(t) d\nu + \int_{\Gamma_{ab}} |S_{\Gamma_{ca}} f(t)|^p w(t) d\nu \Big) \leq$$
$$\leq 2^p \Big(\int_{\Gamma_{ac}} |S_{\Gamma_{ac}} (f \chi_{\Gamma_{bc}}(t)|^p w(t) d\nu + \int_{\Gamma_{cb}} |S_{\Gamma_{cb}} (f \chi_{\Gamma_{ca}})(t)|^p w(t) d\nu \Big).$$

Taking into account the above arguments, we can see that the operator S is continuous in $L^p_w(\Gamma_{ac})$ and $L^p_w(\Gamma_{cb})$. Thus we conclude that

$$\int_{\Gamma_{ab}} |S_\Gamma f(t)|^p w(t) d\nu \leq c \int_{\Gamma} |f(t)|^p w(t) d\nu.$$

The inequality

$$\int_{\Gamma_{ba}} |S_\Gamma f(t)|^p w(t) d\nu \leq c \int_{\Gamma} |f(t)|^p w(t) d\nu$$

is derived analogously.

Consequently, the theorem is proved in the general case for that part concerns the sufficiency.

For the necessity it suffices now to remark that there exists a constant $c > 0$ such that for any $\Gamma(z_1, r_1)$ there exists another $\Gamma_2(z_2, r_2)$ such that the inequalities

$$S_\Gamma(\chi_{\Gamma_1} f)(z) \geq c \Big(\frac{1}{\nu \Gamma_1} \int_{\Gamma_1} f(t) d\nu \Big) \chi_{\Gamma_2}(z)$$

and

$$S_\Gamma(\chi_{\Gamma_2} f)(z) \geq c \Big(\frac{1}{\nu \Gamma_2} \int_{\Gamma_2} f(t) d\nu \Big) \chi_{\Gamma_1}(z)$$

hold for any real f and $z \in \Gamma$. ■

4.3. Weight inequalities for singular integrals on smooth curves. Let $t = t(s)$ be an equation of the curve Γ with respect to the arc abscissa. The arc length is assumed to be equal to 2π. We extend the function $f(t)$ and the weight function w periodically on $\mathbb{R}$ and consider the integral

$$S_\Gamma f(t(s)) = \int_0^{2\pi} \frac{f(t(\sigma))t'(\sigma)}{t(\sigma) - t(s)} d\sigma.$$

Theorem 4.3. *Let $1 < p < \infty$, Γ be a smooth curve. Then the inequality*

$$\int_0^{2\pi} |S_\Gamma f(t(s))|^p w^p(s) ds \le c \int_0^{2\pi} |f(t(s))|^p w^p(s) ds \tag{4.21}$$

holds for all $f \in L^p(\Gamma, w)$ with a constant $c > 0$ independent of f if and only if the condition

$$\sup \Big(\frac{1}{|I|} \int_I w^p(s) ds \Big)^{\frac{1}{p}} \Big(\frac{1}{|I|} \int_I w^{-p'}(s) ds \Big)^{\frac{1}{p'}} < \infty \tag{4.22}$$

is fulfilled, where the least upper bound is taken over all intervals of the length less than 2π.

Proof. The sufficiency follows from Theorem 4.2. We dwell on the proof of the necessity. Let the inequality (4.21) hold.

Consider on $[-\pi, 3\pi; -\pi, 3\pi]$ the function

$$\phi(s, \sigma) = \begin{cases} x'(s)\frac{x(s)-x(\sigma)}{s-\sigma} + y'(s)\frac{y(s)-y(\sigma)}{s-\sigma} & \text{for} \quad s \ne \sigma, \\ 1, & \text{for} \quad s = \sigma, \end{cases} \tag{4.23}$$

where $t(s) = x(s) + iy(s)$.

Because of the smoothness of the curve,

$$\lim_{s \to s_0, \sigma \to \sigma_0} \Phi(s, \sigma) = 1$$

for an arbitrary $s_0 \in [-\pi, 3\pi]$. Therefore, for every $s_0 \in [-\pi, 3\pi]$ there exists its neighbourhood Δ_{s_0} such that $\phi(s, \sigma) \ge \frac{1}{2}$ for any $s \in \Delta_{s_0}$, $\sigma \in \Delta_{s_0}$.

Choose from the covering $\{\Delta_{s_0}\}$ of the segment $[-\pi, 3\pi]$ a finite covering $\{\Delta'_k\}_{k=1}^m$. Assume

$$c_0 = \min_{1 \le i,j \le m} |\Delta'_i \cap \Delta'_j|.$$

By the equality

$$\operatorname{Re} \frac{t'(s)(s - \sigma)}{t(s) - t(\sigma)} = \Big(x'(s) \frac{x(s) - x(\sigma)}{s - \sigma} + y'(s) \frac{y(s) - y(\sigma)}{s - \sigma} \Big) \Big| \frac{t(s) - t(\sigma)}{s - \sigma} \Big|^{-2}$$

for an arbitrary interval I, $|I| < \min(\frac{1}{2}c_0, \frac{\pi}{4})$ and any s and σ from I we have the inequality

$$\operatorname{Re}\frac{t'(s)(s-\sigma)}{t(s)-t(\sigma)} \geq \frac{1}{2}. \tag{4.24}$$

To prove the necessity of the condition (4.22), it suffices, as is easily seen, to show that it is fulfilled for any I, $|I| < \min(\frac{1}{2}c_0, \frac{\pi}{4})$.

Let $I \cap (0, 2\pi) \neq \varnothing$ and $|I| < \min(\frac{1}{2}c_0, \frac{\pi}{4})$. Denote by I_1 any of the neighbouring intervals of the same length as the interval I. Without restriction of generality, we assume that the interval I_1 is on the left of I. Let I_0 be an interval of length 2π containing a set $I \cup I_1$.

Let φ, a non-negative summable function, be equal to zero outside I. Suppose $f(t) = \varphi(s(t))$.

We will have

$$|S_\Gamma f(t)| = \Big|\int_{I_0} \frac{f(t(s))t'(s)}{t(s)-t(\sigma)}ds\Big| \geq \Big|\int_I \frac{\varphi(s)}{s-\sigma}\operatorname{Re}\frac{t'(s)(s-\sigma)}{t(s)-t(sg)}ds\Big|.$$

By virtue of (4.24), for any $\sigma \in I_1$ we have

$$|S_\Gamma f(t(\sigma))| \geq \frac{1}{2}\int_I \frac{\varphi(s)ds}{s-\sigma} \geq \frac{1}{4|I|}\int_I \varphi(s)ds. \tag{4.25}$$

Thus, for an arbitrary, non-negative, 2π-periodic function $\varphi \in L(I_0)$ vanishing outside I we have the inequality

$$|(S_\Gamma f)(t(\sigma))| \geq \Big(\frac{1}{4|I|}\int_I \varphi(s)ds\Big)\chi_{I_1}(\sigma) \tag{4.26}$$

for any σ, where $f(t) = \varphi(s(t))$.

Analogously, for any non-negative, 2π-periodic function $\varphi_1 \in L(I_0)$ vanishing outside I_1, we have

$$|S_\Gamma f_1(t(\sigma))| \geq \Big(\frac{1}{4|I_1|}\int_I \varphi_1(s)ds\Big)\chi_{I_1}(\sigma), \tag{4.27}$$

where $f_1(t) = \varphi_1(s(t))$.

In (4.27) we assume that $\varphi_1(s) = w^{-p'}(s)$ for $s \in I_1$. Then from the inequality (4.21) we conclude that

$$\Big(\frac{1}{|I_1|}\int_{I_1} w^{-p'}(s)ds\Big)^{\frac{1}{p'}}\int_I w^p(\sigma) \leq b\int_I w^{-p'}(\sigma)d\sigma. \tag{4.28}$$

It follows from (4.28) that under our assumption

$$\int_I w^p(\sigma)d\sigma < \infty.$$

On the other hand, if we put $\varphi(s)=1$ for $s\in I$, then because of (4.26) from (4.21) we find that

$$\int\limits_{I_1} w^p(\sigma)d\sigma \le b_1 \int\limits_I w^p d\sigma.$$

Interchanging the intervals I and I_1, similarly to the above-proved we conclude that

$$\int\limits_I w^p(\sigma)^p d\sigma \le b_2 \int\limits_{I_1} w^p(\sigma)d\sigma.$$

Thus

$$\frac{1}{b_1}\int\limits_{I_1} w^p(\sigma)d\sigma \le \int\limits_I w^p(\sigma)d\sigma \le b_2 \int\limits_{I_1} w^p(\sigma)d\sigma.$$

By (4.28), from the latter inequality it immediately follows that (4.22) is fulfilled. ■

Theorem 4.4. *Let Γ be a closed smooth curve. Then for the inequality*

$$\int\limits_{\{\sigma\in(0,2\pi):|S_\Gamma f(t(\sigma))|>\lambda\}} w(\sigma)d\sigma \le \frac{c}{\lambda}\int\limits_0^{2\pi} |f(t(s))|w(s)ds \tag{4.29}$$

with the constant $c>0$ to exists for any $\lambda>0$ and $f\in L(\Gamma,w)$, it is necessary and sufficient that the inequality

$$\frac{1}{|I|}\int\limits_I w(\sigma)d\sigma \operatorname{ess\,sup}_{s\in I}\frac{1}{w(s)} < c_1 \tag{4.30}$$

with the constant $c_1>0$ for every interval I of the length less than 2π be fulfilled.

Proof. We dwell on the proof of the necessity, since the sufficiency is proved in Theorem 4.2. Given (4.29), let I be an arbitrary interval possessing the properties $\operatorname{ess\,inf}_{s\in I} w(s)<\infty$, $I\cap(0,2\pi)\neq\varnothing$ and $|I|<\min(\frac{1}{2}c_0,\frac{\pi}{4})$, where c_0 is the constant from the previous theorem.

Let I_1 be one of the neighbouring intervals of I having the same length.

For an arbitrary $\varepsilon>0$ there exists a set E of positive measure such that

$$w(s)\le \operatorname{ess\,inf}_{\sigma\in I} w(\sigma)+\varepsilon, \quad s\in E. \tag{4.31}$$

Let now $\varphi(s)=\chi_E(s)$, $f(t)=\varphi(s(t))$. By arguing as in the previous theorem, we have

$$|S_\Gamma f(t(\sigma))| \ge \frac{1}{2}\left|\int\limits_E \frac{ds}{\sigma-s}\right| \ge \frac{1}{10}\frac{mE}{mI} \tag{4.32}$$

for an arbitrary $\sigma\in I_1$.

By the condition of the theorem,

$$\mu_w\Big\{\sigma \in (0,2\pi) : |S_\Gamma f(t(\sigma))| \geq \frac{mE}{10mI}\Big\} \leq \frac{bmI}{mE}\int\limits_E w(s)ds, \qquad (4.33)$$

where μ_w is the Borel measure defined as

$$\mu_w e = \int\limits_e w^p(\sigma)d\sigma.$$

By (4.30) and (4.32),

$$\mu_w\{\sigma : (0,2\pi) : |(S_\Gamma f)(t(\sigma))| \geq \frac{mE}{10mI}\} \leq b|I| \operatorname*{ess\,inf}_{\sigma\in I} w(\sigma) + \varepsilon.$$

Further, by virtue of (4.31) and from the arbitrariness of ε we obtain

$$\int\limits_{I_1} w(\sigma)d\sigma \leq b|I| \operatorname*{ess\,inf}_{s\in I} w(s). \qquad (4.34)$$

It is evident that $\operatorname*{ess\,inf}_{s\in I} w(s) > 0$ or otherwise w would vanish on I, and then the condition (4.30) would be fulfilled.

Analogous reasoning (after interching I and I_1) results in the conclusion that

$$\int\limits_{I} w(s)ds \leq b|I_1| \operatorname*{ess\,inf}_{\sigma\in I_1} w(\sigma). \qquad (4.35)$$

Next we have

$$\frac{1}{|I|}\int\limits_I w(\sigma)d\sigma \leq \frac{1}{|I|}\frac{1}{\operatorname*{ess\,inf}_{s\in I_1} w(s)}\int\limits_{I_1} w(s)ds \int\limits_I w(\sigma)d\sigma,$$

whence, by virtue of (4.34) and (4.35), we conclude that (4.30) is valid. ∎

4.4. Some two-weight estimates for a singular operator. For a 2π-periodic summable function f on $(-\pi,\pi)$ we put

$$\widetilde{f}(x) = \int\limits_{-\pi}^{\pi} \frac{f(y)dy}{e^{ix} - e^{iy}}.$$

It will be assumed that $1 < p < \infty$ and the positive number α is so large that the function $\psi(x) = x^{p-1}\ln^p \frac{\alpha}{x}$ increases on $(0,\pi)$, $\alpha > e\pi$.

Theorem 4.5. *Let $1 < p < \infty$ and $x_0 \in (-\pi, \pi)$. Then there exists a constant $M(p) > 0$ such that the inequality*

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x - x_0|^{p-1} dx \leq M(p) \int_{-\pi}^{\pi} |f(x)|^p |x - x_0|^{p-1} \ln^p \frac{\alpha}{|x - x_0|} dx \quad (4.36)$$

holds for arbitrary f for which the integral on the right-hand side is finite.

Moreover, the exponent p on the right-hand side with the logarithm is sharp, that is, it cannot be replaced by any $p_1 < p$.

The proof will be based on the following Hardy type two-weight inequality.

Theorem A ([71]). *Let $1 \leq p < q < \infty$ and functions v, w defined on $(0, \pi)$ be positive. Then for the equality*

$$\int_0^{\pi} v(x) \left| \int_0^x F(y) dy \right|^p dx \leq N(p) \int_0^{\pi} w(x) |F(x)|^p dx \quad (4.37)$$

to hold with a constant $N(p)$ not depending on F, it is necessary and sufficient that the condition

$$\sup_{x>0} \left(\int_x^{\pi} v(y) dy \right) \left(\int_0^x w^{1-p'}(y) dy \right)^{p-1} < \infty \quad (4.38)$$

be fulfilled.

Proof of Theorem 4.5. It can be assumed without loss of generality that $x_0 = 0$. Note that if the integral on the right-hand side of (4.36) is finite, then the function f is summable on $(-\pi, \pi)$ and therefore $\widetilde{f}(x)$ exists almost everywhere. Indeed,

$$\int_{-\pi}^{\pi} |f(x)| dx = \int_{-\pi}^{\pi} |f(x)| |x|^{1-\frac{1}{p}} \ln \frac{\alpha}{|x|} |x|^{\frac{1}{p}-1} \ln^{-1} \frac{\alpha}{|x|} dx \leq$$

$$\leq \left(\int_{-\pi}^{\pi} |f(x)|^p |x|^{p-1} \ln^p \frac{\alpha}{|x|} dx \right)^{\frac{1}{p}} \left(\int_{-\pi}^{\pi} \frac{dx}{|x| \ln^{p'} \frac{\alpha}{|x|}} \right)^{\frac{1}{p'}} < \infty.$$

Further we have

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x|^{p-1} dx = (p-1) \int_{-\pi}^{\pi} |\widetilde{f}(x)|^p \left(\int_0^{|x|} \tau^{p-2} d\tau \right) dx =$$

$$= (p-1) \int_0^{\pi} \tau^{p-2} \left(\int_{\pi > |x| > \tau} |\widetilde{f}(x)|^p dx \right) d\tau \leq$$

$$\leq 2^{p-1}(p-1)\Big[\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big|\int_{\pi>|y|>\frac{\tau}{2}}\frac{f(y)}{e^{ix}-e^{iy}}dy\Big|^p dx\Big)d\tau+$$

$$+\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big|\int_{0<|y|<\frac{\tau}{2}}\frac{f(y)}{e^{ix}-e^{iy}}dy\Big|^p dx\Big)d\tau\Big]=$$

$$=2^{p-1}(p-1)(I_1+I_2).$$

By the Riesz theorem, we conclude that

$$I_1=\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big|\int_{-\pi}^{\pi}\frac{f(y)\chi\{y:\pi>|y|>\frac{\tau}{2}\}}{e^{ix}-e^{iy}}dy\Big|^p dx\Big)d\tau\leq$$

$$\leq R_p\int_0^{\pi}\tau^{p-2}\Big[\int_{-\pi}^{\pi}(|f(y)|\chi\{y:\pi>|y|>\frac{\tau}{2}\})^p dy\Big]d\tau\leq$$

$$\leq R_p\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|y|>\frac{\tau}{2}}|f(y)|^p d\tau\Big).$$

Changing in the latter expression the order of integration, we obtain

$$I_1\leq M_1\int_{-\pi}^{\pi}|f(y)|^p\Big(\int_0^{2|y|}\tau^{p-2}d\tau\Big)dy\leq$$

$$\leq M_2\int_{-\pi}^{\pi}|f(y)|^p|y|^{p-1}dy\leq M_2\int_{-\pi}^{\pi}|f(y)|^p|y|^{p-1}\ln^p\frac{\alpha}{|y|}dy. \tag{4.39}$$

Let us now estimate I_2. For $0<\tau<\pi$, $\pi>|x|>\tau$, $0<|y|<\frac{\tau}{2}$ we have $|x-y|\leq|x|+|y|\leq\pi+\frac{\pi}{2}=\frac{3\pi}{2}$. Moreover $|x|\leq|x-y|+|y|\leq|x-y|+\frac{\tau}{2}\leq|x-y|+\frac{1}{2}|x|$, and hence $|x-y|\geq\frac{1}{2}|x|>\frac{1}{2}|\tau|$. Also,

$$|e^{ix}-e^{iy}|=2\Big|\sin\frac{x-y}{2}\Big|\geq\frac{2}{\pi}|x-y|$$

for $\frac{1}{2}|\tau|\leq|x-y|\leq\pi$, and $|e^{ix}-e^{iy}|\geq 2\sin\frac{3\pi}{4}$ for $\pi\leq(x-y)\leq\frac{3\pi}{2}$.

By virtue of all the inequalities obtained above, we have

$$I_2\leq\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big(\int_{\{|y|<\frac{\tau}{2}\}\cap\{\frac{\tau}{2}<|x-y|<\pi\}}|f(y)|\frac{1}{|e^{ix}-e^{iy}|}dy\Big)^p dx\Big)d\tau+$$

$$+\int_0^{\pi}\tau^{p-2}\Big(\int_{\pi>|x|>\tau}\Big(\int_{\{y:|y|<\frac{\tau}{2}\}\cap\{\pi<|x-y|<\frac{3\pi}{2}\}}|f(y)|\frac{dy}{|e^{ix}-e^{iy}|}\Big)^p dx\Big)d\tau\leq$$

$$\leq M_3\Bigg[\int\limits_0^{\pi}\tau^{p-2}\Bigg(\int\limits_{\pi>|x|>\tau}\frac{dx}{|x|^p}\Bigg(\int\limits_{|y|<\frac{\tau}{2}}|f(y)|dy\Bigg)^p\Bigg)d\tau+$$

$$+\int\limits_0^{\pi}\tau^{p-2}\Bigg(\int\limits_{\pi>|x|>\tau}dx\int\limits_{|y|<\frac{\pi}{2}}|f(y)|dy\Bigg)^p d\tau\Bigg].$$

Furthermore,

$$I_2\leq M_4\int\limits_0^{\pi}\frac{1}{\tau}\Bigg(\int\limits_{|y|<\frac{\tau}{2}}|f(y)|dy\Bigg)^p d\tau+$$

$$+M_3\int\limits_0^{\pi}\tau^{p-2}\Bigg(\int\limits_{\pi>|x|>\tau}dx\int\limits_{|y|<\frac{\pi}{2}}|f(y)|dy\Bigg)^p d\tau=I_{21}+I_{22}. \qquad (4.40)$$

Let us verify whether (4.38) is fulfilled for the pair of weights $v(t)=\frac{1}{\tau}$ and $w(\tau)=\tau^{p-1}\ln^p\frac{\alpha}{\tau}$. We have

$$\int\limits_x^{\pi}\frac{d\tau}{\tau}\Bigg(\int\limits_0^{x}\frac{1}{\tau}\ln^{p(1-p')}\frac{\alpha}{\tau}d\tau\Bigg)^{p-1}=$$

$$=M_5\int\limits_x^{\pi}\frac{d\tau}{\tau}\Bigg(\int\limits_0^{x}\frac{d\ln\frac{\alpha}{\tau}}{\ln^{p'}\frac{\alpha}{\tau}}\Bigg)^{p-1}=c\ln\frac{\pi}{x}\frac{1}{\ln\frac{\alpha}{x}}\leq M_6.$$

Therefore, to estimate I_{21}, we use Theorem A and obtain

$$I_{21}\leq M_7\int\limits_{-\pi}^{\pi}|f(x)|^p|x|^{p-1}\ln^p\frac{\alpha}{|x|}dx. \qquad (4.41)$$

Using Theorem A, we estimate I_{22} as follows:

$$I_{22}\leq M_8\int\limits_0^{\pi}\tau^{p-2}\int\limits_{\pi>|x|>\tau}dx\Bigg(\int\limits_{|y|<\frac{|x|}{2}}|f(y)|dy\Bigg)^p d\tau\leq$$

$$\leq M_9\int\limits_{-\pi}^{\pi}|x|^{p-1}\Bigg(\int\limits_{|y|<\frac{|x|}{2}}|f(y)|dy\Bigg)^p dx\leq$$

$$\leq M_9\int\limits_{-\pi}^{\pi}|x|^{p-1}\Bigg(\int\limits_{|y|<|x|}|f(y)|dy\Bigg)^p dx\leq$$

$$\leq M_{10}\int\limits_{-\pi}^{\pi}|f(x)|^p|x|^{p-1}\ln\frac{\alpha}{|x|}dx. \tag{4.42}$$

By (4.39), (4.41) and (4.42) we conclude that

$$\int\limits_{-\pi}^{\pi}|\widetilde{f}(x)|^p|x|^{p-1}dx\leq M(p)\int\limits_{-\pi}^{\pi}|f(x)|^p|x|^{p-1}\ln\frac{\alpha}{|x|}dx.$$

It remains to show that in inequality (4.36) exponent p with the logarithm cannot be replaced by a smaller number. Assume the contrary. Let $\varepsilon\in(0,1)$, $x_0=0$. Fix the number $t>0$ and put

$$f_t(y)=\begin{cases}\frac{\alpha}{y}\ln^{(p-\varepsilon)(1-p')}\frac{1}{y} & \text{for } \ 0<y<\frac{t}{2},\\ 0 & \text{for } \ y\notin(0,\frac{t}{2}).\end{cases}$$

Substituting the function f into inequality (4.36), where the exponent p with the logarithm is replaced by $p-\varepsilon$, we obtain

$$\int\limits_{-\pi}^{\pi}\left|\int\limits_{0}^{\frac{t}{2}}\frac{f_t(y)}{x-y}dy\right|^p|x|^{p-1}dx\leq$$

$$\leq M\int\limits_{0}^{\frac{t}{2}}\frac{1}{y}\ln^{(p-\varepsilon)p(1-p')}\frac{\alpha}{y}\ln^{p-\varepsilon}\frac{\alpha}{y}dy=M\int\limits_{0}^{\frac{t}{2}}\frac{1}{y}\ln^{(p-\varepsilon)(1-p')}\frac{\alpha}{y}dy.$$

Hence

$$\int\limits_{t}^{\pi}\left(\int\limits_{0}^{\frac{t}{2}}\frac{f_t(y)}{x-y}dy\right)^p|x|^{p-1}dx\leq M\int\limits_{0}^{\frac{t}{2}}\frac{1}{y}\ln^{(p-\varepsilon)(1-p')}\frac{\alpha}{y}dy. \tag{4.43}$$

On the other hand, it is obvious that

$$\int\limits_{t}^{\pi}\left(\int\limits_{0}^{\frac{t}{2}}\frac{f_t(y)}{x-y}dy\right)^p|x|^{p-1}dx\geq\int\limits_{t}^{x}\frac{1}{x}\left(\int\limits_{0}^{\frac{t}{2}}f_t(y)dy\right)^p dx. \tag{4.44}$$

By virtue of (4.43) and (4.44) we must have

$$\int\limits_{t}^{\pi}\frac{dx}{x}\left(\int\limits_{0}^{\frac{t}{2}}f_t(y)dy\right)^p\leq M\int\limits_{0}^{\frac{t}{2}}\frac{1}{y}\ln^{(p-\varepsilon)(1-p')}\frac{\alpha}{y}dy,$$

that is, the inequality

$$\ln\frac{\pi}{t}\left(\int\limits_{0}^{t}\frac{1}{y}\ln^{(p-\varepsilon)(1-p')}\frac{\alpha}{y}dy\right)\leq M$$

must be fulfilled for $0 < t < \pi$. But this is impossible, since

$$\left(\int_0^t \frac{1}{y} \ln^{(p-\varepsilon)(1-p')} \frac{\alpha}{y} dy\right)^{p-1} \sim \ln^{\varepsilon-1} \frac{\alpha}{y}.$$

Thus we have proved the validity of the last part of the theorem. ■

Theorem 4.5′. *Let $1 < p < \infty$, $c \in \gamma$. Then the operator*

$$T : f \to Tf, \quad (Tf)(\zeta_0) = (\zeta_0 - c)^{\frac{1}{p'}} \int_\gamma \frac{f(\zeta)}{(\zeta - c)^{\frac{1}{p'}} \ln|\zeta - c|} \frac{d\zeta}{\zeta - \zeta_0}$$

is continuous in $L^p(\gamma)$.

Theorem 4.6. *Let $1 < p < \infty$ be a $\varphi - 2\pi$-periodic, continuous on $\mathbb{R}$ function and let x_0 be a point on $(-\pi, \pi)$. Assume*

$$\rho(x) = e^{\widetilde{\varphi}(x)}.$$

Then there exists a constant $c > 0$ such that the inequality

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x - x_0|^{p-1} \rho(x) dx \leq$$

$$\leq c \int_{-\infty}^{\infty} |f(x)|^p |x - x_0|^{p-1} \rho(x) \ln^p \frac{2\pi}{|x - x_0|} dx \tag{4.45}$$

holds for an arbitrary f for which the integral on the right-hand side of (4.45) *is finite.*

Proof. Choose arbitrarily $p_2 > p$ and assume

$$t = \frac{p-1}{p_2 - 1}. \tag{4.46}$$

Obviously, $0 < t < 1$. By Theorem 4.5 we have

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^{p_2} |x - x_0|^{p_2 - 1} dx \leq$$

$$. \leq c_1 \int_{-\pi}^{\pi} |f(x)|^{p_2} |x - x_0|^{p_2 - 1} \ln^q \frac{2\pi}{|x - x_0|} dx, \quad q \geq p_2. \tag{4.47}$$

Now we put $q = \frac{p}{t}$ in (4.47), what by (4.46) comes to the same thing that $q = \frac{p(p_2-1)}{p-1}$. Since $p_2 > p$, it is obvious that $q > p_2$. Further, we choose p_1 from the equality

$$\frac{1}{p} = \frac{t}{p_2} + \frac{1-t}{p_1}.$$

Since the function φ is continuous, we have for any t, $0 < t < 1$,

$$\int_{-\pi}^{\pi} |f(x)|^{p_1} (\rho(x))^{\frac{1}{1-t}} dx \leq c_2 \int_{-\pi}^{\pi} |f(x)|^{p_1} (\rho(x))^{\frac{1}{1-t}} dx. \tag{4.48}$$

The validity of this inequality follows from the results connected with the boundary value problem of linear conjugation with a continuous coefficient obtained in [140], [90] (see also Theorem 2.2, Ch. II). In particular, (4.48) is also valid for t defined by (4.46). Using Stein's interpolation theorem (0.20) and taking into account the inequalities (4.47) and (4.48), we obtain

$$\int_{-\pi}^{\pi} |\widetilde{f}(x)|^p |x - x_0|^{(p_2-1)t} \rho(x)\, dx \leq$$

$$\leq \int_{-\pi}^{\pi} |f(x)|^p |x - x_0|^{(p_2-1)t} \ln^{qt} \frac{2\pi}{|x - x_0|} \rho(x)\, dx, \tag{4.49}$$

Since $t = \frac{p-1}{p_2-1}$ and $q = \frac{p}{t}$, the inequality (4.49) proves the equality (4.45).

As far as $\widetilde{f}(x) = -\pi i S_\gamma(f(e)e^{-iy})(e^{ix})$, from the above result it immediately follows ■

Theorem 4.7. *Let $c \in \gamma$ and $\rho(\zeta) = \exp(K_\gamma \varphi)(\zeta)$, where φ is a function continuous on γ. Then the operator*

$$T : f \to Tf,$$

$$(Tf)(\zeta_0) = (\zeta_0 - c)^{\frac{1}{p'}} \rho(\zeta_0) \int_\gamma \frac{f(\zeta)}{(\zeta - c)^{\frac{1}{p'}} \rho(\zeta) \ln|\zeta - c|} \frac{d\zeta}{\zeta - \zeta_0}, \tag{4.50}$$

is continuous in $L^p(\gamma)$.

Remark. For the validity of Theorem 4.7 it is sufficient to assume that φ is continuous in a neighborhood of the point c and $\rho \in W_p(\Gamma)$.

4.5. An another necessary condition for belonging to the class $W_p(\Gamma)$.

Theorem 4.8. *If Γ is a closed Lyapunov curve and $\rho \in W_p(\Gamma)$, then $\rho = \exp[u + iS_\Gamma v]$, where u and v are bounded functions, and $\operatorname{Im} v = 0$, $\|v\|_\infty < \frac{\pi}{2\min(p,p')}$.*

Proof. The length of the curve Γ is assumed to be equal to 2π.

Since Γ is a smooth curve and $\rho \in W_p(\Gamma)$, according to Theorem 4.3, $\rho(t(s)) = \rho_0(e^{is})$ belongs to $W_p(\gamma)$. But then by Corollary 5 of Theorem 4.1, $\rho = \exp(u_1 + \widetilde{v})$. where u_1 and v are bounded real functions, and $\|v\|_\infty < \frac{\pi}{2\min(p,p')}$.

On the other hand, using (0.6) and the estimate

$$\Big|\frac{t'(\sigma)}{t(s)-t(\sigma)} - \frac{ie^{is}}{e^{is}-e^{i\sigma}}\Big| \le \frac{M}{|s-\sigma|^{1-\alpha}}$$

which is valid in the case of the Lyapunov curve for which $t' \in H(\alpha)$ (see, for e.g., [66]), we obtain

$$|\widetilde{v}(\sigma) - i(S_\Gamma v)(t(\sigma))| = |i\widetilde{v} + S_\Gamma v| =$$

$$= | - i(S_\Gamma v)(e^{i\sigma}) + \frac{i}{2}v(\sigma) + (S_\Gamma v)(t(\sigma))| \le$$

$$\le \frac{1}{2}\|v\|_\infty + \int\limits_0^{2\pi} |v|\Big|\frac{t'(s)}{t(s)-t(\sigma)} - \frac{ie^{is}}{e^{is}-e^{i\sigma}}\Big|ds \le$$

$$\le \|v\|_\infty + M\int\limits_0^{2\pi} |v|\frac{ds}{|s-\sigma|^{1-\alpha}} \le c\|v\|_\infty.$$

Now we have

$$\rho = \exp(u_1 + \widetilde{v}) = \exp[u_1 + \widetilde{v} - iS_\Gamma v + iS_\Gamma v] = \exp(u + iS_\Gamma v),$$

where $\|v\|_\infty < \frac{\pi}{2\min(p,p')}$, $u = u_1 + \widetilde{v} - iS_\Gamma v$ and the function u, as we have just proved, is bounded. ■

4.6. Some properties of Cauchy type integrals with densities from the classes $L^p(\Gamma;\rho)$. Let Γ be a closed curve of the class R bounding a finite domain D, $z = z(w)$ be a function which conformally maps U onto D and let $w = w(z)$ be the inverse to it function. Moreover, let $f \in L^p(\Gamma;\rho)$ and $\phi(z) = (K_\Gamma f)(z)$.

Consider the function

$$\Psi(w) = \phi(z(w)) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{f(t)dt}{t - z(w)}, \quad |w| < 1. \tag{4.51}$$

At this stage it is naturally to pose a problem and to show under which conditions this function is representable by the Cauchy integral in the domain U. It is also of some interest to find the conditions under which $\Psi(z) = (K_\gamma f)(w(z))$ is representable by the Cauchy integral in the domain D. These problem arise, for example, in considering the Riemann-Hilbert problem in domains with non-smooth boundaries (see §7, Ch. II).

Proposition 4.1. *Let $\Gamma \in R$, $\rho \in W_p(\Gamma)$, $(\rho\sqrt[p]{z'})^{-1} \in W_p(\gamma)$, $f \in L^p(\Gamma;\rho)$. Then there exists a function $\psi \in L^p(\gamma;\rho\sqrt[p]{z'})$ such that for the function Ψ given by* (4.51) *the representation*

$$\Psi(w) = \frac{1}{2\pi i}\int\limits_\gamma \frac{\psi(\zeta)d\zeta}{\zeta - w} \tag{4.52}$$

is valid, and $\Psi \in H^1$*.*

Proof. For the function $\rho \in W_p(\Gamma)$, there exists $\varepsilon > 0$ such that

$$\rho \in W_{p+\varepsilon}(\Gamma) \tag{4.53}$$

(see, e.g., [29]). But then, by Lemma 4.2, we have $\frac{1}{\rho} \in L^{p'+\varepsilon_1}(\Gamma)$ for some $\varepsilon_1 > 0$. Since $f\rho = f_0 \in L^p(\Gamma)$, we have that $f = \frac{f_0}{\rho} \in L^{1+\delta}(\Gamma)$ for some $\delta > 0$. This implies that $\phi(z) = (K_\Gamma f)(z) \in E^{1+\delta}(D)$ as $\Gamma \in R$. Consequently,

$$\phi_0(w) = \sqrt[1+\delta]{z'(w)}\phi(z(w)) \in H^{1+\delta}$$

(see, e.g., [43], p. 422). Therefore

$$\phi(z(w)) = \Psi(w) = \phi_0(w)\frac{1}{\sqrt[1+\delta]{z'(w)}}, \quad \phi_0 \in H^{1+\delta}. \tag{4.54}$$

Show that

$$\frac{1}{z'} \in H_{\eta_1}, \quad \eta_1 > 0. \tag{4.55}$$

First we will show that $[z'(\zeta)]^{-1} \in L^{\eta_1}(\gamma)$. We have

$$\begin{aligned}[z'(\zeta)]^{-\alpha} &= [\rho(z(\zeta))\sqrt[p]{z'(\zeta)}]^{-1}\rho(z(\zeta))[z'(\zeta)]^{\frac{1}{p}-\alpha} = \\ &= (\rho\sqrt[p]{z'})^{-1}[z']^{\frac{1-p\alpha}{p}}\rho.\end{aligned} \tag{4.56}$$

Here the multiplier $(p\sqrt[p]{z'})^{-1}$ belongs to $L^{p'}(\Gamma)$. By (4.53), $\rho^{p+\varepsilon}(\zeta)z'(\zeta) \in L^{p+\varepsilon}(\gamma)$, i.e., $\rho = \frac{\rho_0}{(z')^{\frac{1}{p+\varepsilon}}}$, $\rho_0 \in L^{p+\varepsilon}(\gamma)$. Therefore $\rho(z')^{\frac{1-p\alpha}{p}} = \rho_0(z')^{\frac{\varepsilon-\varepsilon p\alpha-p^2\alpha}{p(p+\varepsilon)}}$. Taking α sufficiently small, we find that $\frac{\varepsilon-\varepsilon p\alpha-p^2\alpha}{p(p+\varepsilon)} = \varepsilon_0 > 0$. Then $\rho(z')^{\varepsilon_0}$ is integrable in power $\lambda = \frac{(p+\varepsilon)\varepsilon_0}{p+\varepsilon+\varepsilon_0}$. It follows from (4.56) that $(z')^{-\alpha}$ is integrable in power $\beta = \frac{\lambda p'}{\lambda+p'}$. This means that $(z')^{-1} \in L^{\eta_1}(\gamma)$, $\eta_1 = \alpha\beta$. On this basis, we are able to establish that there exists $\nu > 0$ such that

$$w' \in E^{1+\nu}(D).$$

The fact that $z' \in H^1$ implies that $w' \in E^1(D)$ and, since Γ is a Smirnov curve, it suffices to show that $w'(t) \in L^{1+\nu}(\Gamma)$. We have

$$\begin{aligned}\int_\Gamma |w'(t)|^{1+\nu}|dt| &= \int_\gamma |w'(z(\zeta))|^{1+\nu}|z'(\zeta)||d\zeta| = \\ &= \int_\gamma \frac{|z'(\zeta)|}{|z'(\zeta)|^{1+\nu}}d\zeta = \int_{gm} \frac{|d\zeta|}{|z'(\zeta)|^\nu} < \infty\end{aligned}$$

for $\nu \leq \eta_1$. Thus we have proved that $w' \in E^{1+\nu}(D)$. This means that $w_0(\zeta) = \sqrt[1+\nu]{z'(\zeta)}w'(z(\zeta))$ belongs to $H^{1+\nu}$. On the other hand,

$$w_0(\zeta) = \frac{\sqrt[1+\nu]{z'(\zeta)}}{z'(\zeta)} = \frac{1}{[z'(\zeta)]^{\nu(1+\nu)^{-1}}}.$$

Therefore $(z'(\zeta))^{-1} = [w_0(\zeta)]^{\frac{1+\nu}{\nu}} \in H^\nu$, $\nu \leq \eta_1$ and hence (4.55) is proved.

Owing to (4.54) and (4.55), we can conclude that

$$\Psi(w) \in H^\eta \tag{4.57}$$

for some $\eta > 0$.

From (4.51) we have

$$\Psi^+(\zeta) = \phi^+(z(\zeta)) = \frac{1}{2}f(z(\zeta)) + \frac{1}{2}(S_\Gamma f)(z(\zeta)).$$

Since $f \in L^p(\Gamma;\rho)$ and $\rho \in W_p(\Gamma)$, we find that $\phi^+(t) \in L^p(\Gamma;\rho)$. Hence

$$\int_\gamma |\phi^+(z(s))\rho(z(\zeta))|^p |z'(\zeta)||d\zeta| < \infty. \tag{4.58}$$

This inequality makes it possible to estimate the norm $\phi^+(z(\zeta))(\equiv \Psi^+(\zeta))$ in $L(\gamma)$. Indeed,

$$\int_a^b |\phi^+(z(\zeta))||d\zeta| = \int_\gamma |\phi^+(z(\zeta))\rho(z(\zeta))\sqrt[p]{z'(\zeta)}||\rho(z(\zeta))\sqrt[p]{z'(\zeta)}|^{-1}|d\zeta| \leq$$

$$\leq \left(\int_\gamma |\phi^+(z(\zeta))\rho(z(\zeta))\sqrt[p]{z'(\zeta)}|^p|d\zeta|\right)^{\frac{1}{p}} \left(\int_\gamma |\rho(z(\zeta))\sqrt[p]{z'(\zeta)}|^{-p'}|d\zeta|\right)^{\frac{1}{p'}} < \infty,$$

by virtue of (4.58) and of the assumption $\rho\sqrt[p]{z'} \in W_p(\gamma)$. Thus, taking into account (4.57), we conclude that $\Psi \in H^1$, and therefore

$$\Psi(w) = \frac{1}{2\pi i}\int_\gamma \frac{\Psi^+(\zeta)d\zeta}{\zeta - w} = \frac{1}{2\pi i}\int_\gamma \frac{\phi^+(z(\zeta))d\zeta}{\zeta - w}.$$

The function $\psi(\zeta) = \phi^+(z(\zeta))$, due to (4.58), belongs to $L^p(\gamma;\rho\sqrt[p]{z'})$. Hence the equality (4.52) is valid. ■

Now we will show the validity of the inverse assertion.

Proposition 4.2. *Let* $\Gamma \in R$, $\psi \in L^p(\gamma;\rho(z(\zeta))\sqrt[p]{z'(\zeta)})$, $\rho(t) \in W_p(\Gamma)$, $\rho\sqrt[p]{z'} \in W_p(\gamma)$ *and let the function* Ψ *be given by* (4.52). *Then the function*

$$\phi(z) = \Psi(w(z)) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(\zeta)d\zeta}{\zeta - w(z)} \tag{4.59}$$

is representable by the Cauchy integral in the domain D:

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)dt}{t-z}, \tag{4.60}$$

where $f(t) = \phi^+(t) = \Psi^+(\zeta(t)) \in L^p(\Gamma; \rho)$.

Proof. The Cauchy type integral in a circle belongs to $\cap_{\delta<1} H^\delta$ ([133], p. 94), and $z' \in H^1$. Therefore $\sqrt[p]{z'(w)}\Psi(w) \in H^{\eta_0}$ for some $\eta_0 > 0$, and hence $\Psi(w(z)) = \phi(z) \in E^{\eta_0}(D)$.

Further,

$$\int_{\Gamma} |\Psi^+(w(t))||dt| = \int_{\gamma} |\Psi^+(\zeta)||z'(\zeta)||d\zeta| =$$

$$= \int_{\gamma} |\Psi^+(\zeta)\rho(z(\zeta))||z'(\zeta)|^{\frac{1}{p}}|z'(\zeta)|^{\frac{1}{p'}}|\rho(z(\zeta))|^{-1}|d\zeta| \leq$$

$$\leq \Big(\int_{\gamma} |\Psi^+(\zeta)\rho(z(\zeta))\sqrt[p]{z'(\zeta)}|^p|d\zeta|\Big)^{\frac{1}{p}} \Big(\int_{\gamma} |\rho(z(\zeta))|^{-p'}|z'(\zeta)||d\zeta|\Big)^{\frac{1}{p'}} < \infty.$$

The last conclusion follows from the assumptions that $\rho\sqrt[p]{z'} \in W_p(\gamma)$ and $\rho \in W_p(\Gamma)$ (which imply that $\frac{1}{\rho} \in L^{p'}(\Gamma)$, and hence $\frac{1}{\rho(z(\zeta))} \in L^{p'}(\gamma; \sqrt[p]{z'(\zeta)})$).

Thus we have shown that $\phi(z) \in E^1(D)$, i.e.,

$$\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi^+(t)dt}{t-z} =$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi^+(w(t))dt}{t-z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)dt}{t-z}.$$

Here $f \in L^p(\Gamma; \rho)$. Indeed,

$$\int_{\Gamma} |f(t)||\rho(t)|^p|dt| = \int_{\Gamma} |\Psi^+(w(t))\rho(t)|^p|dt| =$$

$$= \int_{\gamma} |\Psi^+(\zeta)\rho(z(\zeta))|^p|z'(\zeta)||d\zeta| = \int_{\gamma} |\Psi^+(\zeta)\rho(z(\zeta))\sqrt[p]{z'(\zeta)}|^p|d\zeta|. \tag{4.61}$$

According to the assumptions that $\rho\sqrt[p]{z'} \in W_p(\gamma)$, and $\psi \in L^p(\gamma; \rho\sqrt[p]{z'})$, we find that $\Psi^+ \in L^p(\Gamma; \rho)$, and therefore on the basis of (4.61) we conclude that $f \in L^p(\Gamma; \rho)$. ∎

§ 5. On Singular Integrals in the Mean

5.1. Definition of an integral in the mean and some of its properties. Let Γ be a rectifiable Jordan curve and let $t = t(s)$ be its equation with respect to the arc abscissa. As usual, we extend this function and the functions given on Γ with the period l to $\mathbb{R}$. If $0 < \varepsilon < \frac{l}{4}$, then $m_\varepsilon = \inf |t(s) - t(s_0)| > 0$, where the greatest lower bound is taken on the set $\{(s, s_0) : 0 < s_0 < l, s_0 + \varepsilon < s < s_0 - l + \varepsilon\}$ (see §3). Therefore, if $\varphi \in L^p(\Gamma)$, $p \geq 1$, then the function

$$(M_\varepsilon \varphi)(t_0) = \frac{1}{\pi i} \int\limits_{s_0+\varepsilon}^{s_0+l-\varepsilon} \frac{\varphi(t(s))t'(s)ds}{t(s) - t(s_0)} \tag{5.1}$$

belongs also to $L^p(\Gamma)$, and

$$||M_\varepsilon \varphi||_p \leq \frac{l^{\frac{1}{p'}}}{m_\varepsilon} ||\varphi||_p. \tag{5.2}$$

Definition. We will say that there exists a p-mean singular integral of the function $\varphi \in L^p(\Gamma)$, if there exists a function $\psi \in L^p(\Gamma)$ such that

$$\lim_{\varepsilon \to 0} \int\limits_0^l |(M_\varepsilon \varphi)(t(s)) - \psi(t(s))|^p ds = 0. \tag{5.3}$$

The function ψ will be denoted by $S_\Gamma^{(p)} f$, and let $S_\Gamma^{(p)}$ be an operator which puts the function φ into correspondence with the function $S_\Gamma^{(p)} \varphi$.

It is evident that if $S_\Gamma^p \varphi$ exists, then it coincides with $S_\Gamma \varphi$ almost everywhere. However, it may happen that $S_\Gamma^{(p)} \varphi$ does not exist for some φ, whereas $S_\Gamma \varphi$ exists for all $\varphi \in L^p(\Gamma)$, $p \geq 1$. In particular, we will show that such a situation takes place for $p = 1$. First we prove the validity of the following

Lemma 5.1. *If $S_\Gamma^{(p)} \varphi$ exists for all $\varphi \in L^p(\Gamma)$, $p \geq 1$, then the operator $S_\Gamma^{(p)}$ is continuous in $L^p(\Gamma)$.*

Proof. Let $\mathcal{M}_n \varphi = M_{\frac{1}{n}} \varphi$ and let $\mathcal{M}_n : \varphi \to \mathcal{M}_n \varphi$ be an operator defined on $L^p(\Gamma)$. Since $L^p(\Gamma)$ is a Banach space, the operator $\mathcal{M}_n$, by (5.2), is continuous on it and, by definition, for every function φ the sequence $\mathcal{M}_n \varphi$ converges to $S^{(p)} \varphi$, the well-known theorem on the continuity of the limiting operator, that is of the operator S_Γ^p, is applicable here. ■

Let now $\Gamma = \gamma$. Then $S_\gamma^{(1)}$ fails to exist for all $\varphi \in L(\gamma)$, since otherwise the operator $S_\gamma^{(1)}$ and hence the operator S_γ would be continuous in $L(\gamma)$. But it is not true.

A set of those functions φ from $L^p(\Gamma)$ for which $S_\Gamma^{(p)}\varphi$ exists will be denoted by $D(S_\Gamma^{(p)})$.

Lemma 5.2. *If Γ is a closed curve of the class R bounding the domain D, then for any $\varphi \in D(S_\Gamma^{(p)})$, $1 \le p < \infty$, the Cauchy type integral $\phi = K_\Gamma\varphi$ belongs to $E^p(D)$.*

Proof. If $p > 1$, then $\phi = K_\Gamma\varphi$ belongs to $E^p(D)$ according to Theorem 3.3. Let $p = 1$. Since $\Gamma \in \bigcap\limits_{q<1} R_{1,p}$ (see subsection 3.4), $K_\Gamma\varphi$ belongs to the class $\bigcap\limits_{q<1} E^q(D)$ (according to corollary of Theorem 3.3). As $\varphi \in D(S_\Gamma^{(1)})$, we have $(S_\Gamma\varphi) \in L(\Gamma)$. Taking into consideration that the curve from R is a Smirnov curve, we conclude that $(K_\Gamma\varphi) \in E^1(D)$. ■

In addition to the above-said, we note that the equality $S_\Gamma(S_\Gamma^{(p)}\varphi) = \varphi$ is valid. Indeed, if Γ bounds the domains D^+ and D^-, then $\phi = K_\Gamma\varphi \in E^p(D^\pm) \subset E^1(D^\pm)$. For $z \in D^+$ we then have

$$\begin{aligned} 0 = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\phi^-(\zeta)d\zeta}{\zeta - z} = \frac{1}{4\pi i}\int\limits_\Gamma \frac{-\varphi + S_\Gamma\varphi}{\zeta - z}\, d\zeta = \\ = -\frac{1}{4\pi i}\int\limits_\Gamma \frac{\varphi - S_\Gamma\varphi}{\zeta - z}\, d\zeta, \end{aligned} \tag{5.4}$$

whence

$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{S_\Gamma^{(p)}\varphi d\zeta}{\zeta - z} = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(\zeta)d\zeta}{\zeta - z} = \phi(z), \quad z \in D^+. \tag{5.5}$$

Taking into consideration the equality $S_\Gamma\phi^+ = \phi^+$, we obtain the desired equality $S_\Gamma(S_\Gamma^{(p)}\varphi) = \varphi$.

5.2. Connection between the singular integral in the mean and the Cauchy type integral. The existence almost everywhere of a singular integral $(S_\Gamma f)(t_0)$ is, by I. I. Privalov's theorem, equivalent to that of angular boundary values of the Cauchy type integral $(K_\Gamma f)(z)$. It turns out that in the case of smooth curves, the existence of the mean singular integral $S_\Gamma^{(p)} f$ is equivalent to the belonging to the Smirnov class E^p of the function $K_\Gamma f$.

Theorem 5.1. *Let Γ be a closed smooth Jordan curve bounding the domains D^+ and D^-, and $f \in L^p(\Gamma)$, $p \ge 1$. In order for the integral $K_\Gamma f$ to belong to the class $E^p(D^+)$, $(E^p(D^-))$ it is necessary and sufficient that the p-mean singular integral $S_\Gamma^{(p)} f$ exist.*

Proof. Sufficiency. Let there exist $S_\Gamma^{(p)} f$. We construct a sequence of the curves $\Gamma_{\lambda_n} \subset D^+(D^-)$ such that $\sup_n \int_{\Gamma_{\lambda_n}} |K_\Gamma f|^p |dz| < \infty$.

Denote by $\varphi(\sigma)$ the angle formed by the tangent to Γ at $t(\sigma)$ and the real axis. The function φ is continuous on $[0, l]$ and $\varphi(l) = \varphi(0) + 2\pi$. Therefore there exists a real polynomial $q(\sigma)$ such that $|\varphi(\sigma) - q(\sigma)| < \theta \frac{\pi}{2}$, $\theta \in (0,1)$, $\varphi(0) = q(0)$, $\varphi(l) = q(l)$. Denote by ρ the standard radius of the curve Γ corresponding to the angle $\alpha_0 < (1-\theta)\frac{\pi}{2}$ (for definition and for the properties of the standard radius see [106], pp. 18–20). Since Γ is a smooth curve, $|t(s) - t(\sigma)| \geq ms(t(s), t(\sigma))$, $m > 0$. Let $0 < \lambda < \min(\rho, m) = \lambda_0$. Consider the curve Γ_λ given parametrically by the equation

$$z_\lambda(\sigma) = t(\sigma) + i\lambda \exp iq(\sigma), \quad 0 \leq \sigma \leq l.$$

Obviously, Γ_λ is a closed rectifiable curve. Show that if $z_\lambda(0)$ is in $D^+(D^-)$, then Γ_λ lies in $D^+(D^-)$. Indeed, the point $z_\lambda(\sigma^*)$ belonging to Γ would otherwise lie on the curve Γ_λ. But then the ends of the segment $[t(\sigma^*), z_\lambda(\sigma^*)]$ lie on the standard arc, since $|t(\sigma^*) - z_\lambda(\sigma^*)| = \lambda < \rho$. The vector with the origin at $t(\sigma^*)$ and with the end at $z_\lambda(\sigma^*)$ forms with the tangent at the point $t(\sigma^*)$ the angle $\frac{\pi}{2} - [\varphi(\sigma^*) - q(\sigma^*)]$ lesser than $\frac{\pi}{2} - \theta\frac{\pi}{2} > \alpha_0$. (This follows from the equality $z_\lambda(\sigma^*) - t(\sigma^*) = \lambda \exp(\frac{i}{2}\pi + q(\sigma^*)) = \lambda \exp[i\varphi(\sigma^*) + i(\frac{\pi}{2} + q(\sigma^*) - \varphi(\sigma^*))]$ and also from the condition $|\varphi(\sigma) - q(\sigma)| < \theta\frac{\pi}{2}$). But this contradicts the property of the standard radius. Thus, Γ_λ lies in either of the domains D^+ or D^-. For definiteness we assume that $\Gamma_\lambda \subset D^+$.

Show the existence of such sequences λ_n, $\lambda_n \to 0$ for which Γ_{λ_n} are Jordan curves. Assume the contrary. Let for any $\lambda \in (0, \lambda_0)$ Γ_λ intersect itself, that is, there is a pair of numbers s_λ and σ_λ such that $s_\lambda \neq \sigma_\lambda$ but $z_\lambda(s_\lambda) = z_\lambda(\sigma_\lambda)$, i.e.,

$$t(s_\lambda) + i\lambda \exp[iq(s_\lambda)] = t(\sigma_\lambda) + i\lambda \exp[iq(\sigma_\lambda)]. \tag{5.6}$$

From (5.6) follows

$$\frac{t(s_\lambda) - t(\sigma_\lambda)}{s_\lambda - \sigma_\lambda} = i\lambda \frac{\exp iq(\sigma_\lambda) - \exp iq(s_\lambda)}{s_\lambda - \sigma_\lambda}. \tag{5.7}$$

The expression $[\exp iq(\sigma_\lambda) - \exp iq(s_\lambda)](s_\lambda - \sigma_\lambda)^{-1}$ is bounded. On the other hand, $|t(s_\lambda) - t(\sigma_\lambda)| \geq m|s_\lambda - \sigma_\lambda|$, and from (5.7) we obtain

$$m \leq \overline{\lim}_{\lambda \to 0} \left| \frac{t(s_\lambda) - t(\sigma_\lambda)}{s_\lambda - \sigma_\lambda} \right| =$$
$$= \lim_{\lambda \to 0} \left| i\lambda \frac{\exp iq(\sigma_\lambda) - \exp iq(s_\lambda)}{s_\lambda - \sigma_\lambda} \right| = 0.$$

This contradiction shows that the assumption for Γ_λ to be non-Jordan for all λ is invalid. Hence there exists at least one value $\lambda_1 \in (0, \lambda_0)$ for which Γ_{λ_1} is a Jordan curve.

On the basis of the above arguments, we can state that the same is valid for the segment $(0, \lambda_1)$ which proves the possibility to distinguish Jordan curves Γ_{λ_n}.

Suppose $d(z, \Gamma) = \inf_{\zeta \in \Gamma} |z - \zeta|$, $d(\Gamma_\lambda, \Gamma) = \inf_{z \in \Gamma_\lambda} d(z, \Gamma)$ and let $D^+_{\Gamma_\lambda}$ and $D^-_{\Gamma_\lambda}$ be the domains bounded by Γ_λ. It is obvious that $d(\Gamma_\lambda, \Gamma) \to 0$ as $\lambda \to 0$. If z is an arbitrary point from D^+ and $d(\Gamma_\lambda, \Gamma) < d(z, \Gamma)$, then z falls into the domain $D^+_{\Gamma_\lambda}$. Therefore the domains $D^+_{\Gamma_\lambda}$ exhaust D^+, and hence $\Gamma_\lambda \to \Gamma$.

Consider the difference

$$\int_0^l \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - [t(\sigma_0) + i\lambda \exp iq(\sigma_0)]} - \int_{\sigma_0+\varepsilon}^{\sigma_0+l-\varepsilon} \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - t(\sigma_0)}.$$

Let $\psi(\sigma_0) = q(\sigma_0) - \varphi(\sigma_0)$. Then $|\psi(\sigma_0)| \le \frac{\theta\pi}{2}$, $0 < \theta < 1$. Now we use theorem from [130]: If Γ is a smooth curve, $f \in L^p(\Gamma)$, $p \ge 1$ then

$$\lim_{\varepsilon \to 0} \int_0^l \Bigg| \int_0^l \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - z_\varepsilon(\sigma_0)} - i\pi f(t(\sigma_0)) - \\ - \int_{I(\sigma_0,\varepsilon)} \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - t(\sigma_0)} \Bigg|^p d\sigma_0 = 0, \tag{5.8}$$

where $z_\varepsilon(\sigma_0) = t(\sigma_0) + i\varepsilon \exp[i(\varphi(\sigma_0) - \psi(\sigma_0))]$, $|\psi(\sigma_0)| < \theta\frac{\pi}{2}$, $0 < \theta < 1$, $I(\sigma_0, \varepsilon) = (\sigma_0 + \varepsilon, \sigma_0 + l - \varepsilon)$.

Taking into account the fact that $q(\sigma_0) = \varphi(\sigma_0) + \psi(\sigma_0)$, provided $|\psi(\sigma_0)| < \frac{\theta\pi}{2}$, we obtain

$$\lim_{n \to \infty} \int_0^l \Bigg| \int_0^l \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - [t(\sigma_0) + i\lambda_n \exp iq(\sigma_0)]} - \pi i f(t(\sigma_0)) - \\ - \int_{I(\sigma_0,\lambda_n)} \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - t(\sigma_0)} \Bigg|^p d\sigma_0 = 0. \tag{5.9}$$

By assumption of the theorem, $S^{(p)}_\Gamma f$ exists. Therefore the last equality results in

$$\int_0^l \Bigg| \int_0^l \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - [t(\sigma_0) + i\lambda_n \exp iq(\sigma_0)]} \Bigg|^p d\sigma_0 \le \pi(\|f\|_p + \|S^{(p)}_\Gamma f\|_p) = C.$$

Now we have

$$\int_{\Gamma_{\lambda_n}} |\phi(z)|^p |dz| =$$

$$= \int_0^l \left| \int_0^l \frac{f(t(\sigma))t'(\sigma)d\sigma}{t(\sigma) - [t(\sigma_0) + i\lambda_n \exp iq(\sigma_0)]} \right|^p |t'(\sigma_0) + iq'(\sigma)\lambda_n \exp iq(\sigma_0)| d\sigma_0 \leq$$

$$\leq MC, \quad M = 1 + \lambda_0 \max_{0 \leq \sigma_0 \leq l} |q'(\sigma_0)|$$

which implies that $\phi \in E^p(D^+)$.

If $z_\lambda(0)$ falls into D^-, then considering the curves $z_\lambda^- = t(\lambda) - i\lambda \exp iq(\sigma)$, we analogously construct a sequence of curves $\Gamma_{\lambda_n}^- \subset D^-$ and show that $\phi \in E^p(D^-)$.

Necessity. Let $p > 1$ and $f \in L^p(\Gamma)$. Since $\Gamma \in R$, we have that $\phi = K_\Gamma f$ belongs to the classes $E^p(D^\pm)$ (see, e.g., Theorem 3.4). Therefore the equalities

$$\frac{1}{4\pi i} \int_\Gamma \frac{f + S_\Gamma f}{t - z} dt = \begin{cases} \phi(z), & z \in D^+, \\ 0, & z \in D^- \end{cases} \tag{5.10}$$

and

$$\frac{1}{4\pi i} \int_\Gamma \frac{f - S_\Gamma f}{t - z} dt = \begin{cases} 0, & z \in D^+, \\ \phi(z), & z \in D^- \end{cases} \tag{5.11}$$

are valid.

Using (5.8) we obtain

$$\lim_{\lambda \to 0} \int_0^l \left| \frac{1}{2\pi i} \int_0^l \frac{[f(t(\sigma)) - (S_\Gamma f)(t(\sigma))]t'(\sigma)d\sigma}{t(\sigma) - [t(\sigma_0) + i\lambda \exp i(\varphi(\sigma_0) + \psi(\sigma_0))]} - \right.$$

$$- \frac{f(t(\sigma_0)) - (S_\Gamma f)(t(\sigma_0))}{2} -$$

$$\left. - \frac{1}{2\pi i} \int_{I_0(\sigma_0, \lambda)} \frac{f(t(\sigma)) - (S_\Gamma f)(t(\sigma))}{t(\sigma) - t(\sigma_0)} t'(\sigma) d\sigma \right|^p d\sigma_0 = 0. \tag{5.12}$$

But because of (5.11), the first summand of the sum under the integral sign in (5.12) equals zero, and therefore

$$\lim_{\lambda \to 0} \int_\Gamma \left| \frac{1}{2\pi i} \int_{I(\sigma_0, \lambda)} \frac{(f - S_\Gamma f)dt}{t - t_0} - \frac{(S_\Gamma f)(t_0) - f(t_0)}{2} \right|^p d\sigma_0 = 0. \tag{5.13}$$

Note that the equality similar to (5.9) is also valid when substituting λ_n by $(-\lambda_n)$, and we use it in the case when the density of the integral equals $f + S_\Gamma f$. Then, taking into account (5.10), we arrive at

$$\lim_{\lambda \to 0} \int_\Gamma \left| \frac{1}{2\pi i} \int_{I(\sigma_0, \lambda)} \frac{f + S_\Gamma f}{t - t_0} dt - \frac{f(t_0) + (S_\Gamma f)(t_0)}{2} \right|^p d\sigma_0 = 0. \tag{5.14}$$

From (5.13) and (5.14) follows the existence of p-mean singular integrals of the functions $f \pm S_\Gamma f$, and

$$S_\Gamma^{(p)}(f - S_\Gamma f) = S_\Gamma f - f, \quad S_\Gamma^{(p)}(f + S_\Gamma f) = S_\Gamma f + f.$$

This implies the existence of the integral $S^{(p)} f$.

If $p = 1$ and $\phi = K_\Gamma f \in E^1(D^+)$, then $S_\Gamma \phi^+ = \phi^+$, and we can easily conclude that $(K_\Gamma f)(z)$ belongs to $E^1(D^-)$ (for details see the proof of Theorem 3.4), and therefore the equalities (5.10)–(5.11) are valid. On the basis of these equalities we can as above show the existence of the integral $S_\Gamma^{(1)} f$, since (5.8) is valid for $p = 1$. ■

Corollary 1. *If $p > 1$ and Γ is a closed smooth curve, then $S_\Gamma^{(p)}$ exists for all $f \in L^p(\Gamma)$.*

Corollary 2. *If Γ is a closed smooth curve, then there exists a constant C_p such that for all $\varepsilon > 0$ simultaneously*

$$\|M_\varepsilon f\|_p \le C_p \|f\|_p, \quad p > 1.$$

Since the sequence of the operators $M_{\frac{1}{n}} f$, due to the above arguments, converges for every $f \in L^p(\Gamma)$, from the Banach-Steinhaus theorem we obtain the assertion of the corollary.

Remark. As far as an unclosed smooth curve Γ can always be supplemented to a closed Jordan smooth curve, the assertions of Corollaries 1 and 2 hold valid for such curves as well.

§ 6. Application of Lebesgue Integral Generalizations to Cauchy Type Integrals

In the present section some properties of conjugate functions and connected with them Cauchy type integrals will be studied using a generalization of a Lebesgue integral.

Let f be a 2π-periodic summable on $(0, 2\pi)$ function,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \quad \text{and} \quad \sum_{k=1}^{\infty}(-b_k \cos kx + a_k \sin kx)$$

be the Fourier series of f and its conjugate, respectively.

$$\widetilde{f}(x) = -\frac{1}{\pi}\int_0^{2\pi} f(t)\frac{1}{2}\operatorname{ctg}\frac{1}{2}(t - x)dt \tag{6.1}$$

is the function conjugate to f.

It is known (Kolmogorov [84], Smirnov [145], Titchmarsh [154], see also [169], pp. 153–154 and [5] pp. 585–591) that if $\widetilde{f} \in L(0, 2\pi)$, then the series conjugate to the Fourier series of $f \in L(0, 2\pi)$ is the Fourier series of the function $\widetilde{f}$. It is also known (Smirnov [145], Privalov [133], p. 116) that a Cauchy-Lebesgue type integral with summable boundary value is representable in a circle by a Cauchy integral. But as far as the conjugate functions and the boundary value of the Cauchy type integral are not always summable, there arises the problem of extending the notion of the Lebesgue integral so that these functions would be integrable in the new sense.

A.P. Kolmogorov [84] was the first who treated this problem and proved that for any $f \in L(0, 2\pi)$ the function $\widetilde{f}$ is B-integrable on $[0, 2\pi]$ and the conjugate series is the Fourier series (B) for $\widetilde{f}$ (for definition of a B-integral as well as for the proof of the theorem see [169], pp. 153–154). Titchmarsh [154] has obtained an analogous result for an A-integral. Later, by means of the A-integral P. L. Ul'yanov has established a number of significant properties of conjugate functions [156] (for definition of an A-integral and the proof of results obtained by Titchmarsh and Ul'yanov concerning conjugate functions see also [5], pp. 585–591) and of the Cauchy type integrals. In particular, P. L. Ul'yanov [157], [158] has shown that under certain assumptions regarding the lines of integration, the boundary value of the Cauchy-Lebesgue type integral is A-integrable, while the function representable by the Cauchy-Lebesgue type integral is representable by the Cauchy A-integral as well. The paper [59] is devoted to application of an A-integral to the theory of a Cauchy type integral and to treatment of a non-homogeneous boundary value problem of linear conjugation.

From what has been said above it is obvious that the A-integral turned out to be a rather useful tool for investigation of some questions of trigonometric Fourier series and Cauchy type integrals. On the other hand, the A-integral, because of its generality, possesses specific disadvantages (see, e.g., [155] and [162] and bibliography given in [162]) which sometimes make its application difficult. Therefore it is much better to define a more simple integral which would answer the same purpose. Hence it is of interest to illustrate these specific properties of the A-integral as well as of conjugate functions and Cauchy type integrals which lead to the above-mentioned results.

As is shown in [58], [60], [61] specific properties of these integrals do not affect most of the results obtained for conjugate functions and Cauchy-Lebesgue type integrals by means of the A– and B-integrals. They hold valid for any generalization of the Lebesgue integral in a sense of which a conjugate function is integrable and its integral equals zero. That is, any integral, being a linear functional ϕ defined on a linear family of functions given on $[a, b]$, containing all summable functions and their conjugates $\widetilde{f}$, and satisfying the condition: if $f \in L(0, 2\pi)$, then

$$\phi(f) = \int_0^{2\pi} f(x)dx \quad \text{and} \quad \phi(\widetilde{f}) = 0,$$

is fitted for this aim.

In this section we will give the definition of a rather simple and convenient for application functional (integral).

Bearing in mind the condition $\phi(\widetilde{f}) = 0$, we note (by V.I. Smirnov's theorem (see, e.g., [5], p. 583)) that if $\widetilde{f}$ conjugate to $f \in L(0, 2\pi)$ is summable, then

$$\int_0^{2\pi} \widetilde{f}(x)dx = 0. \tag{6.2}$$

This means that the integrals in the iterated integral $\int_0^{2\pi} dx \int_0^{2\pi} f(t)$ ctg $\frac{t-x}{2} dt$ can be interchanged.

Quote here one assertion following from Theorem 6.2 which will be proved below.

Let $f \in L(0, 2\pi)$. There exists a measurable set $E \subset [0, 2\pi]$ of measure 2π such that if a, $b \in E$, then the function

$$h(x) = \int_0^{2\pi} f(t)\frac{1}{2} \operatorname{ctg} \frac{t-x}{2} dt - \int_a^b f_1(t)\frac{1}{2\lambda} \operatorname{ctg} \frac{t-x}{2\lambda} dt,$$

where f_1 is the restriction of f on $[a, b]$, and $\lambda = \frac{b-a}{2\pi}$, is summable on $[a, b]$. The above mentioned and the next results of this section follows from the two last facts.

6.1. $\widetilde{L}$-integral and conjugate functions. We say that a function f is $\widetilde{L}$-integrable on $[a, b]$ if it can be represented as

$$f = g + \widetilde{h}, \quad \text{where} \quad g, h \in L(a, b), \tag{6.3}$$

$$\widetilde{h}(x) = -\frac{1}{\pi} \int_a^b h(t)\frac{1}{2\lambda} \operatorname{ctg} \frac{1}{2\lambda}(t-x)dt \quad \text{with} \quad \lambda = (b-a)/2\pi \tag{6.4}$$

is conjugate to h on $[a, b]$. The number

$$\widetilde{L}(f) = (\widetilde{L}) \int_a^b f(x)dx \equiv \int_a^b g(x)dx$$

will be termed an $\widetilde{L}$-integral of f on $[a, b]$. The quantity $\widetilde{L}(f)$ does not depend on the manner how f is represented in terms of (6.3). Indeed, let besides (6.3) we have

$$f = g_1 + \widetilde{h}_1, \quad \text{where} \quad g_1, h_1 \in L(a, b). \tag{6.5}$$

Then, by definition,

$$(\widetilde{L})\int\limits_a^b f(x)dx = \int\limits_a^b g_1(x)dx.$$

From (6.3) and (6.5) we find that $g - g_1 + (\widetilde{h - h_1}) = 0$. Further, according to (6.2),

$$\int\limits_a^b (\widetilde{h - h_1})(x)dx = \int\limits_a^b (g_1 - g)(x)dx = 0$$

and hence

$$\int\limits_a^b g(x)dx = \int\limits_a^b g_1(x)dx.$$

Remark. It is evident that the function conjugate the summable function is $\widetilde{L}$-integrable, and its $\widetilde{L}$-integral equals zero. Obviously, any integral being an extension of the Lebesgue integral in whose sense the conjugate function is integrable and its integral equals zero, contains also the $\widetilde{L}$-integral. Therefore all the results obtained by means of the $\widetilde{L}$-integral are valid for the above mentioned integrals (in particular, for A- and B-integrals which are the extensions of the Lebesgue integral).

Let a 2π-periodic bounded function φ satisfy the condition

$$\int\limits_0^{2\pi} |\varphi(t) - \varphi(x)||\operatorname{ctg}\frac{1}{2}(t-x)|dt < C \quad (C \text{ is a constant}) \tag{6.6}$$

(such, for example, are the functions φ for which $w(\varphi, \sigma)\sigma^{-1} \in L(0, \pi)$, where $w(\varphi; \sigma)$ is the module of continuity of φ).

Theorem 6.1. *If $f \in L(0, 2\pi)$ and φ satisfies condition (6.6), then $\varphi\widetilde{f} \in \widetilde{L}(0, 2\pi)$, and we have the equality*

$$(\widetilde{L})\int\limits_0^{2\pi} \varphi(x)\widetilde{f}(x)dx = -\int\limits_0^{2\pi} \widetilde{\varphi}(x)f(x)dx. \tag{6.7}$$

Proof. Indeed, $\varphi(x)\widetilde{f}(x) = (\widetilde{\varphi f})(x) + \frac{1}{\pi}\int_0^{2\pi} f(t)[\varphi(x) - \varphi(t)]\frac{1}{2}\operatorname{ctg}\frac{1}{2}(t-x)dt$, where the function $f(t)[\varphi(x) - \varphi(t)]\frac{1}{2}\operatorname{ctg}\frac{1}{2}(t-x)$ is, by Fubini's theorem and condition (6.6), summable on the square $[0, 2\pi, 0, 2\pi]$. Hence $\varphi\,\widetilde{f} \in \widetilde{L}(0, 2\pi)$. Taking the $\widetilde{L}$-integral on the both sides of the last equality and changing the order of integration, we obtain the equality (6.7). ■

Corollary 1. *A series conjugate to Fourier trigonometric series of function $f \in L(0,2\pi)$ is the Fourier $\widetilde{L}$-series of the function $\widetilde{f}$ if coefficients defined by the $\widetilde{L}$-integral (in particular, the Fourier $A-$ and B-series with coefficients defined respectively).*

This follows from equality (6.7) taking into consideration that the functions $\cos nx$ and $-\sin nx$ $(n = 0, 1, 2, \dots)$ are self-conjugate.

Corollary 2. *If $f \in L(0,2\pi)$, $u(r,\theta)$ is its Poisson integral and $v(r,\theta)$ is harmonically conjugate to $u(r,\theta)$ function, i.e.,*

$$v(r,\theta) = -\frac{1}{\pi}\int_0^{2\pi} f(t)\frac{r\sin(t-\theta)}{1-2r\cos(t-\theta)+r^2}dt,$$

then $v(r,\theta)$ is representable by the Poisson $\widetilde{L}$-integral (in particular, by of A integral, of B integral) of the function $\widetilde{f}$,

$$v(r,\theta) = \frac{1}{2\pi}(\widetilde{L})\int_0^{2\pi} \widetilde{f}(t)\frac{1-r^2}{1-2r\cos(t-\theta)+r^2}dt.$$

Theorem 6.2. *Let $f \in L(0,2\pi)$ and φ satisfies the condition (6.6). Then there exists the set $E \subset [0,2\pi]$ of measure 2π depending only on f such that if $a, b \in E$, then $\varphi\widetilde{f} \in \widetilde{L}(a,b)$ and*

$$(\widetilde{L})\int_a^b \varphi(x)\widetilde{f}(x)dx = -\frac{1}{\pi}\int_0^{2\pi} f(t)dt\int_a^b \varphi(x)\frac{1}{2}\operatorname{ctg}\frac{1}{2}(t-x)dt. \tag{6.8}$$

Proof. Show that we can take as E the set of all x for which $f(\cdot)\ln|\cdot - x| \in L$.

Let $a, b \in E$, $x \in (a,b)$. Consider the equality

$$\varphi(x)\widetilde{f}(x) - \varphi(x)\Big(-\frac{1}{\pi}\int_a^b f_1(t)\frac{1}{2\lambda}\operatorname{ctg}\frac{t-x}{2\lambda}dt\Big) =$$

$$= -\frac{1}{\pi}\int_0^{2\pi} f(t)\frac{\varphi(t)-\varphi(x)}{2\operatorname{tg}\frac{t-x}{2}}dt + (\widetilde{\chi_1\varphi f})(x) + (\widetilde{\chi_3\varphi f})(x) +$$

$$+\Big[(\widetilde{\chi_2\varphi f})(x) - \varphi(x)\Big(-\frac{1}{\pi}\int_a^b f_1(t)\frac{1}{2\lambda}\operatorname{ctg}\frac{t-x}{2\lambda}dt\Big)\Big], \tag{6.9}$$

where f_1 is the restriction of f on (a,b), and χ_1, χ_2, χ_3 are the characteristic functions of the segments $[0,a]$, $[a,b]$, $[b,2\pi]$ respectively.

Using Fubini's theorem, we can easily verify that the right-hand side of (6.9) is summable on $[a,b]$. Hence $\varphi\widetilde{f} \in \widetilde{L}(a,b)$. Further, taking the $\widetilde{L}$-integral on the both sides of the last equality and replacing the iterated integrals, we obtain (6.8). ■

Corollary. *There exists the set $E \subset [0,2\pi]$ of measure 2π depending only on f such that if $a,b \in E$, then the function $\widetilde{f}$ is $\widetilde{L}$-integrable (A–, B-integrable) on (a,b).*

Remark. It has been shown in [58] that the function $\widetilde{f}$ is, generally speaking, non-integrable for all $[a,b] \subset [0,2\pi]$ for none of the extensions of the L-integral being a positive functional (i.e., for non-negative functions taking non-negative values).

Theorem 6.3 ([58]). *If $f \in L(0,2\pi)$, then for almost all $x \in [0,2\pi]$,*

$$f(x) - \frac{a_0}{2} = \operatorname*{limas}_{\varepsilon\to 0}(\widetilde{L})\Bigg(\int\limits_{\varepsilon}^{x-\varepsilon} + \int\limits_{x+\varepsilon}^{2\pi+\varepsilon}\Bigg)\widetilde{f}(t)\frac{1}{2}\operatorname{ctg}\frac{1}{2}(t-x)\,dt, \qquad (6.10)$$

where $a_0/2$ is the mean-value of the function f on $[a,b]$.

Here limas denotes an asymptotic (approximate) limit (see, e.g., [169], Ch. IV, §2). Obviously, if $\widetilde{f} \in L(0,2\pi)$, then the asymptotic limit can be replaced by the usual one, while the $\widetilde{L}$-integral by the Lebesgue integral.

The equality (6.10) together with (6.1) is a generalization of Hilbert's inversion formula.

Theorem 6.4. *Let $f \in L(0,2\pi)$ and let φ be an absolutely continuous function such that $\varphi' \in L_p(0,2\pi)$, $p > 1$. Then*

(1) $(\widetilde{L})\displaystyle\int\limits_0^{2\pi} \varphi(x)\widetilde{f}(x)dx = -\int\limits_0^{2\pi} \varphi'(x)G(x)dx$, *where*

$$G(x) = \frac{1}{\pi}\int\limits_0^{2\pi} f(t)\lg\Big|\sin\frac{1}{2}(t-x)\Big|dt;$$

(2) $(\widetilde{L})\displaystyle\int\limits_a^b \varphi(x)\widetilde{f}(x)dx = \varphi(b)G(b) - \varphi(a)G(a) - \int\limits_a^b \varphi'(x)G(x)dx$, *for a, $b \in E$ and E is the set from Theorem 6.2.*

Proof. We will prove only the equality (2), since the equality (1) can be proved analogously.

Let $a, b \in E$. Using the integration by part, we can easily verify that the equality

$$\int_a^b \varphi(x) \operatorname{ctg} \frac{1}{2}(t-x)dx = -\varphi(b) \lg|\sin \frac{1}{2}(t-b)| + \varphi(a) \lg|\sin \frac{1}{2}(t-a)| + $$
$$+ \int_a^b \varphi'(x) \lg|\sin \frac{1}{2}(t-x)|dx$$

is valid for almost all $t \in [0, 2\pi]$. The latter equality and Theorem 6.2 (which can be used, since $\varphi \in H(\alpha)$) allows us to write

$$(\widetilde{L}) \int_a^b \varphi(x)\widetilde{f}(x)dx = -\frac{1}{\pi}\int_0^{2\pi} f(t)dt \int_a^b \varphi(x)\frac{1}{2} \operatorname{ctg} \frac{1}{2}(t-x)dx =$$
$$= \varphi(b)G(b) - \varphi(a)G(a) - \frac{1}{\pi}\int_0^{2\pi} f(t)dt \int_a^b \varphi'(x) \lg|\sin \frac{1}{2}(t-x)|dx.$$

Changing the order of integration in the iterated integral of the above equality, we obtain the equality (2). ■

6.2. $\widetilde{L}$-integral and Cauchy singular integrals. Here and in what follows, Γ is assumed to be a simple, closed, rectifiable curve. Without loss of generality we also assume that the length of the curve Γ is equal to 2π and write the equality in the form $t = t(s)$, $0 \le s \le 2\pi$ where s is the arc coordinate. Moreover, Γ is supposed to satisfy the condition

$$\int_0^{2\pi} \Big|\frac{t'(\sigma)}{t(\sigma)-t(s)} - \frac{1}{2} \operatorname{ctg} \frac{1}{2}(\sigma - s)\Big| d\sigma < C, \quad (C \ \text{ is a constant}) \tag{6.11}$$

(the last condition is satisfied, for example, by for piecewise Lyapunov curves without cusps and those of bounded rotation without cusps (see subsection 3.1).

We say that the function f is $\widetilde{L}$-integrable on Γ if $t'(s)f[t(s)] \in \widetilde{L}(0, 2\pi)$ and write $f(t) \in \widetilde{L}(\Gamma)$. Under the $\widetilde{L}$ -integral of f along Γ is meant a number

$$(L)\int_\Gamma f(t)dt = (\widetilde{L})\int_0^{2\pi} f[t(s)]t'(s)ds.$$

Remark. It is evident that the following two conditions are equivalent:
(1) $f(t) = f_1(t) + S_\Gamma(f_2)(t)$; where f_1, $f_2 \in L(\Gamma)$;
(2) $t'(s)f[t(s)] \in \widetilde{L}(0, 2\pi)$.

Moreover, the equality

$$(\widetilde{L})\int_\Gamma f(t)dt \equiv (\widetilde{L})\int_0^{2\pi} f[t(s)]t'(s)ds = \int_\Gamma [f_1(t)-f_2(t)]dt \qquad (6.12)$$

holds.

Indeed, by virtue of (1) we have

$$t'(s)f[t(s)] = t'(s)f_1(t(s)) + \frac{1}{\pi i}\int_0^{2\pi} f_2(t(\sigma))t'(\sigma)\frac{t'(s)}{t(\sigma)-t(s)}d\sigma =$$

$$= t'(s)f_1(t(s)) + \frac{1}{\pi i}\int_0^{2\pi} f_2(t(\sigma))t'(\sigma)\Big[\frac{t'(s)}{t(\sigma)-t(s)} - \frac{1}{2}\operatorname{ctg}\frac{1}{2}(\sigma-s)\Big]d\sigma +$$

$$+\frac{1}{\pi i}\int_0^{2\pi} f_2(t(\sigma))t'(\sigma)\frac{1}{2}\operatorname{ctg}\frac{\sigma-s}{2}d\sigma. \qquad (6.13)$$

The second summand on the right-hand side of the last equality is, by the condition (6.11) and Fubini's theorem, summable on $[0,2\pi]$, and hence $t'(s)f[t(s)] \in \widetilde{L}(0,2\pi)$.

Let now the condition (2) be fulfilled, i.e., there exist $f_1,\ f_2 \in L(0,2\pi)$ such that

$$t'(s)f[t(s)] = f_1(s) + \int_0^{2\pi} f_2(\sigma)\frac{1}{2}\operatorname{ctg}\frac{\sigma-s}{2}d\sigma =$$

$$= f_1(s) + \int_0^{2\pi} f_3(\sigma)t'(\sigma)\Big[\frac{1}{2}\operatorname{ctg}\ \frac{\sigma-s}{2} - \frac{t'(s)}{t(\sigma)-t(s)}\Big]d\sigma +$$

$$+t'(s)\int_0^{2\pi} f_3(\sigma)\frac{t'(\sigma)}{t(\sigma)-t(s)}d\sigma, \quad \text{where} \quad f_3(\sigma) = f_2(\sigma)/t'(\sigma).$$

Analogously, using Fubini's theorem, we can easily see that the second summand on the right-hand of the last equality is summable on $[0,2\pi]$. Hence $t'(s)f[t(s)] = g_1(s) + t'(s)\int_\Gamma \frac{g_2(\sigma)d\tau}{\tau-t(s)}$, where $g_1,\ g_2 \in L(\Gamma)$ and $f(t) = g_1(s)/t'(s) + \int_\Gamma \frac{g_2(\sigma)d\tau}{\tau-t}$, $\tau = t(\sigma)$.

The equality (6.12) can be obtained by integrating the equality (6.13) and interchanging the integrals in the iterated integral. Indeed, by definition of the $\widetilde{L}$-integral along Γ, we find that

$$(\widetilde{L})\int_\Gamma f(t)dt = (\widetilde{L})\int_\Gamma f[t(s)]t'(s)ds = \int_\Gamma t'(s)f_1(t)ds +$$

$$+\frac{1}{\pi i}\int\limits_0^{2\pi} ds \int\limits_0^{2\pi} f_2(t(\sigma))t'(\sigma)\Big[\frac{t'(s)}{t(\sigma)-t(s)} - \frac{1}{2}\operatorname{ctg}\frac{\sigma-s}{2}\Big]d\sigma =$$

$$= \int\limits_\Gamma [f_1(t) - f_2(t)]dt.$$

It follows from what has been said above that if $f(t) = f_1(t) + S_\Gamma(f_2)(t)$, where $f_1,\, f_2 \in L(\Gamma)$, we can define its $\widetilde{L}$- integral along Γ by the equality

$$(\widetilde{L})\int\limits_\Gamma f(t)dt = \int\limits_\Gamma [f_1(t) - f_2(t)]dt.$$

Theorem 6.5. *Let* $f \in L(\Gamma)$. *Let*

$$\varphi \in L^\infty(\Gamma) \quad and \quad C = \sup_{t\in\Gamma}\int\limits_\Gamma \frac{|\varphi(\tau)-\varphi(t)|}{|\tau - t|}\,|d\tau| < \infty. \tag{6.14}$$

Then $\varphi S_\Gamma(f) \in \widetilde{L}(\Gamma)$ *and*

$$(\widetilde{L})\int\limits_\Gamma \varphi(t)S_\Gamma(f)(t)dt = -\int\limits_\Gamma f(t)S_\Gamma(\varphi)(t)dt. \tag{6.15}$$

Proof. Consider the identity

$$t'(s)\varphi(t)S_\Gamma(f)(t) = \frac{\varphi(t)t'(s)}{\pi i}\int\limits_\Gamma \frac{f(\tau)d\tau}{\tau - t} =$$

$$= \frac{t'(s)}{\pi i}\int\limits_\Gamma \frac{\varphi(t)-\varphi(\tau)}{\tau - t} f(\tau)d\tau + \frac{t'(s)}{\pi i}\int\limits_\Gamma \frac{\varphi(\tau)f(\tau)}{\tau - t}d\tau =$$

$$= \frac{t'(s)}{\pi i}\int\limits_\Gamma \frac{\varphi(t)-\varphi(\tau)}{\tau - t} f(\tau)d\tau +$$

$$+\frac{1}{\pi i}\int\limits_0^{2\pi} \varphi(\tau)f(\tau)t'(\sigma)\Big[\frac{t'(s)}{t(\sigma)-t(s)} - \frac{1}{2}\operatorname{ctg}\frac{\sigma-s}{2}\Big]d\sigma +$$

$$+\frac{1}{\pi i}\int\limits_0^{2\pi} \varphi(\tau)f(\tau)t'(\sigma)\frac{1}{2}\operatorname{ctg}\frac{\sigma-s}{2}d\sigma. \tag{6.16}$$

Because of (6.11), (6.14) and Fubini theorem on the inversion of the order of integration, the first and the second summands on the right-hand side of the last equality are functions summable on Γ. Hence $\varphi S_\Gamma(f) \in \widetilde{L}(\Gamma)$.

Further, integrating (6.16) and using Fubini's theorem and Theorem 6.1, we obtain (6.15). ∎

Remark 1. For $\varphi \in H_\alpha(\Gamma)$, $0 < \alpha \le 1$, and $\Gamma \in R$ it is easy to verify that the condition (6.14) is satisfied.

Remark 2. By a line of integration is meant as above a closed curve satisfying certain conditions. But as is seen from the proof of the theorem, it is also valid when the line of integration consists of a finite number of non-intersecting curves of the type mentioned above.

This remark concerns all the results obtained for the $\widetilde{L}$-integral.

Theorem 6.6. *Let $f \in \Gamma$ and conditions* (6.11), (6.14) *are fulffiled. There exists a measurable depending only on f set $E \subset [0, 2\pi]$ of measure 2π such that if s', $s'' \in [0, 2\pi]$ then $\varphi S_\Gamma(f) \in \widetilde{L}(\Gamma_{t't''})$ and*

$$(\widetilde{L}) \int\limits_{\Gamma_{t't''}} \varphi(t)dt \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - t} = \int\limits_{\Gamma} f(\tau)d\tau \int\limits_{\Gamma_{t't''}} \frac{\varphi(t)dt}{\tau - t}, \tag{6.17}$$

where $\Gamma_{t't''}$ is a portion of the contour Γ with the ends $t' = t(s')$ and $t'' = t(s'')$.

The theorem is proved in the same way as Theorem 6.5 with the only difference that instead of Theorem 6.1 we use Theorem 6.2.

Theorem 6.7 ([58]). *Let Γ satisfy the condition $t''(s) \in H(\alpha)$, $f \in L(\Gamma)$ and φ satisfy the conditions* (6.14). *Then*

$$\begin{aligned} &\operatorname*{limas}_{\varepsilon \to 0} (\widetilde{L}) \int\limits_{\Gamma_\varepsilon} \frac{\varphi(t)dt}{t - t_0} \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - t} = \\ &= -\pi^2 \varphi(t_0) f(t_0) + \int\limits_{\Gamma} f(\tau)d\tau \int\limits_{\Gamma} \frac{\varphi(t)dt}{(\tau - t)(t - t_0)} \end{aligned} \tag{6.18}$$

for almost all $t_0 \in \Gamma$

The equality (6.18) is a generalization of the well-known Poincaré–Bertrani equality (see, e.g., [66], p. 30). If $\varphi(x) \equiv 1$, then we obtain the generalization of the inversion formula of a singular Cauchy integral

$$\operatorname*{limas}_{\varepsilon \to 0} (\widetilde{L}) \int\limits_{\Gamma_\varepsilon} \frac{dt}{t - t_0} \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - t} = -\pi^2 f(t_0). \tag{6.19}$$

Obviously, if $S_\Gamma(f)$ is summable on Γ, the then asymptotic limit in formula (6.19) can be replaced by the ordinary one and the $\widetilde{L}$-integral by the Lebesgue integral.

6.3. Cauchy type $\widetilde{L}$-integrals. Let Γ satisfy the conditions (6.11) and let $f \in \widetilde{L}(\Gamma)$. Then, according to Theorem 6.5, the analytic function

$$F(z) = \frac{1}{2\pi i}(\widetilde{L}) \int\limits_{\Gamma} \frac{f(t)dt}{t-z} \tag{6.20}$$

is defined in the domain $z \notin \Gamma$ and is called a Cauchy type $\widetilde{L}$ -integral. For almost all $t \in \Gamma$, there exist boundary values $F^{+}(t)$ and $F^{-}(t)$ belonging to $\widetilde{L}(\Gamma)$ and almost everywhere on Γ

$$F^{+}(t) - F^{-}(t) = f(t). \tag{6.21}$$

Indeed, without restriction of generality, we may assume that $f(t) = S_{\Gamma}(f_1)(t)$, where $f_1 \in L(\Gamma)$. Then, by Theorem 6.5, for $z \notin \Gamma$ we have

$$\begin{aligned} F(z) &= \frac{1}{2(\pi i)^2}(\widetilde{L}) \int\limits_{\Gamma} \frac{dt}{t-z} \int\limits_{\Gamma} \frac{f_1(\tau)d\tau}{\tau - t} = \\ &= \frac{1}{2(\pi i)^2} \int\limits_{\Gamma} f_1(\tau)d\tau \int\limits_{\Gamma} \frac{dt}{(t-z)(\tau-t)} = \\ &= \frac{1}{2(\pi i)^2} \int\limits_{\Gamma} \frac{f_1(\tau)d\tau}{\tau - z} \int\limits_{\Gamma} \frac{dt}{t-z} + \frac{1}{2(\pi i)^2} \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - z} \int\limits_{\Gamma} \frac{dt}{\tau - t} = \\ &= \begin{cases} \frac{1}{2\pi} \int_{\Gamma} \frac{f_1(\tau)d\tau}{\tau - z} & \text{for} \quad z \in D^{+}, \\ -\frac{1}{2\pi} \int_{\Gamma} \frac{f_1(\tau)d\tau}{\tau - z} & \text{for} \quad z \in D^{-}. \end{cases} \end{aligned} \tag{6.22}$$

Moreover, by the Sokhotskiĭ-Plemelj formulas,

$$F^{+}(t) - F^{-}(t) = S_{\Gamma}(f_1)(t) = f(t) \quad \text{and} \quad F^{+}(t) + F^{-}(t) = f_1(t). \tag{6.23}$$

If, in addition, we have $t''(s) \in H(\alpha)$, then by virtue of (6.19) and (6.23) in

$$F^{+}(t) + F^{-}(t) = \frac{1}{\pi i} \operatorname*{limas}_{\varepsilon \to 0} (\widetilde{L}) \int\limits_{\Gamma_{\varepsilon}} \frac{f(\tau)d\tau}{\tau - t}, \tag{6.24}$$

where Γ_{ε} is the largest arc of the contour Γ with the ends $t(s_0 - \varepsilon)$ and $t(s_0 + \varepsilon)$.

The equalities (6.21) and (6.24) generalize the Sokhotskiĭ-Plemelj formulas.

Theorem 6.8. *Let $f \in L(\Gamma)$, $F(z) = K_{\Gamma}(f)(z)$. Then $F^{+} \in \widetilde{L}(\Gamma)$ and $F(z)$ is representable in D^{+} by the Cauchy $\widetilde{L}$-integral. Moreover,*

$$(\widetilde{L}) \int\limits_{\Gamma} \beta^{+}(t)F^{+}(t)dt = 0 \tag{6.25}$$

$$\left(\textit{in particular,} \quad (\widetilde{L}) \int\limits_{\Gamma} t^n F^+(t)dt = 0, \quad n = 0, 1, 2, \dots \right),$$

where $\beta^+(t)$ *is the boundary value of the bounded in* D^+ *analytic function satisfying* (6.14).

Proof. Let $z \notin \Gamma$. By Theorem 6.5 and the Sokhotskiĭ-Plemelj formulas,

$$\frac{1}{2\pi i}(\widetilde{L}) \int\limits_{\Gamma} \frac{F^+(t)dt}{t-z} = \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{f(t)dt}{t-z} +$$

$$+\frac{1}{(2\pi i)^2}(\widetilde{L}) \int\limits_{\Gamma} \frac{dt}{t-z} \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - t} =$$

$$= \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{f(t)dt}{t-z} + \frac{1}{(2\pi i)^2} \int\limits_{\Gamma} f(\tau)d\tau \int\limits_{\Gamma} \frac{dt}{(t-z)(\tau - t)} =$$

$$= \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{f(t)dt}{t-z} + \frac{1}{(2\pi i)^2} \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - z} \int\limits_{\Gamma} \frac{dt}{t-z} +$$

$$+\frac{1}{(2\pi i)^2} \int\limits_{\Gamma} \frac{f(\tau)d\tau}{\tau - z} \int\limits_{\Gamma} \frac{dt}{\tau - t} = \begin{cases} F(z) & \text{for} \quad z \in D^+, \\ 0 \quad \text{for} & z \in D^-. \end{cases}$$

The proof of the equality (6.25) is performed analogously to the previous one by using Theorem 6.5 and the fact that if β is a bounded in D^+ analytic function, then $S_\Gamma(\beta^+)(t) = \beta^+(t)$. ■

Corollary. *If* $f \in L(\Gamma)$ *and the boundary value of Cauchy type integral* $K_\Gamma(f)$ *is summable on* Γ, *then* $K_\Gamma(f)$ *is representable in* D^+ *by the Cauchy-Lebesgue integral.*

The assertion of the corollary in the case where Γ is a circumference represents by itself well-known V. I. Smirnov's theorem [146], (see also [133], p. 116).

Theorem 6.9. *A class of functions representable in* $D^+ \cup D^-$ *by the Cauchy type* $\widetilde{L}$*-integral coincides with the a of functions representable in the form*

$$F(z) = \begin{cases} F_1(z) & \textit{for} \quad z \in D^+, \\ F_2(z) & \textit{for} \quad z \in D^-, \end{cases} \tag{6.26}$$

where $F_i(z) = K_\Gamma(f_i)(z)$, $f_i \in L(\Gamma)$, $i = 1, 2$.

Proof. Let F be representable in the form (6.26). Consider on Γ the function $f(t) = F_1^+(t) - F_2^-(t)$, where

$$F_1^+(t) = \frac{1}{2} f_1(t) + \frac{1}{2} S_\Gamma(f_1)(t) \quad \text{and} \quad F_2^-(t) = -\frac{1}{2} f_2(t) + \frac{1}{2} S_\Gamma(f_2)(t).$$

Obviously, $f \in \widetilde{L}(\Gamma)$, and owing to the previous theorem and the formula (6.21), for $z \notin \Gamma$ we can write

$$\frac{1}{2\pi i}(\widetilde{L})\int\limits_{\Gamma}\frac{f(t)dt}{t-z}=\frac{1}{2\pi i}(\widetilde{L})\int\limits_{\Gamma}\frac{F_1^+(t)-F_2^-(t)}{t-z}dt=$$
$$=\frac{1}{2\pi i}(\widetilde{L})\int\limits_{\Gamma}\frac{F_1^+(t)dt}{t-z}-\frac{1}{2\pi i}(\widetilde{L})\int\limits_{\Gamma}\frac{F_2^-(t)}{t-z}dt=$$
$$=\begin{cases}F_1(z) & \text{for} \quad z\in D^+,\\ F_2(z) & \text{for} \quad z\in D^-.\end{cases}$$

The inverse assertion follows from the equality (6.22). ■

Corollary. *The function representable in D^+ (in D^-) by the Cauchy type $\widetilde{L}$-integral is representable by the Cauchy $\widetilde{L}$-integrals as well.*

Indeed, by Theorem 6.9, the function representable, for example, in D^+ by the Cauchy type $\widetilde{L}$-integral is representable in D^+ by the Cauchy-Lebesgue type integral and, hence, by the Cauchy $\widetilde{L}$-integral, according to Theorem 6.8.

Remark. The function F representable in the form (6.26) cannot always be representable in $D^+\cup D^-$ by a single Cauchy-Lebesgue type integral (see, e.g., Remark 2 to Theorem 6.10).

Theorem 6.10. *Assume that the functions ϕ and F are representable in $D^+(D^-)$ by the Cauchy-Lebesgue type integrals with densities φ and $f \in L(\Gamma)$, respectively, where φ satisfies the conditions (6.14), then the product ϕF is also representable in $D^+(D^-)$ by a Cauchy-Lebesgue type integral.*

Proof. Let $z \in D^+(D^-)$. Then, since $S_\Gamma(\varphi)$ is bounded on Γ, by Theorem 6.5,

$$(\widetilde{L})\int\limits_{\Gamma}\frac{\varphi(t)S_\Gamma(t)}{t-z}dt=\frac{1}{\pi i}\int\limits_{\Gamma}f(\tau)d\tau\int\limits_{\Gamma}\frac{\varphi(t)dt}{(\tau-t)(t-z)}=$$
$$=-\int\limits_{\Gamma}\frac{\varphi(\tau)S_\Gamma(\varphi)(\tau)}{\tau-z}d\tau+\frac{1}{\pi i}\int\limits_{\Gamma}\frac{f(\tau)}{\tau-z}d\tau\int\limits_{\Gamma}\frac{\varphi(t)dt}{t-z}.$$

Hence

$$\phi(z)F(z)=\frac{1}{4\pi i}\int\limits_{\Gamma}\frac{f(\tau)S_\Gamma(\varphi)(\tau)}{\tau-z}d\tau+\frac{1}{4\pi i}(\widetilde{L})\int\limits_{\Gamma}\frac{\varphi(t)S_\Gamma(t)}{t-z}dt,$$

that is, ϕF is representable in D^+ (D^-) as a sum of a Cauchy-Lebesgue type integral and a Cauchy type $\widetilde{L}$-integral. But this implies the validity of the theorem, since by Theorem 6.9, a Cauchy type $\widetilde{L}$-integral is also representable in $D^+(D^-)$ by a Cauchy-Lebesgue type integral. ■

Corollary. *If the functions ϕ and F are representable in $D^+ \cup D^-$ by the Cauchy type integrals with densities φ and $f \in \widetilde{L}(\Gamma)$, respectively, φ satisfying the conditions* (6.14)*, then the product ϕF is also representable in $D^+ \cup D^-$ by a Cauchy type $\widetilde{L}$-integral.*

Indeed, due to the assertion of the theorem, ϕF is representable by the Cauchy-Lebesgue type integral both in D^+ and in D^-. Then, by Theorem 6.9, ϕF is representable in $D^+ \cup D^-$ by the Cauchy type $\widetilde{L}$-integral.

Remark 1. Note that the assertion of the theorem is about the representability in D^+ (or in D^-) by a Cauchy type integral, whereas the corollary states the representability in $D = D^+ \cup D^-$.

Remark 2. As it follows from the assertion of the corollary, the $\widetilde{L}$-integral, in general, cannot be replaced by the Lebesgue integral even for $f \in L(\Gamma)$.

Example. Let Γ be a unit circumference, $\varphi(t) \equiv 1$ on Γ and $f \in L(\Gamma)$ be such that $S_\Gamma(f) \notin L(\Gamma)$. Then the function

$$\psi(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{dt}{t-z}\frac{1}{2\pi i}\int\limits_\Gamma \frac{f(t)dt}{t-z} = \begin{cases} \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z} & \text{for} \quad z \in D^+, \\ 0 \quad \text{for} \quad z \in D^-, \end{cases}$$

representable in $D^+ \cup D^-$ by a Cauchy type $\widetilde{L}$-integral, neverthless cannot be represented by a Cauchy–Lebesgue integral. Otherwise, we would have $\psi^+(t) - \psi^-(t) = \frac{1}{2}f(t) + \frac{1}{2}S(f)(t) \in L(\Gamma)$ which is impossible since $S(f) \notin L(\Gamma)$.

6.4. An extension to more general curves. Let Γ be a simple, rectifiable, closed curve and let, moreover, for Γ the following analogue of Smirnov theorem is valid:

If the boundary value $F^+(t)$ of the Cauchy-Lebesgue type integral $F(z) = K_\Gamma(f)(z)$ is summable on Γ, then $F(z)$ is representable in D^+ by a Cauchy integral (such are, for example, regular curves).

The above-formulated theorem is equivalent to the following assertion: if F^+ is summable on Γ, then $\int_\Gamma F^+(t)dt = 0$, which implies that for $f \in L(\Gamma)$ and $S_\Gamma(f) \in L(\Gamma)$, the equality

$$\int\limits_\Gamma S_\Gamma(f)(t)dt = \frac{1}{\pi i}\int\limits_\Gamma dt \int\limits_\Gamma \frac{f(\tau)d\tau}{\tau - t} = -\int\limits_\Gamma f(t)dt \tag{6.27}$$

holds.

On the basis of the last equality, in exactly the same way as in subsection 6.2, we introduce the following definition.

We say that the function f is $\widetilde{L}$-integrable on Γ if it can be represented in the form

$$f(t) = f_1(t) + S_\Gamma(f_2)(t), \quad f_1, f_2 \in L(\Gamma). \tag{6.28}$$

The number

$$\widetilde{L}(f) = (\widetilde{L}) \int_\Gamma f(t)dt = \int_\Gamma [f_1(t) - f_2(t)]dt$$

is termed an $\widetilde{L}$-integral of f on Γ.

The correctness of the definition of the $\widetilde{L}$-integral (that is, the independence of $\widetilde{L}(f)$ on the representation (6.28)) is proved in the same way as in subsection 6.1, using equality (6.27) instead of (6.2).

Note also that Theorems 6.5, 6.8, 6.9 and 6.10 are valid in the case in which the line of integration satisfies the conditions of the present section (i.e., when equality (6.27) is valid for it). They are proved in exactly the same way as in subsections 6.2, 6.3.

Notes and Comments to Chapter I

The notion of singular integrals used in §1 is accepted, for example, in monograph [105], [106], [66]. Independence of the definition of a singular integral on parametrization of curve is proved in [124].

In connection with the assertion from the remark to Theorem 1.1 the reader can be referred to [98], [38].

For the results of §2 in non-weighted case see [112].

The boundedness of singular operators over curves in L^p $(1 < p < \infty)$ comes from S. Mikhlin (the curves of continuous curvature) [98], B. Khvedelidze (Lyapunov curves) [66], I. Danilyuk and V. Shelepov (curves of bounded rotation) [20]. Such boundedness for piecewise Lyapunov contours with cusps has been proved by E. Gordadze [44]. The problem remained open for smooth curves.

In 1976, A. P. Calderon proved the boundedness of Cauchy singular integral operators in L^2 over Lipschitz curves under the assumption that Lipschitz constant is sufficiently small. This additional condition was later removed by R. Coifman, A. McIntosch, and Y. Meyer [14]. The other proofs can be found in R. Coifman, P. Jones, and S. Semmes [13], G. David [23], G. David and S. Semmes [24], T. Murai [101], M. Melnikov and T. Verdera [97], etc.

It has been shown by V. Paatashvili and G. Khuskivadze [122] that if S_Γ is bounded in $L^p(\Gamma)$ for some p, $1 < p < \infty$, then the curve Γ is regular, i.e., satisfies the condition (3.2). In the same paper it is shown that this condition is sufficient in a class of broken lines and also of those over which this boundedness fails and the hypothesis on its sufficiency is stated in the general case.

In 1982, G. David solved this problem completely; he proved that condition (3.2) is necessary and sufficient for S_Γ to be continuous in L^p, $1 < p < \infty$.

At his lectures A. Zygmund noted that the boundedness of the operator S_Γ in $L^p(\Gamma)$ for any smooth curves has a consequence the existence almost everywhere of the integral $S_\Gamma f(t)$ for arbitrary rectifiable curves and summable on them functions f. Subsequently, this fact has been proved by V. Havin (see [28], pp. 248–249).

The proof of Theorem 3.5 follows the method used by M. Cotlar for the Hilbert transform [15].

The proof of the boundedness of S_Γ over a closed curve from J_0 has been performed by V. Kokilashvili [72]. The case of open arcs from J_0 was considered later in [125].

The equivalence of the boundedness of S_Γ from $L^p(\Gamma)$ into $L^s(\Gamma)$ ($p \geq s > 1$) and the belonging to Smirnov class E^s of the Cauchy type integral of $K_\Gamma(f)$ for any $f \in L^p(\Gamma)$ is proved by V. Havin [51] and V. Paatashvili [114]. For individual functions, some sufficient conditions for such an inclusion were obtained in [113], [119]. In the case of smooth curves the similar problem is investigated in §5.

The fact that the condition A_p with respect to the arcs is necessary and sufficient in the case of smooth curves (but not only for Lyapunov contours as is incorrectly cited in [7]) was proved by V. Kokilashvili [73]. I. Simonenko has constructed an example of a function which satisfies the Muckenhoupt condition over arcs but is not a weight function for the Cauchy singular integral in the case of a contour with cusps.

The conventional exposition of one-weight norm inequalities for singular integrals on curves based on the well-known Calderon-Zygmund theory as well as on Coifman's concept can be found in [7]. In fact, all these results can be considered as a particular case of the weight theory of singular integrals defined on homogeneous type spaces, the comprehensive investigation of which is presented in the monograph of I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec [40].

Exposition of Theorem 4.5 on two-weighted estimates for conjugate functions is an amalgam of the proof presented in [80]. For more general pair of weights see D.E. Edmunds and V. Kokilashvili [30]. For optimal conditions for two-weight strong and weak type inequalities for singular integrals of homogeneous type spaces, in particular, for fractal sets, the reader can be referred to [40].

As for the p-mean singular integrals in connection with the belonging to Smirnov classes $E^p(D)$ of Cauchy type integrals for individual functions, they have been used V. Paatashvili [114].

An example of the curve Γ and of a continuous on it function φ for which $(S_\Gamma\varphi)\overline{\in} L(\Gamma)$, is given in [51]. Curves from Examples (3) and (4) of subsection 3.4 were constructed by G. Khuskivadze [62]. Moreover, it has been shown therein that the curves satisfying conditions (3.44) ((3.48)) can be constructed so as to have $\omega_r(\theta,\delta) \le C\delta^\alpha$, $r>1$, $\alpha > r^{-1}$, $r>1$, $\alpha > r^{-1}$ where $\omega_r(\theta,\delta)$ is an integral module of continuity of the function $\theta = \theta(s)$, $t'(s) = \exp i\theta(s)$.

Some properties of the Cauchy type integral, when a line of integration is an infinite set of curves, were investigated for the first time in [2] and [55–56]. The questions of the continuity of S_Γ in the Lebesgue spaces are studied in [1], [64], [126], [127].

Theorem 6.1 (under more general assumptions with respect to φ), the assertion of Corollary 2 of Theorem 6.1 and Theorem 6.3, for the A-integral, have been proved by P. Ul'yanov [156]. The assertion of Corollary 1 of Theorem 6.1, for the B-integral, is due to A. Kolmogorov [84], and for the A-integral, to E.C. Titchmarsh [154]. The assertion of corollary of Theorem 6.2, and Theorem 6.4, for the A-integral,have been obtained by E.C. Titchmarsh [154]. Theorem 6.8 in the case of $\widetilde{A}$-integrals and Lyapunov curves has been obtained earlier in [158]. The main results of §6 in the case where Γ is a Lyapunov curve were obtained by G. Khuskivadze [58], [62].

CHAPTER II
THE DISCONTINUOUS BOUNDARY VALUE PROBLEM IN CLASS OF CAUCHY TYPE INTEGRALS

Let Γ be a closed, rectifiable Jordan curve bounding a finite domain $D^+$and an infinite domain D^-. In §§1–3, 5 we consider the problem of linear conjugation formulated as follows: define a function $\phi \in \mathcal{K}^p(\Gamma, w)$, whose boundary values ϕ^+ (from D^+) and ϕ^- (from D^-) satisfy almost everywhere on Γ the condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \quad t \in \Gamma, \tag{I}$$

where G and g are functions given on Γ, and $g \in L^p(\Gamma, w)$.

It is assumed that $\Gamma \in J^*$ and the function G (or G_w which will be defined by means of G and w in §5) belongs to the class $\widetilde{A}(p)$ introduced in Section 1.2. The assumptions for w are adopted (see (5.1)) which in the case of Lyapunov contours cover all admissible in this problem weight functions.

Along with the problem (I) in the class $L^p(\Gamma, w)$ we consider a singular integral equation of the type

$$a\varphi + bS_\Gamma\varphi + V\varphi = f, \tag{II}$$

where a, b, f are functions given on Γ, $f \in L^p(\Gamma, w)$, $(a-b)(a+b)^{-1} \in \widetilde{A}(p)$ and V is a compact operator in $L^p(\Gamma, w)$.

Comprehensive investigation of the character of solvability of the problem (I) allows one to obtain Noetherian theorems for the equation (II).

As far as the problem (I) is, generally speaking, unsolvable in the class $\mathcal{K}^1(\Gamma)$, it is advisable in this case either to narrow this class and to consider those subsets in which the character of solvability is similar as in case of $\mathcal{K}^p(\Gamma)$, $1 < p < \infty$, or to extend, within reasonable limits, the class of unknown functions containing $\mathcal{K}^1(\Gamma)$ and to clarify the picture of solvability. Both possibilities are realized in §4 and §8, respectively.

§ 1. The Problem of Linear Conjugation in the Class $\mathcal{K}^p(\Gamma)$

1.1. Assumption regarding the boundary curve. We will assume that Γ belongs to the class J^* (see §3, Chapter I). Recall the definition.

The curve $\Gamma \in K$ belongs to the class J^* if it is divided into a finite number of arcs belonging to the class J and having tangents at the ends. The Jordan curve $\Gamma \in K$ with the equation $t = t(s)$, $0 \le S \le l$ is assumed to belong to the class J if there exists for it a Jordan smooth curve μ of the same length with the equation $\mu = \mu(s)$, $0 \le s \le l$, such that

$$\operatorname*{ess\,sup}_{0\le\sigma\le l} \int_0^l \Big| \frac{t'(s)}{t(s)-t(\sigma)} - \frac{\mu'(s)}{\mu(s)-\mu(\sigma)} \Big| ds < \infty. \tag{1.1}$$

1.2. The Class $\widetilde{A}(p)$ of functions. The index of the function $G \in \widetilde{A}(p)$. We will say that a measurable on Γ function G belongs to the class $\widetilde{A}(p)$, $1 < p < \infty$, if the following conditions are fulfilled:

(1) $0 < \operatorname{ess\,inf} |G|$; $\operatorname{ess\,sup} |G| < \infty$

(2) for all $t \in \Gamma$, with the possible exception of a finite number of points $c_k = t(s_k)$, $s_k < s_{k+1}$ $(k = \overline{1,n})$, there exists on Γ a neighbourhood in which the values of G lie in some sector with the vertex at the origin and the angle less than $a(p) = \frac{2\pi}{\max(p,p')}$, $p' = \frac{p}{p-1}$;

(3) there exist at the points c_k the limits $G(c_k-)$ and $G(c_k+)$; let the angles δ_k between the vectors corresponding to $G(c_k-)$ and $G(c_k+)$ be such that

$$\frac{2\pi}{p} < \delta_k \le \frac{2\pi}{p'}, \text{ for } p > 2 \text{ and } \frac{2\pi}{p'} \le \delta_k \le \frac{2\pi}{p}, \ \ 1 < p < 2, \ \ k = \overline{1,n}.$$

The points c_k will be called p-points of discontinuity of the function G.

By analogy with the class $A(p)$, the set $\widetilde{A}(p)$ has been introduced in [78]. Recall, that the class $A(p)$ has been introduced and applied to the boundery value problem of linear conjugation by I. Simonenko [141]. The subset of the functions from $\widetilde{A}(p)$ which do not posses p-points of discontinuity, coincides with $A(p)$. The existence of p-points of discontinuity makes it possible to cover by the class $\widetilde{A}(p)$ the most part of those functions satisfying condition (1) which were considered in terms of the coefficients of the problem of conjugation with a finite index. In particular, it can be easily verified that $\widetilde{A}(p)$ contains any admissible piecewise continuous coefficients ([168]) and the functions whose argument φ is representable in the form $\varphi = \varphi_0 + \varphi_1$, where φ_0 is continuous and φ_1 is of a bounded variation, i.e., $\widetilde{A}(p)$ contains the class of coefficients considered in [18], [21].

Combining the definitions of the argument for piecewise continuous functions and for a function from $A(p)$, we can for a given p determine the argument for $G(t)$ at every point $t \in \Gamma$ so that the increment of the argument resulting of going around Γ appears to be exactly the same characteristic for the problem with the coefficient G as the increment of the argument is for the continuous coefficient.

$\boldsymbol{\arg_p G(t)}$. Suppose that $\Gamma_k = [c_k, c_{k+1})$, $c_{n+1} = c_1$, $k = \overline{1,n}$, are half-open arcs of the curve Γ connecting the points c_k and c_{k+1}, where $c_{n+1} = c_1$. Given $\varepsilon > 0$, there exist arcs $[t(s_k), t(s_k + \eta_k))$ and $(t(s_{k+1} - \eta_{k+1}), t(s_{k+1}))$, η_k, $\eta_{k+1} \in (0, \frac{1}{2}(s_{k+1} - s_k))$ such that the values of G lie in a sector with the vertex at the origin and the angle less than ε. Assume that the numbers η_k correspond to the choice of $\varepsilon = a(p)$, and let τ_k and τ_{k+1} be some points from these intervals. Every point of the closed arc $[\tau_k, \tau_{k+1}]$ possesses a neighbourhood in which the values of the function G are located in a sector with the angle less than $a(p)$. These neighbourhoods cover $[\tau_k, \tau_{k+1}]$, and therefore we can choose a finite covering. Adding to the set of intervals of this covering the intervals $[t(s_k), t(s_k + \eta_k)]$ and $(t(s_k - \eta_{k+1}), t(s_{k+1}))$, we obtain the arc covering $[c_k, c_{k+1})$. Since $\Gamma = \cup \Gamma_k$, we finally conclude that there exists a finite covering B of the curve Γ consisting either of open arcs or of those of the kind $[t(s_k), t(s_k + \eta_k))$; on every arc, the values of the functions are located in a sector with the vertex at the origin and the angle less than $a(p)$.

Let now c be an arbitrary point on Γ. If c is not a p-point of discontinuity of G, then we form a new covering B' of the curve Γ replacing the arc $(a, b) \ni c$ in the covering B by the arcs $(a, c]$ and $[c, b)$. But if $c = c_k$, then we put $B' = B$. For a fixed covering B', we select arbitrarily an argument of the number $G(c)$ and denote it by $\arg_p G(c)^+$. Going along Γ in the positive direction, we define the argument $\arg_p G(t)$ on all arcs from B' so that if t_1 and t_2 belong to the same arc, then $|\arg_p G(t_1) - \arg_p G(t_2)| < a(p)$. Thus we reach the arc whose right end is the nearest to c point c_k, and there exists $\lim\limits_{s \to s_k -} \arg G(t(s)) = \alpha_k$. Define $\arg_p G(c_k)$ according to the rule

$$\arg_p G(c_k) = \begin{cases} \alpha_k + \delta_k & \text{for} \quad \delta_k < \frac{2\pi}{p}, \\ \alpha_k + \delta_k - 2\pi & \text{for} \quad \delta_k > \frac{2\pi}{p}. \end{cases} \tag{1.2}$$

Continuing the process of defining the argument, after going around the curve we come at the point c to a new value of $\arg_p G(c)^-$.

The integer

$$\varkappa = \varkappa_p = \varkappa_p(G) = \frac{1}{2\pi}[\arg_p G(c)^- - \arg_p G(c)^+]$$

does not depend on the choice of the covering B' and on the point c; we will call it the index of the function G in the class $\mathcal{K}_p(\Gamma)$ and denote by $\operatorname{ind}_p G$.

Note here that $\arg_p G(c_k)$ for all k is defined by the first equality from (1.2) if $1 < p < 2$, and by the second one if $p > 2$. There are no p-points of discontinuity for $p = 2$.

1.3. Decomposition of the function $G \in \widetilde{A}(p)$. . From the definition of the function $\varphi(t) = \arg_p G(t)$ it follows that for its oscillation

$$\Omega(\varphi, t) = \inf_{l(t) \subset \Gamma} \{ \sup_{\tau \in l(t)} \varphi(\tau) - \inf_{\tau \in l(t)} \varphi(\tau) \} \tag{1.3}$$

the inequality

$$\sup_{t \in \Gamma} \Omega(\varphi, t) < 2\pi \tag{1.4}$$

is valid. Therefore, by Lemma 1 from [141] we can determine a real function $\varphi_1(t)$ such that $\varphi_1(t(s))$ satisfies the Lipschitz condition on $[0, l)$, $\lim\limits_{s \to l} \varphi_1(t(s)) = \varphi_1(t(0)) + 2\pi\varkappa_p$, and $|\varphi(t) - \varphi_1(t)| < \pi$. The function $G(t)\exp(-i\varphi_1(t))$ belongs to $\widetilde{A}(p)$.

Put

$$\varphi_2(t) = \varphi(c_k) + \frac{s(t, c_k)}{s(c_{k+1}, c_k)}[\varphi(c_{k+1}) - \varphi(c_k)],$$

$$t \in [c_k, c_{k+1}), \quad c_{n+1} = c_1, \quad k = \overline{1, n}, \tag{1.5}$$

where $s(t, c_k)$ is the length of the least arc of the curve Γ with the ends t and c_k.

Then the function $\varphi_3 = \varphi - \varphi_1 - \varphi_2$ is continuous at the points c_k and $\sup_{t \in \Gamma} \Omega(\varphi_3, t) < a(p)$.

Thus the following lemma is valid.

Lemma 1.1. *Let $G \in \widetilde{A}(p)$ and c_k, $k = \overline{1, n}$, be its p-points of discontinuity. Then*

$$G(t) = |G(t)| G_1(t) G_2(t) G_3(t), \tag{1.6}$$

where $G_k(t) = \exp i\varphi_k(t)$, $k = 1, 2, 3$; G_1 satisfies the Lipschitz condition and $\varkappa(G_1) = \varkappa_p(G)$; φ_2 is a piecewise continuous function given by (1.5), *φ_3 is continuous at the points c_k and $\sup_{t \in \Gamma} \Omega(\varphi_3, t) < a(p)$. If $G \in A(p)$, then in decomposition* (1.6) *we take $G_2 \equiv 1$.*

In view of the remark of the previous section, the jump of the function $\arg_p G(t)$ at the points c_k or, which is the same thing, of the function φ_2 will be δ_k for $1 < p < 2$ and $\delta_k - 2\pi$ for $2 < p < \infty$. Denoting this jump by $2\pi\mu_k$, we obtain

$$\begin{aligned} \frac{1}{p'} \le \mu_k < \frac{1}{p} \quad &\text{for} \quad 1 < p < 2, \\ -\frac{1}{p'} < \mu_k < -\frac{1}{p} \quad &\text{for} \quad 2 < p < \infty. \end{aligned} \tag{1.7}$$

1.4. Statement of the result. The aim of §1–3 is to prove the validity of the following

Theorem 1.1. *Let Γ be a closed curve of the class J^*, $G \in \widetilde{A}(p)$, $1 < p < \infty$ and $\varkappa = \varkappa_p(G)$ be its index. Then*

I. *For the problem* (I) *in the class $\mathcal{K}^p(\Gamma)$ the following assertions are valid:*

(i) *if $\varkappa \geq 0$, then the problem is solvable for any $g \in L^p(\Gamma)$, and its general solution is given by the equality*

$$\phi(z) = X(z) \mathcal{K}_\Gamma \Big(\frac{g}{X^+}\Big)(z) + X(z) P_{\varkappa - 1}(z), \tag{1.8}$$

where the function X is constructed in quadratures in terms of G (*by formulas* $3.30, 3.27 - 3.29$) *and $P_{\varkappa-1}$ is an arbitrary polynomial of degree not higher than $\varkappa - 1$, $P_{-1}(z) \equiv 0$;*

(ii) *if $\varkappa < 0$, then the homogeneous problem has only the zero solution, while the inhomogeneous problem is solvable only for the functions g satisfying the condition*

$$\int_\Gamma \frac{t^k}{X^+(t)} g(t) d = 0, \quad k = 0, 1, \ldots, |\varkappa| - 1. \tag{1.9}$$

If this condition is fulfilled, then the solution is given by (1.8), *where* $P_{\varkappa-1} \equiv 0$.

II. *For the singular integral equation* (II) *in the space* $L^p(\Gamma)$, $1 < p < \infty$, *the Noether theorems are valid under the assumption that* $(a-b)(a+b)^{-1} \in \widetilde{A}(p)$.

Moreover,

$$l - l' = \varkappa_p(G), \tag{1.10}$$

where l *and* l' *is the number of linearly independent solutions of the homogeneous equation* (II) *and its adjoint.*

Solutions of the singular integral equation (II) with $V = 0$, *are given by the formula*

$$\varphi = \phi^+ - \phi^-, \tag{1.11}$$

where ϕ *is the solution of the problem of conjugation*

$$\phi^+ = (1-b)(a+b)^{-1}\phi^- + f(a+b)^{-1}.$$

1.5. On the method of proving Theorem 1.1. The first assertion of Theorem 1.1 is a generalization of the well-known results [66], [18], [141]. In those works, by developing the method of solving the problem (I) in continuous and piecewise continuous posing, the authors justified the theory of solving the problem of conjugation in the classes $\mathcal{K}^p(\Gamma)$ using the method of factorization.

The essence of this method consists in the following: if for a function G, $\inf |G| > 0$, one can construct a function X which is analytic on the plane cut along Γ, and for which the relations: (i) $X \in \widetilde{\mathcal{K}}^p(\Gamma)$; (ii) $\frac{1}{X} \in \widetilde{\mathcal{K}}_{p'}$, (Γ); (iii) $X^+ = GX^-$, (iv) $X^+ \in W_p(\Gamma)$ are valid, then the assertion I of Theorem 1.1 with $\varkappa$ defined by the relation $\lim_{z\to\infty} X(z)z^{\varkappa} = \text{const} \neq 0$ is valid (see, e.g., [68]).

The function X satisfying the condition (i)–(iv) is called a factor-function for G in the class $\mathcal{K}^p(\Gamma)$. Since the solution of the characteristic singular integral equation in the class $L^p(\Gamma)$ reduces equivalently to the solution of the problem of linear conjugation ([66]) with the coefficient $G = (a-b)(a+b)^{-1}$, from the characterizability of the function follows neotherianness of the characteristic singular integral equation, and hence neotherianness of the operator $A : a\varphi + bS_\Gamma\varphi$ in $L^p(\Gamma)$. As far as the neotherianness and the index do not vary by adding a compact operator to a Noetherian operator [3], we immediately arrive at the assertion II of Theorem 1.1.

Proof of Theorem 1.1 is performed by means of the factor-function for $G \in \widetilde{A}(p)$. At this step the results of Chapter I, §2–4, are of importance.

Construction of a factor-function first for particular cases of curves and coefficients allows one to investigate the problem in the general case.

§ 2. On the Belonging to the Smirnov Class of the Function $\exp \mathcal{K}_\Gamma\varphi$

2.1. The belonging to the Smirnov class of the function $\exp \mathcal{K}_\Gamma\varphi$. When a function is factored in the class $\mathcal{K}^p(\Gamma)$, there naturally arises the problem of finding the conditions on Γ and φ under which the function

$$X(z) = \exp\Big[\frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(t)dt}{t-z}\Big]$$

belongs to some Smirnov class in domains bounded by the curve Γ. In this section we present sufficient conditions for this.

First of all, we prove the following

Lemma 2.1. *If $\Gamma \in R$, then for the norm of the operator S_Γ in $L^p(\Gamma)$ the inequality*

$$||S_\Gamma|| \le Cp \tag{2.1}$$

is valid for $p \ge 2$, where C is independent of p.

Proof. By induction we prove that for an arbitrary natural k,

$$||S_\Gamma||_{2^k} \le ||S_\Gamma||_2 \operatorname{ctg}\frac{\pi}{2^{k+1}}.$$

For $k > 1$ we use the inequality (see Chapter I, subsection 3.2, inequality (3.38))

$$||S_\Gamma||_{2p} \le ||S_\Gamma||_p + \sqrt{1 + ||S_\Gamma||_p^2}.$$

Then for $p = 2^k$ we will have

$$\begin{aligned}||S_\Gamma||_{2^{k+1}} &\le ||S_\Gamma||_2 \operatorname{ctg}\frac{\pi}{2^{k+1}} + \sqrt{1 + ||S_\Gamma||_2^2 \operatorname{ctg}^2\frac{\pi}{2^{k+1}}} \le \\ &\le ||S_\Gamma||_2\Big[\operatorname{ctg}\frac{\pi}{2^{k+1}} + \sin^{-1}\frac{\pi}{2^{k+1}}\Big] = ||S_\Gamma||_2 \operatorname{ctg}\frac{\pi}{2^{k+2}}.\end{aligned}$$

Let now $2^k < p < 2^{k+1}$ and t be such that $\frac{1}{p} = \frac{t}{2^k} + \frac{1-t}{2^{k+1}}$. Then by virtue of the interpolation theorem we arrive at

$$||S_\Gamma||_p \le (||S_\Gamma||_2 \operatorname{ctg}\frac{\pi}{2^{k+1}})^t(||S_\Gamma||_2 \operatorname{ctg}\frac{\pi}{2^{k+2}})^{1-t} \le \frac{4}{\pi}||S_\Gamma||_{2p} = Cp. \quad \blacksquare$$

Theorem 2.1. *Let Γ be a closed Jordan curve of the class R bounding a finite domain D^+ and and an infinite domain D^-. Then:*

(i) *for any bounded, measurable on Γ function φ there exist numbers $\delta > 0$ and an integer $n_0 \ge 0$ such that*

$$\exp(\mathcal{K}_\Gamma\varphi) = X(z) \in E^\delta(D^+), \quad (z-z_0)^{-n_0}[X(z)-1] \in E^\delta(D^-);$$

(ii) *for an arbitrary continuous on Γ function φ we have*

$$X(z) \in \cap_{p>1}E^p(D) \quad \text{and} \quad [X(z)-1] \in \cap_{p>1}E^p(D^-).$$

Proof. Assume that $\sup|\varphi| < M$ and show that $X \in E^\delta(D^+)$ for $\delta < (CeM)^{-1}$, where C is a constant from the inequality (2.1). Denote by Γ_r the image of the circumference $|\zeta| = r$ under a conformal mapping of the circle U onto the domain D^+. We have

$$\int_{\Gamma_r} |X(z)|^\delta |dz| \le \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_r} \Big| \frac{\delta}{2\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{t-z} \Big|^n |dz|. \tag{2.2}$$

Since $\Gamma \in R$ and φ is bounded, by the corollary of Theorem 3.3 from Chapter I we obtain

$$(\mathcal{K}_\Gamma \varphi)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{t-z} \in \cap_{p>1} E^p(D^+).$$

Hence $[\mathcal{K}_\Gamma\varphi]^n \in E^1(D^+)$ for an arbitrary natural n. Therefore $\int_{\Gamma_r} |(\mathcal{K}_\Gamma\varphi)(z)|^n |dz|$ increases together with r ([43], p. 422), and

$$\int_{\Gamma_r} \Big| \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi dt}{t-z} \Big|^n |dz| \le \int_{\Gamma} \Big| \frac{\varphi(\tau)}{2} + \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{t-\tau} \Big|^n d\sigma \le$$

$$\le 2^n \bigg(\int_{\Gamma} \Big| \frac{\varphi(\tau)}{2} \Big|^n d\sigma + \int_{\Gamma} \Big| \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{t-\tau} \Big|^n d\sigma \bigg). \tag{2.3}$$

Now (2.2) implies

$$\int_{\Gamma_r} |X(z)|^\delta |dz| \le \sum_{n=0}^{\infty} \frac{\delta^n}{n!} \bigg[\int_{\Gamma} |\varphi(\tau)|^n d\sigma + \int_{\Gamma} \Big| \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)dt}{t-\tau} \Big|^n d\sigma \bigg] \le$$

$$\le l e^{\delta M} + l \sum_{n=0}^{\infty} \frac{(\delta M \|S_\Gamma\|_n)^n}{n!}, \tag{2.4}$$

where l is the length of the curve Γ. By lemma 2.1, $\|S_\Gamma\|_n \le Cn$. Taking into account the last inequality we can see that for $\delta < (CMe)^{-1}$ the series converges. Thus we conclude that for such δ, $X \in E^\delta(D^+)$.

In the domain D^- consider now the functions

$$Y_n(z) = (z - z_0)^{-n}[X(z) - 1], \quad z_0 \in D^+.$$

If Γ_r are the images of circumferences $|\zeta| = r$ for conformal mapping of the circle U onto D^- (we mean that the mapping function is of the form $z = \frac{1}{\zeta} + w(\zeta)$, where w is a regular in U function), then we can easily see that for small r the length of the curve Γ_r – $|\Gamma_r| = 0(\frac{1}{r})$, and $|Y(z)|^\delta = 0(r^{n\delta})$, $z \in \Gamma_r$. Then

$$\int_{\Gamma_r} |Y_n(z)|^\delta |dz| \le \sup_{z\in\Gamma_r} |Y_n(z)| \int_{\Gamma_r} |dz| \le N r^{n\delta - 1}. \tag{2.5}$$

Assuming $n \geq \left[\frac{1}{\delta}\right]$, from (2.5) it follows that for some $r_0 \in (0,1)$,

$$\sup_{r \leq r_0} \int_{\Gamma_r} |Y_n(z)|^\delta |dz| < \infty.$$

For $r \in (r_0, 1)$, as in the case of D^+, we have

$$\sup_{r_0 < r < 1} \int_{\Gamma_r} |Y_n(z)|^\delta |dz| < \infty. \tag{2.6}$$

On the basis of the inequalities (2.5)–(2.6) we conclude that for $\delta < (CMe)^{-1}$ and $n \geq \left[\frac{1}{\delta}\right]$ the inclusion

$$(z - z_0)^{-n}[X(z) - 1] \in E^\delta(D^-)$$

is valid.

Assume now that φ is a continuous on Γ function, p arbitrary positive number and φ_0 is a rational function with poles outside Γ such that $M = \sup|\varphi(t) - \varphi_0(t)| < (Cpe)^{-1}$, where C is a number from (2.1). As is proven, $X \in E^\delta(D^+)$, where $\delta < (CMe)^{-1}$, $M = \sup_{t\in\Gamma}|\varphi(t)|$. Consequently, $[\mathcal{K}_\Gamma(\varphi - \varphi_0)(z)]$ belongs to $E^p(D^+)$. On the other hand, $X_0(z) = \exp(\mathcal{K}_\Gamma\varphi_0)(z)$ is continuous. Hence $|X_0(z)| \geq m > 0$, and thus $\exp \mathcal{K}_\Gamma\varphi \in E^p(D^+)$.

The fact that $[\exp \mathcal{K}_\Gamma\varphi - 1] \in \cap_{p>1} E^p(D^-)$ is proved analogously. ■

Remark. As it follows from the proof of the conclusive part of the theorem, the number δ can be taken from the condition $\delta < (Ce\nu(\varphi))^{-1}$, where $\nu(\varphi) = \inf_\psi \sup_{t\in\Gamma}|\varphi(t) - \psi(t)|$, and the lower bound is taken over all rational functions ψ.

2.2. The case of unclosed curves. In the case where Γ is an open curve of the class R with the tangents at the end points, then complementing it with respect to the closed curve $\widetilde{\Gamma} \in R$ (see Lemma 3.4, Chapter I) and applying Theorem 2.1 to the function $\mathcal{K}_\Gamma\varphi_1$, where

$$\varphi_1(t) = \begin{cases} \varphi(t), & t \in \Gamma, \\ 0, & t \in \widetilde{\Gamma}\backslash\Gamma, \end{cases}$$

we easily find that if $\nu(\varphi_1) < (2Ce)^{-1}$, then the function $X(z) = \exp(\mathcal{K}_\Gamma\varphi)(z)$, $z \overline{\in} \Gamma$ is representable in the form

$$X(z) = \frac{1}{2\pi i} \int_\Gamma \frac{X^+(t) - X^-(t)}{t - z} dt + 1.$$

2.3. On the solution of the problem (I) with a continuous coefficient and $\Gamma \in R$. The above proven theorem allows one to solve the problem in the class $K^p(\Gamma)$, when $\Gamma \in R$ and G is a continuous function. In this case, the function

$$X(z) = \begin{cases} \exp h(z), & z \in D^+, \\ (z - z_0)^{-\varkappa} \exp h(z), & z \in D^-, \quad z_0 \in D^+, \end{cases} \tag{2.7}$$

where

$$\varkappa = \operatorname{ind} G, \quad h(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\ln[(t - z_0)^{-\varkappa} G(t)]}{t - z} dt \tag{2.7$'$}$$

satisfies the conditions (i)–(iii) from the definition of the factor-function, and

$$\lim_{z \to \infty} z^{\varkappa} X(z) = 1. \tag{2.8}$$

Indeed, the fulfilment of the conditions $(i) - (ii)$ is a consequence of Theorem 2.1. The condition (iii) is verified by the Sokhotskiĭ-Plemelj formula; the fulfilment of the condition (2.8) is obvious.

From the above arguments it follows that all the solutions of the problem (I) of the class $\mathcal{K}^p(\Gamma)$ are contained in the set of functions specified by the equality

$$\phi(z) = \frac{X(z)}{2\pi i} \int\limits_{\Gamma} \frac{g(t) dt}{X^+(t)(t - z)} + P_{\varkappa - 1} X(z), \tag{2.9}$$

where $P_{\varkappa-1}$ is an arbitrary polynomial of degree $\varkappa - 1$. It is not also difficult to verify that $\phi \in \widetilde{\mathcal{K}}^r(\Gamma)$, $r \in (1, p)$. But for $\varkappa \geq 0$ any such a solution belongs to the class $\mathcal{K}^p(\Gamma)$ (see [68], Chapter IV, §5, Theorem 1). Hence, if $\varkappa \geq 0$, then the problem (I) is solvable for any $g \in L^p(\Gamma)$, and all its solutions of the class $\mathcal{K}^p(\Gamma)$ are representable in the form (2.9), where X is given by the equalities (2.7)–(2.7$'$).

In spite of the fact that for $\varkappa < 0$ the function ϕ (with $P_{\varkappa-1} = 0$) belongs to $\widetilde{\mathcal{K}}^p(\Gamma)$, it does not belong to $\mathcal{K}^p(\Gamma)$, because X possesses the pole of order $|\varkappa|$ at the point $z = \infty$. As usual, the expansion of the integral multiplier in the neighbourhood of that point results in a solvability condition of the type (1.9).

2.4. Some functions from $W_p(\Gamma)$, $\Gamma \in R$. On the basis of the result obtained in the section 2.3, we can point out some functions from $W_p(\Gamma)$ when $\Gamma \in R$.

Theorem 2.2. *Let Γ be a closed Jordan smooth curve. If φ is a real continuous on Γ function, then*

$$w(t) = \exp\left\{\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau) d\tau}{\tau - t}\right\} \in \cap_{\delta > 1} W_\delta(\Gamma). \tag{2.10}$$

Moreover, if Γ *is a curve such that for some* $p > 1$ *the function*

$$\rho(t) = \prod_{k=1}^{n} |t - c_k|^{h_k}, \quad -\frac{1}{p} < h_k < \frac{1}{p'},$$

belongs to $W_p(\Gamma)$*, then*

$$\rho(t)w(t) = \prod_{k=1}^{n} |t - c_k|^{h_k} \exp\left\{\frac{1}{2\pi} \int_{\Gamma} \frac{\varphi(\tau)d\tau}{\tau - t}\right\} \in W_p(\Gamma). \qquad (2.11)$$

Proof. Let $G(t) = \exp i\varphi(t)$. Then G is continuous and $\operatorname{ind} G = 0$. According to the result obtained from (2.3), the problem with such a coefficient is solvable for any $g \in L^p(\Gamma)$ and its solutions are given by (2.9). But then the operator

$$T : g \to X^+ S_\Gamma \frac{g}{X^+}$$

is defined on the entire $L^p(\Gamma)$ and maps this space into itself. Consequently, the operator T, by Theorem 2.2 in Chapter I, is continuous in $L^p(\Gamma)$. Hence the function $X^+ = w\sqrt{G}$, and thus w (because of $|G| = 1$) belongs to the set $\cap_{\delta>1} W_\delta(\Gamma)$. Next, taking into account the fact that $\rho \in W_{p+\varepsilon}(\Gamma)$ for some $\varepsilon > 0$, by means of the theorem from (0.20) we obtain the inclusion (2.11) as well. ■

Remark. If $\Gamma \in K$ then as $K \subset R$ we have $\rho \in W_p(\Gamma)$ ([68], p.79)

Remark. The result of Theorem 2.2 allows one o construct a factor-function for a piecewise continuous on Γ function G under the assumptions: (i) $\Gamma \in R$ and possesses the tangents at the points of discontinuity of G; (ii) $\frac{1}{2\pi}[\arg G(t_k+) - \arg G(t_k-)] = h_k \neq \frac{1}{p}(\mod 1)$ We omit the details.

§ 3. The Construction of a Factor-Function for $G \in \widetilde{A}(p)$ and $\Gamma \in J$.

3.1. The case where Γ is a smooth curve and $G \in A(p)$. The construction is divided into two steps: first we prove the unique solvability of the problem (I) for $\operatorname{ind} G = 0$ and then, relying on the existence of the solution, we construct the factor-function explicitly.

Step 1. Let $G \in A(p)$, $\operatorname{ind}_p G = 0$. As is shown in [68], any solution of the problem (I) under the assumption $\phi \in K^p(\Gamma)$, $\Gamma \in R$, generates the solution $\varphi = \phi^+ - \phi^-$ of the class $L^p(\Gamma)$ of the linear singular integral equation (3.1),

$$a\varphi + bS_\Gamma\varphi = g, \qquad (3.1)$$

where $a = \frac{1}{2}(1 + G)$ and $b = \frac{1}{2}(1 - G)$. Conversely, to every solution $\varphi \in L^p(\Gamma)$ of the equation (3.1) there corresponds the solution $\phi = \mathcal{K}_\Gamma\varphi$ of the problem (I) (with $G = (a - b)(a + b)^{-1}$ and $g(a + b)^{-1}$ instead of g) corresponding to $\mathcal{K}^p(\Gamma)$. Therefore to prove the unique solvability of the problem (I), it suffices to show the unique solvability of the equation (3.1). Rewrite it in the form

$$M\varphi \equiv a(t(s))\varphi(t(s)) + \frac{b(t(s))}{\pi i}\int_0^l \frac{\varphi(t(\sigma))ie^{i\sigma}d\sigma}{e^{i\alpha\sigma}-e^{i\alpha s}} +$$

$$+\frac{b(t(s))}{\pi i}\int_0^l \varphi(t(s))\Big[\frac{t'(\sigma)}{t(\sigma)-t(s)} - \frac{i\alpha e^{i\alpha}}{e^{i\alpha\sigma}-e^{i\alpha s}}\Big]d\sigma = g(t(s)),$$

$$\alpha = \frac{2\pi}{l}, \quad 0 \le s \le l$$

or

$$M\varphi \equiv a\varphi + bS_\gamma\varphi + b(S_\Gamma - S_\gamma)\varphi = g, \tag{3.2}$$

where γ is a circumference of length l.

Since Γ is the smooth curve, the operator $S_\Gamma - S_\gamma$ is compact in $L^p([0,l])$ [48]. Moreover we have $(a-b)(a+b)^{-1} = G \in A(p)$ and $\operatorname{ind} G = 0$. Therefore for the equation

$$a\varphi + bS_\gamma\varphi = g$$

in the class $L^p(\gamma)$ the Fredholm theorems are valid. But then by virtue of Atkinson's theorem [3], the equation (3.2) is also Fredholmian.

Show that the equation $M\varphi = 0$ in $L^p(\Gamma)$ has only the zero solution. To this end, we consider along with it the equation

$$M'\psi \equiv a\psi - S_\Gamma(b\psi) = 0. \tag{3.3}$$

The operators M' and M are conjugate. (This follows from the fact that for every linear functional μ on $L^p(\Gamma)$ there exists a function $\psi \in L^{p'}(\Gamma)$ such that $\mu(\varphi) = \int_\Gamma \varphi(t)\psi(t)dt$.) If we put

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{b(t)\psi(t)dt}{t-z},$$

where ψ is a solution of the equation (3.3) in $L^{p'}(\Gamma)$, then $\Psi \in \mathcal{K}^{p'}(\Gamma)$ and satisfies the boundary condition

$$\Psi^+(t) = \frac{1}{G}\Psi^-(t). \tag{3.4}$$

But $\phi\Psi \in \mathcal{K}^1(\Gamma)$ (because densities of the corresponding integrals belong to conjugate classes) and $(\phi\Psi)^+ = (\phi\Psi)^-$, and this problem in $\mathcal{K}^1(\Gamma)$ has only the zero solution. Therefore either ϕ or $\Psi = 0$. This implies that either the equation $M\varphi = 0$ in $L^p(\Gamma)$ or the equation $M'\psi = 0$ has only zero solution. As far as the operator M is Fredholmian, both equations have only zero solutions, and the inhomogeneous equations $M\varphi = g$, $g \in L^p(\Gamma)$, $M'\psi = f$,

$f \in L^{p'}(\Gamma)$ are uniquely solvable. Thus we arrive at the conclusion on the solvability of the problem (I).

Remark. Note that after establishing of Fredholm property for the equation (3.1), the smoothness of Γ has no importance. So, if $\Gamma \in R$ and the equation (3.1) is Fredholmian, then it is uniquely solvable (see also [42], p. 258).

Step 2. The construction of a factor-function.

Lemma 3.1. *Let a closed Jordan rectifiable curve $\Gamma \in R$. If for a function G, $\inf|G| > 0$ there exists a function $\ln G$ (i.e., we select a function $\arg G$) such that it is bounded and the boundary values X^+ of the function*

$$X(z) = \exp[\mathcal{K}_\Gamma(\ln G)(z)] \tag{3.5}$$

belong to $W_p(\Gamma)$, then X is the factor-function of G in $\mathcal{K}^p(\Gamma)$.

Proof. Since $\ln G$ is bounded, by Theorem 2.1 there exists $\delta > 0$ such that $X \in E^\delta(D^+)$. But $X^+ \in W_p(\Gamma)$, and therefore $X^+ \in L^p(\Gamma)$, $\frac{1}{X^+} \in L^{p'}(\Gamma)$ (see §4, Chapter I, Lemma 4.2). Since $X^- = \frac{1}{G}X^+$, $X^- \in L^p(\Gamma)$ as well, and $\frac{1}{X^-} \in L^{p'}(\Gamma)$. Hence $(X-1) \in E^p(D^\pm)$, $\left(\frac{1}{X}-1\right) \in E^{p'}(D^\pm)$. From this we can conclude that X possesses all the properties of the factor-function (see section 1.5). ∎

Lemma 3.2. *Let $\Gamma \in R$, $\inf|G| > 0$ and let the problem of linear conjugation* (I) *be solvable for any $g \in L^p(\Gamma)$. If for some function X the conditions* (ii) *and* (iii) *from the definition of a factor-function are fulfilled, then $X^+ \in W_p(\Gamma)$.*

Proof. Under our assumptions, the function given by the formula (2.9) for $P_{\varkappa-1} = 0$ belongs to $L^p(\Gamma)$ for any $g \in L^p(\Gamma)$. The statement of the lemma follows now from Theorem 2.2 of Chapter I. ∎

Theorem 3.1. *Let a closed Jordan rectifiable curve $\Gamma \in R$ be such that for an arbitrary $G \in A(p)$ with $\operatorname{ind} G = 0$, the problem* (I) *is uniquely solvable in $\mathcal{K}^p(\Gamma)$. If $\ln G = \ln|G(t)| + i\arg_p G(t)$, then the function X given by* (3.5) *is the factor-function of G in the class $\mathcal{K}^p(\Gamma)$.*

Proof. Put

$$u(z) = \exp \mathcal{K}_\Gamma(\ln|G|)(z), \quad v(z) = \exp \mathcal{K}_\Gamma(i\arg_p G)(z)$$

and show that $u^+ \in \cap_{\delta>1} W_\delta(\Gamma)$, $v^+ \in W_{\overline{p}+\varepsilon}(\Gamma)$ where $\overline{p} = \max(p, p')$ and ε is a positive number.

Let $H(z) = \mathcal{K}_\Gamma(\ln|G|)(z)$. By Theorem 2.1 we can choose $\lambda_0 > 0$ such that the functions

$$u_\lambda = \begin{cases} \exp\{\lambda H(z)\}, & z \in D^+, \\ \exp\{\lambda H(z)\} - 1, & z \in D^- \end{cases} \tag{3.6}$$

and $u_\lambda^{-1}(z)$ belong to $\mathcal{K}^2(\Gamma)$ for $|\lambda| < \lambda_0$.

As is easily seen, u_λ is a solution of the problem of conjugation

$$\phi^+ = |G|^\lambda \phi^- + |G|^\lambda$$

of the class $\mathcal{K}^2(\Gamma)$.

By the assumption of the theorem, this problem is uniquely solvable in $\mathcal{K}^p(\Gamma)$ for an arbitrary $p > 1$ (as $|G|^\lambda \in \bigcap\limits_{p>1} A(p)$). Therefore $u_\lambda(z)$, being a solution from the class $\mathcal{K}^2(\Gamma)$, is a solution from all the classes $\mathcal{K}^p(\Gamma)$, $p > 1$. Thus $u_\lambda(z) \in \cap_{p>1}\mathcal{K}^p(\Gamma)$ for $|\lambda| < \lambda_0$. From the expansion of $|G|^\pm = |G|^{\pm\lambda_1}|G|^{\pm\lambda_2}\cdots|G|^{\pm\lambda_k}$, where $|\lambda_i| < \lambda_0$, $\sum_{i=1}^k \lambda_i = 1$, we can see that $u^{\pm 1} \in \cap_{p>1}\mathcal{K}^p(\Gamma)$, whence by Lemma 3.2 it follows that $u^+ \in \cap_{\delta>1} W_\delta(\Gamma)$.

Prove that $v^+ \in W_{\overline{p}+\varepsilon}(\Gamma)$.

By virtue of Lemma 1.1, we may assume that $|\arg_p G| < \frac{\pi}{p}$ (because G_1 belongs to the Liepschitz class and it can be factored by a function Z such that $Z^{\pm 1}$ are bounded functions). Then, as it follows from the proof of Theorem 2.1, there exists an absolute constant $d = (Ce\pi)^{-1} < 1$ such that $v^{\pm 1} \in E^\delta(D^+)$ for $\delta < \overline{p}d = \delta_0$, $\overline{p} = \max(p, p')$.

Let first $p > 1 + \frac{1}{d}$. Then $p' < \left(\frac{1+d}{d}\right)' = 1 + d$. From the assumption regarding p we have $1 + d < pd = \overline{p}d = \delta_0$. Therefore $p' < \delta_0$, and hence $\frac{1}{v} \in \mathcal{K}^{p'}(\Gamma)$.

Choose $\varepsilon > 0$ so small as to have $G \in A(p+\varepsilon)$ and $\operatorname{ind}_{p+\varepsilon} G = \operatorname{ind}_p G = 0$ (by the definition of the class $A(p)$ and of its index, such ε exists). By the above proven, $\frac{1}{v} \in \mathcal{K}^{(p+\varepsilon)'}(\Gamma)$, and by the assumption, the problem (I) with the coefficient $G_0 = \exp(i \arg_{p+\varepsilon} G) = \exp(i \arg_p G)$ is uniquely solvable in the class $\mathcal{K}^{p+\varepsilon}(\Gamma)$. On the basis of the above reasoning we apply Lemma 3.2 and conclude that $v^+ \in W_{p+\varepsilon}(\Gamma) = W_{\overline{p}+\varepsilon}(\Gamma)$.

Let now p be an arbitrary number from the interval $[2, +\infty)$ and $|\arg_p G| < \frac{2\pi}{p}$. Assume $p_0 = 2 + \frac{1}{d}$. Then $\frac{p}{p_0}|\arg_p G| < \frac{2\pi}{p_0}$. The functions $G_0^{\pm 1} = \exp(\pm ipp_0^{-1} \arg_p G)$ belong to $A(p_0)$, $p_0 > 1 + \frac{1}{d}$ and, according to the just proven,

$$\exp\left\{\frac{\pm p}{2p_0} S_\Gamma(i \arg_p G)\right\} \in W_{p_0+\varepsilon}(\Gamma),$$

whence we obtain

$$\exp\left\{\pm \frac{1}{2} S_\Gamma(i \arg_p G)\right\} \in L^{p+\varepsilon_1}(\Gamma).$$

Consequently,

$$v^{\pm 1}(z) - 1 = \left[\exp\{\pm \mathcal{K}_\Gamma(i \arg_p G)(z)\} - 1\right] \in E^{p+\varepsilon_1}(D^\pm)$$

and hence, using again Lemma 3.2, $v^+ \in W_{\overline{p}+\varepsilon_1}(\Gamma)$.

The case $p < 2$ can be reduced to the previous one by considering the problem

$$\Psi^+ = \frac{1}{G}\Psi^- + g$$

in the class $\mathcal{K}^{p'}(\Gamma)$ and taking into account that $A(p') = A(p)$.

It follows from the above arguments that:

(i) for an arbitrary $\beta > 0$, $(u^\beta)^+ \in \cap_{p>1} W_p(\Gamma)$;

(ii) there exists $\eta_0 > 0$ such that for $0 < \eta < \eta_0$ we have $(v^+)^{1+\eta} \in W_{\overline{p}}(\Gamma)$.

Using the interpolation theorem from (0.20), we find that $X^+ = u^+ v^+ \in W_{\overline{p}}(\Gamma)$ which, by Lemma 3.1, completes the proof of the theorem. ■

It has been proved that if Γ is a smooth curve, then for an arbitrary $p > 1$ and for $G \in A(p)$ with $\operatorname{ind} G = 0$ the problem (I) is uniquely solvable and therefore from Theorem 3.1 we have

Theorem 3.2. *If Γ is a closed smooth curve, $G \in A(p)$ with $\operatorname{ind}_p G = 0$, $\ln G = \ln |G| + i \arg_p G$, then the function*

$$w(t) = \exp\left\{\frac{1}{2} S_\Gamma(\ln G)(t)\right\}$$

belongs to the class $W_p(\Gamma)$, and the function X given by (3.5) is the factor-function of G in the class $\mathcal{K}^p(\Gamma)$.

3.2. The case $G \in A(p)$, $\Gamma \in J$. Let us first show that if $\Gamma \in J$, then the function

$$\rho_\Gamma(\sigma) = \exp\{\frac{1}{2} S_\Gamma(\ln G)(t(\sigma))\} =$$

$$= \exp\left\{\frac{1}{2\pi i}\int_0^l \frac{[\ln G_0(s)]t'(s)ds}{t(s) - t(\sigma)}\right\}, \quad G_0(s) = G(t(s)), \tag{3.7}$$

belongs to the class $W_p(\Gamma)$.

Since $\Gamma \in J$, there exists a Jordan smooth curve $\mu = \mu(s)$, $0 \le s \le l$, for which the condition (1.1) is fulfilled. By Theorem 3.2, the function

$$\rho_\mu(\sigma) = \exp\left\{\frac{1}{2\pi i}\int_0^l \frac{[\ln G_0(s)]\mu'(s)ds}{\mu(s) - \mu(\sigma)}\right\} \tag{3.8}$$

belongs to the class $W_p(\mu)$. Because μ is a smooth curve, by Theorem 4.3 of Chapter I, we conclude that ρ_μ^p belongs to the Muckenhoupt class A_p. On the other hand, $\Gamma \in K \subset R$. Therefore by Theorem 4.2 of Chapter I we have $\rho_\mu \in W_p(\Gamma)$. It follows from the definition of the class J that $0 < c_1 < \rho_\mu(\sigma)\frac{1}{\rho_\Gamma(\sigma)} < c_2 < \infty$. Consequently $\rho_\Gamma \in W_p(\Gamma)$ as well.

Let now $G \in A(p)$, $\operatorname{ind} G = 0$ and X be a function given by (3.5). Then $X^+ = \sqrt{G}\rho_\Gamma$. Since $0 < m \le |G| \le M$ and $\rho_\Gamma \in W_p(\Gamma)$, we find that

$X^+ \in W_p(\Gamma)$. From this on the basis of Lemma 3.1 we conclude that X is the factor-function for G in $\mathcal{K}^p(\Gamma)$.

3.3. The case $G \in A(p)$, $\Gamma \in J^*$. The construction of the factor-function is performed in two steps. First we construct it for the case where G is a constant outside the arc of small length and then for the general case. Proceeding from the representation (1.6) of the function $G \in A(p)$, without restriction of generality we may assume that $\operatorname{Re} G > 0$, and the intersection of the range of G with a small circle with center at the point $(1, 0)$ is of positive measure.

Let c be an arbitrary point on Γ. By virtue of Proposition 3.1 in Chapter I and by definition of the function from the class $A(p)$, there is an arc $\widetilde{\Gamma}_{ab} \subset \Gamma$ such that $c \in \widetilde{\Gamma}_{ab}$, $\widetilde{\Gamma}_{ab} \in J$ having tangents at the ends and all the values adopted on $\widetilde{\Gamma}_{ab}$ are located in a sector with the vertex at the origin and the angle less than $\frac{2\pi}{p}$.

Complement the curve $\widetilde{\Gamma}_{ab}$ by a broken line δ as in Lemma 3.4 of Chapter I. Then $\Gamma_{ab} = \widetilde{\Gamma}_{ab} \cup \delta$ will, by Proposition 3.2, be a closed curve of the class R.

On $\widetilde{\Gamma}_{ab}$, choose an arc Γ_{de} with the ends d and e lying at positive distance from the ends a and b and consider an auxiliary problem of linear conjugation: to determine a function $\phi \in \mathcal{K}^p(\Gamma_{ab})$ satisfying the condition

$$\phi^+(t) = G_c(t)\phi^-(t) + g(t), \quad t \in \Gamma_{ab}, \tag{3.9}$$

where $g \in L^p(\Gamma_{ab})$.

$$G_c(t) = \begin{cases} G(t), & t \in \Gamma_{de} \\ 1, & t \in \Gamma_{ab} \backslash \Gamma_{de}. \end{cases} \tag{3.10}$$

Since $\operatorname{Re} G > 0$, $G_c \in A(p)$ on the closed curve Γ_{ab}, owing to the result from section 3.2 we can conclude that the function

$$X(z) = \exp\left[\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\ln G_c(t)dt}{t-z}\right] \tag{3.11}$$

is the factor-function for G_c in $\mathcal{K}^p(\Gamma_{ab})$. Therefore the function

$$\rho(t_0) = \exp\left[\frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\ln G_c(t)dt}{t-t_0}\right], \quad \ln G_c = \ln|G_c| + i\arg_p G_c \tag{3.12}$$

belongs to $W_p(\Gamma_{ab})$, and hence to $W_p(\widetilde{\Gamma}_{ab})$. But it follows from (3.10) and the definition of $\arg_p G_c(t)$ that

$$\rho(t_0) = \exp\left[\frac{1}{2\pi i}\int\limits_{\Gamma_{de}} \frac{\ln G(t)dt}{t-t_0}\right]. \tag{3.13}$$

Hence, if $t_0 \in \Gamma \backslash \Gamma_{de}$, then $0 < \inf |\rho(t_0)| \le \sup |\rho(t_0)| < \infty$. Now it is not difficult to verify that $\rho \in W_p(\Gamma)$. This implies that the function X given by (3.11) is the factor-function for G_c in $\mathcal{K}_p(\Gamma)$. Since $\operatorname{ind}_p G_c = 0$, the problem (I) is uniquely solvable.Along with this we also find that the operator

$$A_{G_c} : \varphi \to \frac{1}{2}(1 - G_c)(-\varphi + S_\Gamma \varphi)$$

is Noetherian in $L^p(\Gamma)$.

Owing to the local principle of investigation of singular operators (see [42], Theorem 2.1, Chapter XII) whose validity can be easily verified even when $\Gamma \in R$, we finally conclude that the operator A_G in the class $L^p(\Gamma)$ and the problem of conjugation (I) with the coefficient G in the class $\mathcal{K}^p(\Gamma)$ are Noetherian for $\Gamma \in J^*$ and $G \in A(p)$, respectively.

Lemma 3.3. *If $\Gamma \in J^*$, $G \in A(p)$, $\operatorname{Re} G > 0$, then the problem* (I) *is uniquely solvable.*

Proof. In $L^p(\Gamma)$, consider a family singular operators $A_\alpha : \varphi \to A_\alpha \varphi$, where $A_\alpha \varphi = \phi^+ - G_\alpha \phi^-$, $\phi^\pm = \frac{1}{2}(\pm\varphi + S_\Gamma \varphi)$, $G_\alpha = \alpha + (1-\alpha)G$, $\alpha \in [0,1]$.

From the condition $\operatorname{Re} G > 0$ it follows that $G_\alpha \in A(p)$, $\operatorname{ind} G = 0$. By the above proven, the operators A_α are Noetherian. Moreover, A_α is a continuous on $[0,1]$ operator function and therefore its indices are the same for all α (see, e.g., [42], p. 163). But $A_0 = \phi^+ - G\phi^-$, $A_1 = I$. Consequently, the operator A_0 is Fredholmian. From this, just in the same way as in section 3.1, we obtain the unique solvability in $L^p(\Gamma)$ of the equation $A_0 \varphi = g$, i.e., of the problem (I). ■

Show that for $G \in A(p)$ the factor-function is again given in an ordinary way, that is, by (3.5) with $\ln G = \ln |G| + i \arg_p G$.

Let first $\operatorname{ind}_p G = 0$. Present G as $G = G_1 G_2$, where G_1 is a function from the Lipschitz class with $\operatorname{ind} G_1 = 0$, $G_2 \in A(p)$, $\operatorname{Re} G_2 > 0$ ([141], see also Lemma 1.1). The factor-function for G_1 is bounded from above and separated from zero. Using this fact, we reduce the problem (I) to the problem of same kind with coefficient G_2, $\operatorname{ind}_p G_2 = 0$. The latter is uniquely solvable by Lemma 3.3. Thus the conditions of Theorem 3.1 are fulfilled, and the factor-function of G can be written out by formula (3.5). In particular, we have

$$w(t) = \exp\left\{\frac{1}{2} S_\Gamma (\ln G)(t)\right\} \in W_p(\Gamma). \tag{3.14}$$

Consideration of the case $\operatorname{ind}_p G \neq 0$ reduces in a common way to the case of the non-zero index and then the corresponding factor-function is constructed. We do not write it out for the time being. This will be done in the sequel in a more general case.

Remark. In fact, in the present subsection the following statement was proved: *Let for any $c \in \Gamma$ there exist an arc $\Gamma_c \subset \Gamma$, a closed curve $\widetilde{\Gamma}_c$*

containing Γ_c *and extension* G_c *of* G *onto* $\widetilde{\Gamma}_c$ *which is factorizable in* $\mathcal{K}^p(\widetilde{\Gamma}_c)$. *Then* G *is factorizable in* $\mathcal{K}^p(\Gamma)$.

3.4. Some subclasses of the set $W_p(\Gamma)$, $\Gamma \in J^*$.

Lemma 3.4. *Let* $\Gamma \in J^*$, $c \in \Gamma$, $1 < p < \infty$, $-\frac{1}{p} < h < \frac{1}{p'}$. *The oscillation* $\Omega(\psi, t)$ *of the function* ψ *(see 1.3) is supposed to satisfy the condition*

$$\sup_{t\in\Gamma} \Omega(\psi, t) < \frac{2\pi\lambda(p)}{\overline{p}}, \quad \overline{p} = \max(p, p'),$$

where

$$\lambda(p) = 1 - |h|p'. \tag{3.15}$$

Then the function

$$w(t) = |t-c|^h \exp\left[\frac{1}{2\pi}\int\limits_\Gamma \frac{\psi(\tau)d\tau}{\tau - t}\right]$$

belongs to $W_p(\Gamma)$.

Proof. For $h = 0$, the lemma is valid according to subsection 3.3 (see 3.14), because $\exp i\psi \in A(p)$. Let $0 < h < \frac{1}{p'}$. Choose ε so small that

$$\sup_{t\in\Gamma} \Omega(\psi, t) < \frac{2\pi(1 - hp' - \varepsilon)}{\overline{p}}.$$

Since $\Gamma \in J^* \subset K$, then $w_1(t) = |t-c|^{h(hp'+\varepsilon)^{-1}} \in W_p(\Gamma)$. On the other hand, the function

$$w_2(t) = \exp\left[\frac{1}{2(1-hp'-\varepsilon)}(S_\Gamma\psi)(t)\right]$$

also belongs to $W_p(\Gamma)$ (since under our assumptions $\exp\frac{i\psi(t)}{1-hp'-\varepsilon} \in A(p)$). Using Theorem from (0.20), from which in particular it follows that if $w_i \in W_p(\Gamma)$, $i = 1, 2$, $p > 1$, $\eta \in (0, 1)$ then $w_1^\eta w_2^{1-\eta} \in W_p(\Gamma)$ and putting $\eta = hp' + \varepsilon$, we can easily see that the lemma in the case under consideration is valid.

The case $-\frac{1}{p'} < h < 0$ is treated analogously. ∎

Theorem 3.3. *Let* Γ *be a simple closed curve of the class* J^* *and* ψ *be a real measurable function such that*

$$\Omega(\psi, t) < \frac{2\pi}{\overline{p}}, \quad t \in \Gamma. \tag{3.16}$$

If ψ *is continuous at the points* $c_k \in \Gamma$, $(k = \overline{1, n})$ *and* $-\frac{1}{p} < h_k < \frac{1}{p'}$, *then the function*

$$w(t) = \prod_{k=1}^n |t - c_k|^{h_k} \exp\left[\frac{i}{2}(S_\Gamma\psi)(t)\right]$$

belongs to the class $W_p(\Gamma)$.

Proof. Assume first of all that $n=1$. Put $c_1=c$, $h_1=h$ and prove that

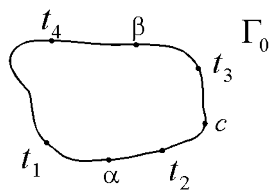
$$w(t)=|t-c|^h\exp\left[\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi(\tau)d\tau}{\tau-t}\right]\in W_p(\Gamma).$$

Since the function ψ is continuous at the point c, there exists an arc $\Gamma_0\subset\Gamma$ such that

$$\sup_{t\in\Gamma_0}\Omega(\psi,t)<\frac{2\pi\lambda(p)}{\overline{p}},$$

where $\lambda(p)$ is defined by the equality (3.15).

Choose the points $t_i=t(s_i)$ $(i=\overline{1,4})$ on Γ_0 such that the point t_i while moving in the positive direction precedes the point t_{i+1} and, moreover, $c\in(t_2,t_3)$. Introduce the notation $(t_i,t_{i+1})=\Gamma_i$, $i=\overline{1,4}$, $t_5=t_1$.



Let now $f\in L^p(\Gamma,w)$. By virtue of the Minkowski inequality, we have

$$\left\{\int\limits_{\Gamma}\left|\int\limits_{\Gamma}\frac{f(\tau)d\tau}{\tau-t}w(t)\right|^p|dt|\right\}^{\frac{1}{p}}\le\sum_{i,j}^{4}\left\{\int\limits_{\Gamma_i}\left|w(t)\int\limits_{\Gamma_j}\frac{f(\tau)d\tau}{\tau-t}\right|^p|dt|\right\}^{\frac{1}{p}}. \quad (3.17)$$

Show first that

$$\int\limits_{\Gamma_2}\left|w(t)\int\limits_{\Gamma_2}\frac{f(\tau)}{\tau-t}\right|^p|dt|\le C_1\int\limits_{\Gamma}|w(t)f(t)|^p|dt|.$$

Let $\alpha\in\Gamma_1$, $\beta\in\Gamma_3$, $\Gamma_2'=(\alpha,\beta)$ and let a function ψ^*, given on Γ and coinciding with ψ on Γ_2' be continuous on $\Gamma\backslash\Gamma_2'$, and $\psi^*(\alpha)=\psi(\alpha)$, $\psi^*(\beta)=\psi(\beta)$. Obviously,

$$\sup_{t\in\Gamma}\Omega(\psi^*,t)\le\sup_{t\in\Gamma_2'\subset\Gamma_0}\Omega(\psi,t)<\frac{2\pi\lambda(p)}{\overline{p}} \quad (3.18)$$

and

$$\sup_{t\in\Gamma_2}\left|\int\limits_{\Gamma}\frac{\psi(\tau)-\psi^*(\tau)}{\tau-t}d\tau\right|=\sup\left|\int\limits_{\Gamma\backslash\Gamma_2'}\frac{\psi(\tau)-\psi^*(\tau)}{\tau-t}d\tau\right|<\infty. \quad (3.19)$$

Assume

$$\phi(t)=\begin{cases}f(t) & \text{for}\quad t\in\Gamma_2,\\ 0 & \text{for}\quad t\in\Gamma\backslash\Gamma_2.\end{cases}$$

We have

$$\int\limits_{\Gamma_2}\left|w(t(s))\int\limits_{\Gamma}\frac{f(\tau)d\tau}{\tau-t(s)}\right|^p ds=\int\limits_{\Gamma_2}\left|w(t(s))\int\limits_{\Gamma}\frac{\phi(\tau)d\tau}{\tau-t(s)}\right|^p ds=$$

$$=\int\limits_{\Gamma_2}\left|\int\limits_{\Gamma}\frac{\phi(\tau)d\tau}{\tau-t}|t-c|^h\exp\left[\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi^*(\tau)}{\tau-t}+\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi(\tau)-\psi^*(\tau)}{\tau-t}d\tau\right]\right|^p ds\le$$

$$\le C_2\int\limits_{\Gamma}\left|\int\limits_{\Gamma}\frac{\phi(\tau)d\tau}{\tau-t}|t-c|^h\exp\left[\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi^*(\tau)d\tau}{\tau-t}\right]\right|^p ds.$$

Owing to (3.18), we use Lemma 3.4 to obtain

$$\int\limits_{\Gamma_2}|w(t(s))\int\limits_{\Gamma_2}\frac{f(\tau)d\tau}{\tau-t(s)}|^p ds\le C_3\int\limits_{\Gamma}\left|f(t)||t-c|^h\exp\left[\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi^*(\tau)d\tau}{\tau-t}\right]\right|^p ds,$$

whence, taking into account (3.19), we get

$$\int\limits_{\Gamma_2}\left|w(t(s))\int\limits_{\Gamma}\frac{f(\tau)d\tau}{\tau-t(s)}\right|^p ds\le$$

$$\le C_3\int\limits_{\Gamma_2}\left|f(t)|t-c|^h\exp\left\{\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi(\tau)d\tau}{\tau-t}+\frac{1}{2\pi}\int\limits_{\Gamma}\frac{\psi^*(\tau)-\psi(\tau)}{\tau-t}d\tau\right\}\right|^p ds\le$$

$$\le C_4\int\limits_{\Gamma_2}|w(t)f(t)|^p ds. \tag{3.20}$$

Let now $i\neq 2$ and

$$F(\tau)=\begin{cases}f(t) & \text{for} \quad \tau\in\Gamma_i,\\ 0 & \text{for} \quad \tau\in\Gamma\backslash\Gamma_i.\end{cases}$$

Since $\Gamma\in J^*$, under the adopted assumptions on ψ we conclude that $\exp\left[\frac{1}{2\pi}\int_\Gamma\frac{\psi(\tau)d\tau}{\tau-t}\right]\in W_p(\Gamma)$ (see Lemma 3.4). On the other hand, for $i\neq 2$ we have

$$0<\inf_{t\in\Gamma_i}|t-c|^{hp}\le\sup_{t\in\Gamma_i}|t-c|^{hp}<\infty.$$

Therefore

$$\int\limits_{\Gamma_i}|w(t)\int\limits_{\Gamma_i}\frac{f(\tau)d\tau}{\tau-t}|^p ds=C_5\int\limits_{\Gamma}\left|\int\limits_{\Gamma}\frac{f(\tau)d\tau}{\tau-t}\right|^p\left|\exp\frac{p}{2\pi}\int\limits_{\Gamma}\frac{\psi(\tau)d\tau}{\tau-t}\right|ds\le$$

$$\le C_6\int\limits_{\Gamma}|F(\tau)|^p\left|\exp\frac{p}{2\pi}\int\limits_{\Gamma}\frac{\psi(\tau)d\tau}{\tau-t}\right|ds=$$

$$= C_6 \int\limits_{\Gamma_i} |f(t)|^p \left| \exp \frac{p}{2\pi} \int\limits_{\Gamma} \frac{\psi(\tau)d\tau}{\tau - t} \right| ds \leq C_7 \int\limits_{\Gamma} |w(t)f(t)|^p ds. \quad (3.21)$$

Further, consider the case in where Γ_i and Γ_j have no common end points. It follows from (3.20)–(3.21) that the restrictions of the function w on the arcs Γ_i belong to $W_p(\Gamma_i)$, $i = \overline{1,4}$. Hence $w \in L^p(\Gamma_i)$, $\frac{1}{w} \in L^{p'}(\Gamma_i)$. If $\tau \in \Gamma_j$, $t \in \Gamma_i$, then $\inf |\tau - t| > 0$, and using Hölder's inequality, we obtain

$$\int\limits_{\Gamma_i} \left| w(t) \int\limits_{\Gamma_j} \frac{f(\tau)d\tau}{\tau - t} \right|^p ds \leq C_8 \int\limits_{\Gamma_i} \Bigl(\int\limits_{\Gamma_j} |f(t(\sigma))| d\sigma \Bigr)^p |w(t(s))|^p ds \leq$$

$$\leq C_9 \Bigl(\int\limits_{\Gamma_j} |w(t)f(t)|^p ds \Bigr) \Bigl(\int\limits_{\Gamma_j} \Bigl| \frac{1}{w^{p'}(t)} \Bigr| ds \Bigr)^{p-1} \leq$$

$$\leq C_{10} \int\limits_{\Gamma} |w(t)f(t)|^p ds. \quad (3.22)$$

Let now Γ_i and Γ_j have common ends. For the sake of definiteness suppose that $j = i + 1$, and let $\Delta_i = \Gamma_i \cup \Gamma_{i+1}$. On Δ_i we define the function

$$\varphi_i(t) = \begin{cases} f(t) & \text{for} \quad t \in \Gamma_{i+1}, \\ 0 & \text{for} \quad t \in \Gamma \backslash \Gamma_{i+1}. \end{cases}$$

Obviously,

$$\int\limits_{\Gamma_i} \Bigl| \int\limits_{\Gamma_{i+1}} \frac{f(\tau)d\tau}{\tau - t} w(t) \Bigr|^p ds \leq \int\limits_{\Delta_i} \Bigl| \int\limits_{\Delta_i} \frac{\varphi_i(\tau)d\tau}{\tau - t} w(t) \Bigr|^p ds.$$

But for the arcs Δ_i, in exactly the same way as it has been proven for the arcs Γ_i, we can state that

$$\int\limits_{\Delta_i} \Bigl| \int\limits_{\Delta_i} \frac{\varphi_i(\tau)d\tau}{\tau - t} w(t) \Bigr|^p ds \leq$$

$$\leq C_{11} \int\limits_{\Delta_i} |\varphi_i(t)w(t)|^p ds \leq C_{11} \int\limits_{\Gamma} |f(t)w(t)|^p ds. \quad (3.23)$$

From the estimates (3.17), (3.20), (3.21) and (3.23) it follows the assertion of the theorem for $n = 1$.

For $n > 1$, we partition the curve Γ into non-intersecting arcs Γ_k, $k = \overline{1,n}$, each containing only one point c_k. If we put

$$w_k(t) = |t - c_k|^{h_k} \exp \Bigl\{ \frac{1}{2\pi} \int\limits_{\Gamma} \frac{\psi(\tau)d\tau}{\tau - t} \Bigr\},$$

then $w_k \in W_p(\Gamma)$. Let

$$F_j(\tau) = \begin{cases} f(\tau), & \tau \in \Gamma_j, \\ 0, & \tau \notin \Gamma \backslash \Gamma_j. \end{cases}$$

Then we have

$$\int_{\Gamma_k} \left| w(t(s)) \int_{\Gamma_j} \frac{f(\tau)d\tau}{\tau - t(s)} \right|^p ds \leq A_1 \int_{\Gamma_k} \left| w_k(t(s)) \int_{\Gamma} \frac{F_j(\tau)d\tau}{\tau - t(s)} \right|^p ds \leq$$
$$\leq A_2 \int_{\Gamma} |F_j(t) w_k(t)|^p ds \leq A_3 \int_{\Gamma} |f(t(s) w(t))|^p ds,$$

and from the inequality

$$\int_{\Gamma} \left| w(t) \int_{\Gamma} \frac{f(\tau)d\tau}{\tau - t} \right|^p ds \leq A \sum_{k,j}^{n} \int_{\Gamma_k} \left| w(t(s)) \int_{\Gamma_j} \frac{f(\tau)d\tau}{\tau - t(s)} \right|^p ds$$

we arrive at the assertion of the theorem. ■

Lemma 3.5. *Let $\Gamma \in J^*$, $1 < p < \infty$, $c \in \Gamma$, $-\frac{1}{p} < h < \frac{1}{p'}$. If*

$$\sup \Omega(\psi, \tau) < \frac{2\pi}{\widetilde{p}}, \quad \widetilde{p} = \max\Big(p, p', \frac{p}{1 + hp}, \frac{p}{p - 1 - hp}\Big),$$

then the function

$$w(t) = |t - c|^h \exp\left[\frac{i}{2}(S_\Gamma \psi)(t)\right]$$

belongs to the class $W_p(\Gamma)$.

Proof. The assertion of the above lemma for $1 < p < 2$ and $0 < h < \frac{1}{p'}$ or for $p > 2$ and $-\frac{1}{p} < h < 0$ is contained in Lemma 3.4. The remaining part of the theorem is proved by the scheme suggested in [36] (see also [42], p. 377) for the proof of an analogous assertion in the case of Lyapunov curves, using Theorem 2.1 and the results from subsection 3.5. In view of the complete analogy, the proof is omitted. ■

In exactly the same way as in Theorem 3.3, using only Lemma 3.5 instead of Lemma 3.4, we prove

Theorem 3.4. *Let Γ be a closed Jordan curve of the class J^*, and ψ be a real measurable function for which the condition* (3.16) *is fulfilled. If $c_k \in \Gamma$, $k = \overline{1, n}$, $-\frac{1}{p} < h_k < \frac{1}{p'}$ and*

$$\Omega(\psi, c_k) < \frac{2\pi}{\widetilde{p}}, \quad \widetilde{p} = \max\Big(p, p', \frac{p}{1 + h_k p}, \frac{p}{p - 1 - h_k p}\Big),$$

then the function

$$w(t)=\prod_{k=1}^{n}|t-c_k|^{h_k}\exp\left[\frac{i}{2}(S_\Gamma\psi)(t)\right]$$

belongs to the class $W_p(\Gamma)$.

Remark 3. Theorems 3.3 and 3.4 are stated for the curves of the class J^*. However, following the thread of the proof, they remain valid for those curves of the class R for which, as is known,

(1) $\rho(t)=\prod_{k=1}^{n}|t-c_k|^{h_k}\in W_p(\Gamma)$, $-\frac{1}{p}<h_k<\frac{1}{p'}$;

(2) $w(t)=\exp\left[\frac{i}{2}(S_\Gamma\psi)(t)\right]$ belongs to the class $W_p(\Gamma)$ if $\sup(\psi,t)<\frac{2\pi}{\overline{p}}$.

Remark 4. If the conditions (1) and (2) take place for some $p>1$, then they may be considered to be fulfilled for $p+\varepsilon$ with any sufficiently small $\varepsilon>0$.

Therefore the function w in the hypotheses of Theorems 3.3 and 3.4 belongs to $W_{p+\varepsilon}(\Gamma)$.

To construct the factor-function for $G\in\widetilde{A}(p)$, we will need, besides the above arguments, an assertion ensuring an estimate of a singular integral whose density is a piecewise linear function.

Lemma 3.6. *Let* Γ *be a closed Jordan curve of the class* K, $c_k=t(s_k)\in\Gamma$, $s_1<s_2<\cdots<s_n$ *and* $\varphi(t(s))=A_ks+B_k$ *for* $t\in[c_k,c_{k+1})$, $c_{n+1}=c_1$; A_k *and* B_k *are real numbers. Then*

$$0<m\le\frac{\exp\left[\frac{i}{2}(S_\Gamma\varphi)(t)\right]}{\prod_{k=1}^{n}|t-c_k|^{h_k}}\le M, \tag{3.24}$$

where $h_k=\frac{1}{2\pi}[\varphi(c_k-)-\varphi(c_k+)]$, $k=\overline{1,n}$.

Proof. Without restriction of generality we assume $n=2$, $c_1=a$, $c_2=b$,

$$\varphi(t)=\begin{cases}As+B, & t\in(a,b)=e,\\ 0, & t\in\Gamma\backslash e,\end{cases}$$

and we have to prove the inequality

$$0<m\le\frac{\exp[\frac{i}{2}(S_\Gamma\varphi)(t)]}{|t-a|^{h_1}|t-b|^{h_2}}\le M, \tag{3.25}$$

where $h_1=-\frac{1}{2\pi}\varphi(a)$, $h_2=\frac{1}{2\pi}\varphi(b)$.

We have

$$\frac{i}{2}(S_\Gamma\varphi)(t)=\frac{1}{2\pi}\int\limits_e\frac{\varphi(\tau)-\varphi(b)}{\tau-t}d\tau+\frac{\varphi(b)}{2\pi}\int\limits_e\frac{d\tau}{\tau-t}=$$

$$= \frac{1}{2\pi} \int\limits_{e} \frac{\varphi(\tau) - \varphi(b)}{\tau - t} d\tau + \frac{\varphi(b)}{2\pi} \Big[i\pi\chi_e(t) + \ln \frac{b-t}{a-t} \Big], \tag{3.26}$$

where χ_e is the characteristic function of the arc (a, b).

Let $c \in \Gamma \backslash [a, b]$ and

$$\psi_b(\tau) = \psi(\tau) = \begin{cases} \varphi(\tau) - \varphi(b) & \text{for} \quad \tau \in (a, b], \\ 0 & \text{for} \quad \tau \in (b, c), \\ (\tau - c)\frac{\varphi(a)-\varphi(b)}{a-b} & \text{for} \quad \tau \in [c, a]. \end{cases}$$

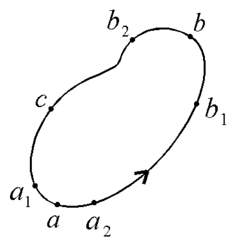
Then the equality (3.26) can be written in the form

$$\Big(\frac{i}{2} S_\Gamma \varphi\Big)(t) = \frac{1}{2\pi} \int\limits_{\Gamma} \frac{\psi(\tau) d\tau}{\tau - t} - \frac{1}{2\pi} \int\limits_{\Gamma \backslash e} \frac{\psi(\tau) d\tau}{\tau - t} + \frac{\varphi(b)}{2\pi} [i\pi\chi_e(t) + \ln \frac{b-t}{a-t}].$$

Since φ satisfies the Lipschitz condition with respect to s and $\Gamma \in K$, the first summand is bounded on Γ. Moreover,

$$\int\limits_{\Gamma \backslash e} \frac{\psi d\tau}{\tau - t} = \int\limits_{ca} \frac{\psi d\tau}{\tau - t},$$

and the distance from the point b to the arc (c, a) is positive. Therefore there exists an arc neighbourhood (b_1, b_2) of the point b such that the second summand in it is bounded. This implies the validity of (3.25) for $t \in (b_1, b_2)$.



Considering the function $\psi_a(\tau)$ similar to $\psi_b(\tau)$, we find that the inequality (3.25) is valid in some neighbourhood (a_1, a_2) of the point a. Its validity in (b_2, a_1) is obvious.

Next, from the equalities

$$\frac{i}{2}(S_\Gamma \varphi)(t) = \frac{1}{2\pi} \int\limits_{e} \frac{A(s - s_0)}{\tau - t} d\tau + \frac{\varphi(t)}{2\pi} \int\limits_{e} \frac{d\tau}{\tau - t}$$

and

$$\frac{1}{2\pi} \int\limits_{e} \frac{d\tau}{\tau - t} = \frac{1}{2\pi} \int\limits_{\Gamma} \frac{d\tau}{\tau - t} - \frac{1}{2\pi} \int\limits_{\Gamma \backslash e} \frac{d\tau}{\tau - t} = \frac{i}{2} - \frac{1}{2\pi} \int\limits_{\Gamma \backslash e} \frac{d\tau}{\tau - t}$$

follows the boundedness of $\frac{i}{2}S_\Gamma\varphi$ on the arc (a_2, b_1), and therefore (3.25) is also valid on the same arc. ■

Remark. If we assume in the hypotheses of the lemma that A_k and B_k are complex numbers and the curve at the point c_k has tangents, then the inequality (3.24) remains valid if we replace h_k by $\operatorname{Re} h_k$.

3.5. The case $\Gamma \in J^*$, $G \in \widetilde{A}(p)$. By Lemma 1.1, we have $G = |G|G_1G_2G_3$, where $G_k = \exp i\varphi_k$, $k = 1, 2, 3$. Moreover, $G_1 \in H(1)$, $\varkappa_p(G_1) = \varkappa_p(G)$, φ_2 is a piecewise linear function with respect to s, φ_3 is continuous at the points of discontinuity of the function φ_2 and $\sup|\varphi_3| < \frac{2\pi}{p}$.

Let

$$h(z) = \mathcal{K}_\Gamma\{\ln[G_1(t)(t - z_0)^{-\varkappa_p}]\}(z), \quad z_0 \in D^+.$$

Assume

$$X_0(z) = \exp[K_\Gamma(\ln|G|)](z), \tag{3.27}$$

$$X_1(z) = \begin{cases} \exp h(z), & z \in D^+, \\ (z - z_0)^{-\varkappa_p} \exp h(z), & z \in D^-, \end{cases} \tag{3.28}$$

$$X_k(z) = \exp[i(\mathcal{K}_\Gamma\varphi_k)(z)], \quad k = 2, 3, \tag{3.29}$$

and

$$X(z) = \prod_{k=0}^{3} X_k(z) \tag{3.30}$$

Prove that X is the factor-function for G in the class $\mathcal{K}^p(\Gamma)$. It is sufficient to show that $X^+ \in W_p(\Gamma)$ (the rest of the properties of the factor-function follow from the this fact, Theorem 2.1 and Smirnov's theorem from (0.19)).

We have $X^+ = \prod_{k=0}^3 X_k^+$. Due to the fact that $G_1 \in H(1)$, the function X_1^+ is bounded and $\inf|X_1^+| > 0$. Moreover $X_0 \in \cap_{\delta>1} W_\delta(\Gamma)$, since $|G| \in \cap_{p>1} A(p)$, $\operatorname{ind}_p |G| = 0$ (see 3.14). Further, from the definition of φ_2 (see 1.5) and by Lemma 3.6 it follows that

$$X_2^+(t) = \prod_{k=1}^{n} |t - c_k|^{h_k} Z(t),$$

where $-\frac{1}{p} < h_k < \frac{1}{p'}$ and Z satisfies the condition $m < |Z| < M$.

Hence

$$X^+(t) = \prod_{k=1}^{n} |t - c_k|^{h_k} \exp\Big[\frac{i}{2}S_\Gamma\varphi_3\Big] Y(t) = w(t)Y(t),$$

where $Y(t) = X_0^+(t)X_1^+(t)Z(t)$ and therefore $Y \in \cap_{\delta>1} W_\delta(\Gamma)$. For w all the conditions of Theorem 3.3 are fulfilled (since $\sup_{t\in\Gamma}\Omega(\varphi_3, t) < a(p)$, by Lemma 1.1) and therefore, taking into account that theorem and Remark 2 to Theorem 3.4, we conclude that $w \in W_{p+\varepsilon}(\Gamma)$. Using the theorem from

(0.20), we obtain $X^+ = (wY) \in W_p(\Gamma)$. Thus the function X given by (3.30) is the factor-function of G in $\mathcal{K}^p(\Gamma)$.

Sum up the results of subsetions 3.1–3.5.

Theorem 3.5. *If a simple closed curve Γ belongs to the class J^*, $G \in \widetilde{A}(p)$, $\varkappa = \varkappa_p(G)$, then the function X given by (3.30) is the factor-function for G in $\mathcal{K}^p(\Gamma)$. It possesses the following properties: (1) $X \in \widetilde{\mathcal{K}}^p(\Gamma)$; (2) $\frac{1}{X} \in \widetilde{\mathcal{K}}^{p'}(\Gamma)$; (3) $X^+(t) = G(t)X^-(t)$, $t \in \Gamma$; (4) $X^+ \in W_p(\Gamma)$; (5) $\lim_{z\to\infty} X(z)(z-z_0)^{\varkappa_p(G)} = 1$, $z_0 \in D^+$.*

Having this theorem we immediately obtain all the assertions of Theorem 1.1.

3.6. The class of functions $M(p)$ and problem (I) in the class $\mathcal{K}^p(\Gamma)$ for $\Gamma \in J^*$, $G \in M(p)$. Denote by $M(p)$ the set of measurable on Γ functions G representable in terms of $G(t) = a(t)b(t)$, where $a \in A(p)$, and b is a piecewise continuous function with a finite number of points of discontinuity of c_k, $k = \overline{1,n}$, $|b(t)| = 1$ and if $b(c_k+)[b(c_k-)]^{-1} = \exp 2\pi i\gamma_k$, then

$$\min(0, 2p^{-1} - 1) \le \gamma_k \le \max(0, 2p^{-1} - 1). \tag{3.31}$$

(For definition of the class $M(p)$ see [36], [42], p. 380.)

Prove that $\widetilde{A}(p) \subset M(p)$.

Let $G \in \widetilde{A}(p)$ and $2\pi\gamma_k$ be jumps of its argument at the points c_k. Denote by ψ a real piecewise linear function with the jumps $2\pi\gamma_k$, where

$$\gamma_k = \begin{cases} \mu_k - \frac{1}{p'} + \varepsilon & \text{for} \quad 1 < p < 2, \\ \mu_k + \frac{1}{p} - \varepsilon & \text{for} \quad p > 2. \end{cases}$$

Since $|\mu_k - \gamma_k| = \frac{1}{\max(p,p')} - \varepsilon$, it is not difficult to verify that the function $G(t)\exp i\psi(t)$ for sufficiently small ε belongs to $A(p)$. Assume $b(t) = \exp i\psi(t)$. Then $G(t) = a(t)b(t)$. If we prove that the conditions (3.31) are fulfilled for $b(t)$, then this will imply that $G \in M(p)$.

If $1 < p < 2$, then $\gamma_k = \mu_k - \frac{1}{p'} + \varepsilon$. By (1.7), $\frac{1}{p'} < \mu_k < \frac{1}{p}$. Therefore $\mu_k = \frac{1}{p'} + \eta_k$, $0 \le \eta_k < \frac{2}{p'} - 1$ and hence $\gamma_k = \eta_k + \varepsilon$. For $p > 2$, (1.7) yields $\mu_k = -\frac{1}{p} - \eta_k'$, whence $0 \le \eta_k' < 1 - \frac{2}{p}$. Obviously, in the both cases one can choose ε such that the inequalities (3.31) will be fulfilled for all γ_k ($k = \overline{1,n}$). Hence $G \in M(p)$.

Let now $\Gamma \in J^*$ and $G \in M(p)$. Then $G(t) = a(t)b(t)$, where a and b satisfy the above-mentioned conditions. Since $\Gamma \in J^*$ and $a \in A(p)$, by Theorem 3.5 we can construct a factor-function $X_a(z)$ with the property $\lim_{z\to\infty} X_a(z)(z-z_0)^{\varkappa_p(a)} = 1$, where $z_0 \in D^+$ and $\varkappa_p(a)$ is the index of the function a. Moreover, we have $b(t) = \exp i\psi(t)$, where ψ is a real piecewise continuous function. Assume $X_b(z) = \exp[i(\mathcal{K}_\Gamma\psi)(z)]$. By Theorem 2.1 we have $(X_b^{\pm 1} - 1) \in E^\delta(D^\pm)$, $\delta > 0$. Suppose $X(z) = X_a(z)X_b(z)$. It is not difficult to see that Theorem 3.4 can be applied to X^+. Hence $X^+ \in W_p(\Gamma)$.

From this, we immediately conclude that X satisfies all the conditions required for the factor-function. From the uniqueness of the factor-function ([68],) it follows that $\varkappa_p(a)$ does not depend on the representation of G in terms of the product of a and b, and now we can state that:

If $\Gamma \in J^*$, $G \in M(p)$ and $\varkappa(G) = \varkappa(a)$ then all statements of Theorem 1.1 *are true.*

This result is more general than that formulated in Theorem 1.1. In this way, the factor-function for G is obtained in terms of $X = X_a X_b$, but to construct a formula of solution for $G \in \widetilde{A}(p)$, it is necessary first to find a representation $G = ab$. There was no necessity in such a representation in subsections 3.1–3.5, so we considered it reasonable to study in detail the problem of conjugation with a coefficient from the significant particular subclass $\widetilde{A}(p)$ of the set $M(p)$.

3.7. The problem of linear conjugation in the classes $\mathcal{K}^p(\Gamma)$ for multiply connected domains and open curves.

1. The case of a finitely connected domain. Let Γ be a finite family of nonintersecting closed curves $\Gamma_i \in J^*$, $i = \overline{1,n}$, bounding a finite domain D^+ and let D^- be the complement of the set $D^+ \cup \Gamma$ with respect to the entire plane. The function G given on Γ will be called a function of the class $\widetilde{A}(p)$ if it belongs to $\widetilde{A}(p)$ on every curve Γ_i. Assume $x_p(G) = \sum x_p^{(i)}(G)$, where $\varkappa_p^{(i)}(G)$ is the index of the restriction of Γ_i on G. If for every curve Γ_i we construct by Theorem 3.5 a factor-function and assume that X is the product of these functions, then, as is easily verified, it will be the factor-function of G in $\mathcal{K}^p(\Gamma)$. Thus Theorem 1.1 is true in the case under consideration.

2. The case of an open curve. Let Γ be a simple, open, oriented, rectifiable curve. We seek for a function $\phi \in \mathcal{K}^p(\Gamma)$, whose boundary values satisfy the condition of conjugation (I). If $\widetilde{\Gamma}$ is a closed curve with Γ on it, then, as is known (see, e.g., [66]), the solution of the above-formulated problem reduces to the solution in the class $\mathcal{K}^p(\widetilde{\Gamma})$ of the following problem of conjugation:

$$\phi^+(t) = \widetilde{G}(t)\phi^-(t) + \widetilde{g}(t), \quad t \in \widetilde{\Gamma},$$

where

$$\widetilde{G}(t) = \begin{cases} G(t), & t \in \Gamma, \\ 1, & t \in \widetilde{\Gamma}\backslash\Gamma, \end{cases} \quad \widetilde{g}(t) = \begin{cases} g(t), & t \in \Gamma, \\ 0, & t \in \widetilde{\Gamma}\backslash\Gamma. \end{cases} \tag{3.32}$$

Thus, if one sets oneself the task of constructing in this a way a solution of the problem on an unclosed curve, the boundary curve in this case must be complemented to such curve $\widetilde{\Gamma}$ for which the problem has already been studied. On this basis we can extent the results of Theorems 3.5 and 1.1 to the case of unclosed from the class J^* curves with the tangents at the ends. For this purpose, applying Lemma 3.4 and assertion 3.2 from Chapter I, we have to complement Γ with respect to a closed curve $\widetilde{\Gamma} \in J^*$ and to impose on G the conditions guaranteeing on Γ the belonging to the class $\widetilde{A}(p)$ of the function $\widetilde{G}$ which is formed by formula (3.32).

As an example, we cite a version of function factorizability on an open curve of the class J without additional assumptions that it has tangents at the ends.

Theorem 3.6. *Let $\Gamma_{ab} \in J$ and $G \in A(p)$ with the condition that the end points a and b possess the neighbourhoods V_a and V_b (on Γ_{ab}) such that all values $G(t)$ for $t \in V_a \cup V_b$ lie in an angle of $\frac{2\pi}{\overline{p}}$ with the vertex at the origin. Then G is factorizable in the class $\mathcal{K}^p(\Gamma)$, $p > 1$.*

Proof. Let μ be the smooth curve for which condition (1.1) is fulfilled. Define on μ the function $G_0(\tau) = G_0(\tau(s)) = G(t(s))$, $0 \le s \le l$ and complement μ with respect to a simple smooth curve $\widetilde{\mu}$. By the assumption on G, there exists a constant h such that the function

$$\widetilde{G}_0(\tau) = \begin{cases} G_0(\tau), & \tau \in \mu, \\ h, & \tau \in \widetilde{\mu} \backslash \mu, \end{cases} \tag{3.33}$$

belongs to $\widetilde{A}(p)$ on $\widetilde{\mu}$. The function $\frac{1}{h}\widetilde{G_0}$ will be the same. For the sake of simplicity, we assume that $\operatorname{ind}_p \frac{1}{h}\widetilde{G}_0 = 0$. (As is seen from the above, to this case easily reduces the case with nonzero index).

Since $\widetilde{\mu}$ is a closed smooth curve, the problem (I) with this coefficient is by Theorem 3.1 uniquely solvable in the class $\mathcal{K}^p(\widetilde{\mu})$, and $X_{\widetilde{\mu}}(z) = \exp\left[\mathcal{K}_{\widetilde{\mu}}(\ln \frac{1}{h}\widetilde{G}_0)(z)\right]$ is its factor-function. But if $\tau \in \widetilde{\mu}\backslash\mu$, then $\frac{1}{h}\widetilde{G}_0 = 1$ and since $\operatorname{ind}_p \frac{1}{h}\widetilde{G}_0 = 0$, we may assume that $\ln \frac{1}{h}\widetilde{G}_0 = 0$ when $\tau \in \widetilde{\mu}\backslash\mu$. Therefore $X_{\widetilde{\mu}}(z) = \exp\left[\mathcal{K}_\mu\left(\ln \frac{1}{h}\widetilde{G}_0\right)(z)\right]$. Moreover, since $X_{\widetilde{\mu}} \in \mathcal{K}^p(\widetilde{\mu})$ and $X_{\widetilde{\mu}}(\infty) = 1$, we find that $X_{\widetilde{\mu}}(z) = \mathcal{K}_{\widetilde{\mu}}(X^+_{\widetilde{\mu}} - X^-_{\widetilde{\mu}}) + 1 = \mathcal{K}_\mu(X^+_\mu - X^-_\mu) + 1$.

Denote restrictions on μ of the functions $X^\pm_{\widetilde{\mu}}$ by $X^\pm$. Obviously, $X^\pm \in L^p(\mu)$, $\frac{1}{X^\pm} \in L^{p'}(\mu)$. We also have $X^\pm_{\widetilde{\mu}} \in W_p(\widetilde{\mu})$ and hence $X^+ \in W_p(\mu)$. In exactly the same way as for the closed curves (see subsection 3.2) we establish that $w(t) = \exp\left[\frac{1}{2}S_\Gamma\left(\frac{1}{h}\ln G\right)\right]$ belongs to the class $W_p(\Gamma)$. Using now a result from subsection 2.2, we can easily establish that the function $\frac{1}{h}G$ is factorizable in the class $\mathcal{K}^p(\Gamma)$ and hence G has the same property. ■

§ 4. The Linear Conjugation Problem in the Class of Functions Representable in Domains $D^\pm$ by a Cauchy Integral

Let Γ be a closed, rectifiable, Jordan curve bounding the domains D^+ and D^-, $z = \infty \in D^-$. Consider the problem (I) in the class of functions which are analytic on a plane cut along Γ and representable in the domains D^+ and D^- by the Cauchy integral (i.e., belonging to $E^1(D^\pm)$). In what follows, the class of such functions will be denoted by $E^1(\Gamma)$. This class is the subset of $\mathcal{K}^1(\Gamma)$. Thus, for example, there exists an integrable on γ function φ_0 such that the boundary values ϕ_0^+ of the Cauchy type integral $\phi_0 = \mathcal{K}_\Gamma\varphi_0$ are not summable. Evidently, $\phi_0 \overline{\in} E^1(\Gamma)$.

Note herewith that if $p > 1$ and boundary of $D^\pm$ belongs to R then the class $E^p(\Gamma)$ coincides with the class $\mathcal{K}^p(\Gamma)$ (see Chapter I, §3, Theorem 3.4).

In the sequel we will assume that

$$G \neq 0, \quad G \in H, \quad \varkappa = \operatorname{ind} G, \quad g, \quad S_g \in L(\Gamma), \quad \Gamma \in K. \tag{4.1}$$

The assumption $S_\Gamma g \in L(\Gamma)$ is necessary if we wish the jump problem

$$\phi^+(t) - \phi^-(t) = g(t)$$

to have a solution in the class $E^1(\Gamma)$.

The following assertions are valid:

(1) if $\Gamma \in K$, $\varphi \in H$ then $(S_\Gamma\varphi)(t)$ coincides almost everywhere with the function of the class H;

(2) if $\Gamma \in R$, $\varphi \in L(\Gamma)$ then $\mathcal{K}_\Gamma\varphi \in \cap_{\delta>1} E^\delta(D^\pm)$ (see Chapter I, Corollary of Theorem 3.3);

(3) if $\mathcal{K}_\Gamma\varphi \in E^1(\Gamma)$, $a \in H$ then $K(a\varphi) \in E^1(\Gamma)$.

The validity of the assertion (1) follows directly from the following reasoning. Let $\varphi \in H(\alpha)$. The second summand in the equality

$$(S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau) - \varphi(t)}{\tau - t} d\tau + \frac{\varphi(t)}{\pi i}\int_\Gamma \frac{d\tau}{\tau - t}$$

coincides almost everywhere with $\varphi(t)$. The first one also belongs to the Hölder class (generally speaking, with the exponent less than α). In order to see that this is so, it suffices to follow the proof of the same assertion (see [106], p.70–73) in the case of smooth curves with the only difference that the estimate of the integral I_1 appearing there should be replaced by

$$|I_1| \leq \frac{1}{2\pi}|\varphi(t_0) - \varphi(t_0 + h)| \int_{L-l} \frac{|dt|}{|t - t_0|} \leq$$

$$\leq \frac{k}{2\pi}|\varphi(t_0) - \varphi(t_0 + h)| \int_{L-l} \frac{ds}{|s - s_0|} \leq c|h|^\alpha |\ln h| \leq c_1 h^{\alpha - \varepsilon}. \tag{4.2}$$

Prove the assertion (3).

By virtue of (2), we have $\mathcal{K}_\Gamma(a\varphi) \in \cap_{\delta<1} E^\delta(D^\pm)$. Further,

$$\int_\Gamma |(\mathcal{K}_\Gamma(a\varphi))^+| ds \leq \frac{1}{2}\int_\Gamma |a\varphi + S_\Gamma(a\varphi)| ds =$$

$$= \frac{1}{2}\int_\Gamma \left| a\varphi + 2aS_\Gamma\varphi + \frac{1}{2\pi i}\int_\Gamma \frac{a(\tau) - a(t)}{\tau - t}\varphi(\tau) d\tau \right| ds \leq$$

$$\leq \frac{1}{2}\int\limits_\Gamma |a\varphi|ds + \max|a|\int\limits_\Gamma |S_\Gamma\varphi|ds + \frac{M}{2\pi}\int\limits_\Gamma\int\limits_\Gamma \frac{|\varphi(\tau)(\sigma)|d\sigma}{|\sigma - s|^{1-\alpha}}ds.$$

Replacing the order of integration in the third summand on the right-hand side, we can see that $[\mathcal{K}_\Gamma(a\varphi)]^+ \in L(\Gamma)$, and hence $(\mathcal{K}_\Gamma(a\varphi)) \in E^1(D^\pm)$.

On the basis of assertions (1)–(3) we can easily prove

Theorem 4.1. *Let a closed curve $\Gamma \in K$ bound the domains D^+ and D^-. If the conditions* (4.1) *are satisfied, then for the linear conjugation problem* (I) *in the class $E^1(\Gamma)$ the assertion* I *from Theorem* 1.1 *is valid.*

§ 5. The Linear Conjugation Problem in the Classes $\mathcal{K}^p(\Gamma, w)$ and Its Applications

Let Γ be a simple, closed, rectifiable curve and

$$w = \exp iS_\Gamma\psi, \quad \operatorname{Im}\psi = 0, \quad \psi \in L^\infty(\Gamma), \quad w \in W_p(\Gamma). \tag{5.1}$$

If $\Gamma \in J^*$, then as an example of such a function may serve $\psi = \arg_p G$, $G \in \widetilde{A}(p)$ (see Theorem 3.5). Under the above assumption regarding w consider the problem: define a function $\phi \in \mathcal{K}^p(\Gamma, w)$ which almost everywhere on Γ satisfies the boundary condition (I).

We will also consider the linear singular integral equation (II) in the class $L^p(\Gamma, w)$.

To preserve the equivalence of the problem (I) in the class $\mathcal{K}^p(\Gamma; w)$ and of the singular equation (II) in the space $L^p(\Gamma; w)$, it is necessary to require for the operator S_Γ to act from $L^p(\Gamma, w)$ to $L^p(\Gamma; w)$. Consequently by Theorem 2.3 of Chapter I, S_Γ is bounded in that space, and hence $w \in W_p(\Gamma)$. We arrive at the same conclusion when we want that the jump problem $\phi^+ - \phi^- = g$ would be solvable for every $g \in L^p(\Gamma; w)$ in the class of those functions from the class $\mathcal{K}^p(\Gamma; w)$ for which $\phi^\pm \in L^p(\Gamma; w)$. Thus, in the above-mentioned cases the necessary condition is: $w \in W_p(\Gamma)$. But in a number of cases, every weighted function from $W_p(\Gamma)$ is equivalent to the function w from (5.1); (e.g., by Theorem 4.8 of Chapter I, for every function $w \in W_p(\Gamma)$ in the case of Lyapunov curves there exist bounded u and v such that $\operatorname{Im} v = 0$, $w = \exp(u + iS_\Gamma v)$.

On the basis of the above arguments, the assumptions (5.1) adopted by us with respect to the weight function w may be considered to be natural.

5.1. Reduction of the problem (I) in the class $\mathcal{K}^p(\Gamma; w)$ to the linear conjugation problem in $\mathcal{K}^p(\Gamma)$.

Theorem 5.1. *Let a closed curve $\Gamma \in R$ and ψ be a real bounded function such that $w = \exp\left(\frac{i}{2}S_\Gamma\psi\right)$ belongs to $W_p(\Gamma)$. Then for an arbitrary solution $\phi \in \mathcal{K}^p(\Gamma; w)$ of the problem* (I) *the function $\Psi(z) = Y(z)\phi(z)$, where*

$$Y(z) = \exp[i(K_\Gamma\psi)(z)],$$

is a solution of the class $\mathcal{K}^p(\Gamma)$ *of the conjugation problem*

$$\Psi^+(t) = G(t)\exp(i\psi)\Psi^-(t) + g_1(t), \quad g_1(t) = g(t)[Y^+(t)]^{-1}. \tag{5.2}$$

Conversely, if Ψ *is a solution of the problem* (5.2) *belonging to* $\mathcal{K}^p(\Gamma)$*, then the function* $\phi(z)Y^{-1}(z)\Psi(z)$ *belongs to the class* $\mathcal{K}^p(\Gamma, w)$ *and satisfies the boundary condition* (I)*.*

Proof. Show first that $Y \in \widetilde{\mathcal{K}}^{p'}(\Gamma, w^{-1})$. By Theorem 2.1, for some $\delta > 0$ we have $Y \in E^\delta(D^+)$. But $y^+ = \exp\left[\frac{i\psi}{2} + \frac{i}{2}S_\Gamma\psi\right] = w\exp\frac{i\psi}{2}$, and since $w \in W_p(\Gamma)$, we have that $Y^+ \in L^p(\Gamma)$ (by Lemma 4.2 of Chapter I). Using the fact that $\Gamma \in R$ and D^+ is a Smirnov's domain, we conclude that $Y \in E^p(D^+)$. From this, following the proof of Theorem 2.1 we notice that one can take in it $n_0 = 0$. Then $(Y-1) \in E^p(D^-)$, and thus $(Y-1) \in E^p(D^\pm)$. Therefore $(Y-1)$ in the domains $D^\pm$ is representable by the Cauchy integral, that is, $Y \in \widetilde{K}(\Gamma)$. The density of the corresponding integral will be

$$Y^+ - Y^- = w\Big[\exp\frac{i\psi}{2} - \exp\frac{-i\psi}{2}\Big].$$

This function, evidently, belongs to the class $L^{p'}(\Gamma, w^{-1})$, and hence $Y \in \widetilde{K}^{p'}(\Gamma, w^{-1})$.

Thus the functions ϕ and Y belong to the adjoint classes $\widetilde{\mathcal{K}}^p(\Gamma, w)$ and $\widetilde{\mathcal{K}}^{p'}(\Gamma, w^{-1})$, respectively, and so $\Psi = \phi Y$ belongs to the class $\widetilde{K}(\Gamma)$ (see, e.g., [68], p. 98–99). Next, it is obvious that $\Psi^\pm \in L^p(\Gamma)$, $\Psi(\infty) = 0$. Consequently, $\Psi \in \mathcal{K}^p(\Gamma)$. It can be easily verified that $g_1 \in L^p(\Gamma)$ and Ψ satisfies the boundary condition (5.2).

The converse assertion is proved analogously. ■

Remark. Theorem 5.1 holds also valid in the case where the condition $w \in W_p(\Gamma)$ from (5.1) is replaced by the condition $w \in L^p(\Gamma)$, $w^{-1} \in L^{p'}(\Gamma)$.

Indeed, Y belongs to $\widetilde{\mathcal{K}}^{p'}(\Gamma)$ since by Theorem 2.1, Smirnov's theorem and also the condition $w^{-1} \in L^{p'}(\Gamma)$. Further $\phi \in \cap_{\delta<1}E^\delta(D^+)$, so as we have $L^p(\Gamma, w) \subset L(\Gamma)$ and $\Gamma \in \cap_{\delta<1}R_{1,\delta}$ (see Chapter I, (3.4)), and we can apply the corollary of Theorem 3.3 in Chapter I. Therefore $\Psi \in E^r(D^\pm)$ for some $r > 0$. But $\Psi^\pm \in L^p(\Gamma)$, and hence $\Psi \in \mathcal{K}^p(\Gamma)$.

In the same way, from the assumptions $\Psi \in \mathcal{K}^p(\Gamma, w)$ and $w \in L^p(\Gamma)$ we establish that $\Phi \in \mathcal{K}^p(\Gamma)$.

5.2. The problem (I) in the class $\mathcal{K}^p(\Gamma, w)$ for $\Gamma \in J^*$. As far as the problem (I) is well studied in the class $\mathcal{K}^p(\Gamma)$, Theorem 5.1 allows one to obtain the appropriate results for this problem in the class $\mathcal{K}^p(\Gamma, w)$ when w is required to satisfy conditions (5.1). As an example, we give here a result based on Theorem 1.1.

Theorem 5.2. *Let $\Gamma \in J^*$, $g \in L^p(\Gamma, w)$ and let ψ be a real, measurable, bounded function such that $w = \exp[\frac{i}{2} S_\Gamma \psi] \in W_p(\Gamma)$. If the function $G_w = G \exp i\psi$ belongs to the class $\widetilde{A}(p)$ (or to $M(p)$) and $\varkappa(G_w)$ is the index of the function G_w in the class $\mathcal{K}^p(\Gamma)$, then for problem* (I) *in the class $\mathcal{K}^p(\Gamma, w)$ assertion* I *of Theorem* 1.1 *is true.*

5.3. On the Noetherianness of a singular integral equation in the spaces $L^p(\Gamma, w)$. The results of subsection 5.2 allow one to obtain an analogue for the second part of Theorem 1.1 regarding singular integral equations in the spaces $L^p(\Gamma, w)$. We will use the results of I.B. Simonenko on the equivalence between the Noetherian characteristic singular integral equation in the Lebesgue spaces and the factorizability of the definite function in the classes representable by the Cauchy type integral.

Theorem 5.3. *Let the closed curve $\Gamma \in R$ and the function w satisfy the conditions* (5.1)*. If a and b are bounded, measurable on Γ functions and*

$$\inf_{t \in \Gamma} |a^2(t) - b^2(t)| > 0, \tag{5.3}$$

then the equation

$$a\varphi + bS_\Gamma \varphi = g \tag{5.4}$$

is Noetherian in the space $L^p(\Gamma, w)$ if and only if the equation

$$a_1 f + b_1 S_\Gamma f = g_1, \tag{5.5}$$

with

$$a_1 = a(1+m) + b(1-m), \quad b_1 = a(1-m) + b(1+m), \quad m = \exp i\psi,$$

is Noetherian in the space $L^p(\Gamma, w)$.

The equations (5.4) *and* (5.5) *have the same indixes.*

Proof. We will rely on the following assertion: - the noetherianness of the equation (5.4) in the space $L^p(\Gamma, w)$, $w \in W_p(\Gamma)$ is, under the condition (5.3), equivalent to the factorizability of the function $G = (a-b)(a+b)^{-1}$ in the class $\mathcal{K}_p(\Gamma; w)$, the index of equation (II) in $L^p(\Gamma; w)$ being equal to $(-\varkappa)$, where $\varkappa$ is the order of the factor-function G at infinity.

This assertion for $w = 1$ is proved in [143]. The case of the power weight has been considered in [42, pp. 272–275]. Making slight modification in the proof of [143] (connected with the properties of the Cauchy type integrals on the curves from the class R), we can see that the above assertion is also valid for the assumptions $\Gamma \in R$, $w \in W_p(\Gamma)$.

Let the equation (5.4) be Noetherian in $L^p(\Gamma; w)$. Show that the equation (5.5) is Noetherian in $L^p(\Gamma)$. By the above, it suffices to show the factorization in $\mathcal{K}^p(\Gamma)$ of the function $G_1 = (a_1 - b_1)(a_1 + b_1)^{-1}$, provided the factorizability in $\mathcal{K}^p(\Gamma; w)$ of the function $G = (a-b)(a+b)^{-1}$.

Let X be a factor-function for G in $\mathcal{K}^p(\Gamma; w)$. Show that the function

$$Z = XY, \tag{5.6}$$

where

$$Y(z) = \exp\left(\frac{1}{2\pi}\int_\Gamma \frac{\psi(\tau)d\tau}{\tau - z}\right), \quad z\overline{\in}\Gamma, \tag{5.7}$$

is the factor-function for G in $\mathcal{K}^p(\Gamma)$.

Since $(a_1 - b_1)(a_1 + b_1)^{-1} = (a - b)(a + b)^{-1}m$ and

$$\frac{Z^+}{Z^-} = \frac{X^+Y^+}{X^-Y^-} = \frac{a-b}{a+b}\exp i\psi = \frac{a-b}{a+b}m,$$

we find that

$$Z^+(Z^-)^{-1} = (a_1 - b_1)(a_1 + b_1)^{-1}.$$

Determine the remaining properties of the factor-function.

In proving Theorem 5.1, we have shown that $Y \in \widetilde{\mathcal{K}}^{p'}(\Gamma; w^{-1})$. The fact that $X \in \widetilde{K}^p(\Gamma; w)$, $w \in W_p(\Gamma)$, implies $Z \in \widetilde{\mathcal{K}}^1(\Gamma)$. Then $Z = \mathcal{K}_\Gamma(Z^+ - Z^-)$, where

$$Z^\pm(t) = X^\pm(t)w(t)m^{\pm\frac{1}{2}}(t). \tag{5.8}$$

It can be easily verified that $Z^\pm \in L^p(\Gamma)$. Moreover, both X and Y belong to the classes $E^\delta(D^\pm)$, $\delta < 1$ and so $Z \in E^{\delta_0}(D^\pm)$, $\delta_0 > 0$. Therefore $Z \in \mathcal{K}^p(\Gamma)$. Analogously we can prove that $Z^{-1} \in \mathcal{K}^{p'}(\Gamma)$, taking into account that $w^{-1} \in L^{p'}(\Gamma)$.

Thus, $Z \in \widetilde{\mathcal{K}}^p(\Gamma; w)$, $Z^{-1} \in \mathcal{K}^{p'}(\Gamma; w^{-1})$. It remains to prove that the operator $Z^+ S_\Gamma \frac{1}{X^+}$ is continuous in $L^p(\Gamma)$. This immediately follows from the continuity in $L^p(\Gamma; w)$ of the operator $X^+ S_\Gamma \frac{1}{X^+}$, if we take into account (5.8).

With regard for (5.6), it is evident that both Z and X have the same order at infinity, and consequently the equations (5.4) and (5.5) have the same index in the classes $L^p(\Gamma; w)$ and $L^p(\Gamma)$, respectively.

In exactly the same way we can prove that the noetherianness of equation (5.5) in $L^p(\Gamma)$ leads to that of the equation (5.4) in $L^p(\Gamma; w)$, and hence we conclude that these equations have the same indexes. ■

In addition to the arguments proved above, we will point out the formulas providing one-to-one correspondence between the solutions of the equations (5.4) and (5.5). For the sake of simplicity we suppose that their index is equal to zero. Since

$$\varphi = \phi^+ - \phi^-, \quad f = F^+ - F^-,$$

where

$$\phi^+ = G\phi^- + g, \quad F^+ = G_1F^- + g_1,$$

we can easily determine that: if φ the solution of equation (5.4) of the class $L^p(\Gamma; w)$ and Y is the function given by (5.7), then the function

$$f = \frac{1}{2}(Y^+ + Y^-)\varphi + \frac{1}{2}(Y^+ - Y^-)S_\Gamma \varphi$$

will be a solution in the class $L^p(\Gamma)$ of equation (5.5) with $g_1 = \frac{g}{Y^+}$ on the right-hand side. If, however, ψ is a solution of (5.5) of the class $L^p(\Gamma)$, then the function

$$\varphi = \frac{1}{2}(\frac{1}{Y^+} + \frac{1}{Y^-})f + \frac{1}{2}(\frac{1}{Y^+} - \frac{1}{Y^-})S_\Gamma f$$

will be a solution in the class $L^p(\Gamma, w)$ of (5.4) with $g = g_1 Y^+$ on the right-hand side.

§ 6. The Linear Conjugation Problem in the Case of a Straight Line

6.1. Functions of the class $\widetilde{A}(p)$ on a straight line. Let t be an arbitrary point on the real D axis and let e be its neighbourhood (the case $t = \infty$ is not excluded; any set of the kind $(t < -N) \cup (t > M)$, $N, M > 0$ is assumed to be the neighbourhood of this point). If ψ is a bounded function, then its oscillation $\Omega(\psi, t_0)$, $t_0 \in D$, can again be defined by the equality (1.3).

Let G be a function given on D and let $\left(\frac{t-i}{t+i}\right)^{-\frac{2}{p}}$ be a boundary value on D of the function $\left(\frac{z-i}{z+i}\right)^{-\frac{2}{p}}$, analytic on the plane cut along the non-intersecting rays coming out the points $z = \pm i$ and lying on the ordinate axis.

We will say that $G \in \widetilde{A}_D(p)$, $1 < p < \infty$ if: (1) $0 < m \leq |G| \leq M$; (2) for all $t \in D$ with the possible exception of the points c_k, $k = \overline{1, n}$ (the case $c_k = \infty$ is not excluded) there exists a neighbourhood in which the values of the function $G^{(1)}(t) = G(t)\left(\frac{t-i}{t+i}\right)^{-\frac{2}{p}}$ lie in an angle less than $\frac{2\pi}{\max(p,p')}$ with the vertex at the origin; (3) at the points c_k there exist the limits $G(c_k\pm)$ (and hence $G^{(1)}(c_k\pm)$). Moreover, $\frac{2\pi}{p} < \delta_k \leq \frac{2\pi}{p'}$ for $p > 2$ and $\frac{2\pi}{p'} \leq \delta_k < \frac{2\pi}{p}$ for $1 < p < 2$, where δ_k is the angle formed by the vectors $G_1^{(1)}(c_k-)$ and $G^{(1)}(c_k+)$ for $|c_k| < \infty$ and by the vectors $\lim_{t\to+\infty} G^{(1)}(t)$ and $\lim_{t\to-\infty} G^{(1)}(t)$ for $c_k = \infty$.

Together with G we will consider the function $G_1(\tau) = G\left(i\frac{1+\tau}{1-\tau}\right)$ defined on the circumference γ and assume $G_2(\tau) = G_1(\tau)\tau^{-\frac{2}{p}}$, where $\tau^{-\frac{2}{p}}$ is the boundary value at the point τ of that branch of the function $z = w^{-\frac{2}{p}}$, analytic in the plane cut along the ray $[0, +\infty)$, which takes the value $\exp(-\frac{\pi i}{p})$ at the point $w = i$.

For the function $G \in \widetilde{A}_D(p)$ we can determine an argument and an index just in the same way as in §1. It is not difficult to verify that

$$\varkappa_p(G, D) = \varkappa_p(G_2, \gamma) + 1.$$

6.2. The class of the functions $m_p(\gamma)$. Let $\phi = \mathcal{K}_D(\varphi)$. Then the function

$$\Psi(w) = \phi\Big(i\frac{1+w}{1-w}\Big), \tag{6.1}$$

analytic in the circle U, is representable in the form

$$\phi\Big(i\frac{1+w}{1-w}\Big) = \frac{1-w}{2\pi i}\int\limits_{\gamma}\frac{\varphi(i\frac{1+\tau}{1-\tau})d\tau}{(\tau-w)(1-\tau)} = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\varphi^*(\tau)d\tau}{\tau-w} + \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\varphi^*(\tau)d\tau}{1-\tau},$$

where

$$\varphi^*(\tau) = \varphi\Big(i\frac{1+\tau}{1-\tau}\Big), \quad \varphi^* \in L^p(\gamma, w), \quad w = |1-\tau|^{-\frac{2}{p}}. \tag{6.2}$$

Denote by $m_p(\gamma)$ the set of the functions Ψ representable in the form

$$\Psi(w) = \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\psi(\tau)d\tau}{\tau-w} + \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\psi(\tau)d\tau}{1-\tau}, \quad |w| \neq 1 \tag{6.3}$$

and such that $\Psi^{\pm} \in L^p(\gamma, w)$ or, which is the same,

$$\psi, S_\gamma\psi \in L^p(\gamma, w), \quad w = |1-\tau|^{-\frac{2}{p}}.$$

Then if $\phi \in \mathcal{K}^p(D)$, then the given by (6.1) function $\Psi \in m_p(\gamma)$, and conversely, if $\Psi \in m_p(\gamma)$, then $\phi(z) = \Psi\Big(\frac{z-i}{z+i}\Big) \in \mathcal{K}^p(D)$.

6.3. Reduction of the problem in the class $\mathcal{K}^p(D)$ to the problem in the class $m_p(\gamma)$. Let one seek for the function $\phi \in \mathcal{K}^p(D)$ satisfying the boundary value

$$\phi^+(t) = G(t)\phi^-(t) + g(t). \tag{6.4}$$

If ϕ is a solution of this problem, then $\Psi(w) = \phi\Big(i\frac{1+w}{1-w}\Big)$ will be a solution in the class $m_p(\gamma)$ of the problem

$$\Psi^+(\tau) = G\Big(i\frac{1+\tau}{1-\tau}\Big)\Psi^-(\tau) + g\Big(i\frac{1+\tau}{1-\tau}\Big).$$

Assuming $G_1(\tau) = G\Big(i\frac{1+\tau}{1-\tau}\Big)$, and $g_1(\tau) = g\Big(i\frac{1+\tau}{1-\tau}\Big)$, the last equality takes the form

$$\Psi^+(\tau) = G_1(\tau)\Psi^-(\tau) + g_1(\tau), \tag{6.5}$$

and we can easily verify that $g_1 \in L^p(\gamma, w)$.

Rewrite (6.5) as

$$\Psi^+(\tau) = G_1(\tau)\tau^{-\frac{2}{p}}\tau^{\frac{2}{p}}\Psi^-(\tau) + g_1(\tau). \tag{6.6}$$

Let now $f(w) = \left(\frac{1-w}{w}\right)^{\frac{2}{p}}$ be a function, analytic in the plane cut along the segment of the real axis $[0,1]$ such that $f(i) = \exp(-\frac{3\pi}{2p}i)$. In the plane cut along the ray $[1,+\infty)$, define the functions $r(w) = (1-w)^{\frac{2}{p}}$, $r(i) = \exp(-\frac{\pi i}{2p})$. They are continuous up to γ except for the point $\tau = 1$ and branches are chosen such that the function

$$\lambda(w) = \begin{cases} (1-w)^{\frac{2}{p}}, & |w| < 1, \\ (\frac{1-w}{w})^{\frac{2}{p}}, & |w| > 1 \end{cases} \tag{6.7}$$

satisfies the condition $\lambda^+(\tau)[\lambda^-(t)]^{-1} = \tau^{\frac{2}{p}}$ on $\gamma\backslash\{1\}$. Now (6.6) can be written as

$$\frac{\Psi^+(\tau)}{\lambda^+(\tau)} = G_1(\tau)\tau^{-\frac{2}{p}}\frac{\Psi^-(\tau)}{\lambda^-(\tau)} + \frac{g_1(\tau)}{(1-\tau)^{2/p}}.$$

Denote

$$F(w) = \frac{\Psi(w)}{\lambda(w)}, \quad G_2(\tau) = G_1(\tau)\tau^{-\frac{2}{p}}, \quad g_2(\tau) = \frac{g_1(\tau)}{(1-\tau)^{2/p}}. \tag{6.8}$$

Then

$$F^+(\tau) = G_2(\tau)F^-(\tau) + g_2(\tau). \tag{6.9}$$

Since $\Psi \in m_p(\gamma)$, then $F \in \widetilde{K}^p(D)$.

Thus if $\phi \in \mathcal{K}^p(D)$ is a solution of the problem (6.4) and $\Psi(w) = \phi\left(i\frac{1+w}{1-w}\right)$, then

$$\Psi(w) = F(w)\lambda(w), \tag{6.10}$$

where $\lambda(w)$ is defined by (6.7), and F is a solution of the problem (6.9) of the class $\widetilde{\mathcal{K}}^p(\gamma)$. However, if it will happen that $\Psi \in m_p(\gamma)$, then $\phi(z) = \Psi\left(\frac{z-i}{z+i}\right)$ will be a solution of the problem (6.4) of the class $\mathcal{K}^p(D)$.

6.4. Solution of the problem. Since $G \in \widetilde{A}_D(p)$, then $G_2 \in \widetilde{A}(p)$ on γ and therefore it is factorizable in $\mathcal{K}^p(\gamma)$. Let X be its factor-function. It can be easily seen that the solution of the class $\widetilde{K}_p(\gamma)$ of the problem (6.9) given by

$$F(w) = \frac{X(w)}{2\pi i}\int\limits_{\gamma}\frac{g_2(\tau)d\tau}{X^+(\tau)(\tau-w)} + X(w)P_{\varkappa}(w),$$

where $\varkappa = \varkappa_p(G_2)$ and $P_{\varkappa} = 0$, if $\varkappa \le -1$.

The function

$$\Psi(w) = \lambda(w)X(w)\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g_2(\tau)d\tau}{X^+(\tau)(\tau-w)} + P_{\varkappa}(w)\right] \tag{6.11}$$

satisfies the conditions $\Psi^{\pm} \in L^p(\gamma, w)$. In order for Ψ to belong to $m_p(\Gamma)$, it is necessary to find a function ψ such that equality (6.3) would hold.

Consider the following cases. I. $\varkappa(G_2) > -1$; II. $\varkappa(G_2) \le -1$.

I. $\varkappa(G_2) > -1$ (that is, $\varkappa(G) > 0$). The set of solutions given by (6.11) involves $\varkappa(G_2) = \varkappa(G) - 1$ constants. So we must choose them in such a manner that the corresponding to them function $\Psi(w)$ would belong to $m_p(\gamma)$. Since $\operatorname{ind}_p G_2 = \varkappa$, there exists $\lim_{w\to\infty} w^{\varkappa} X(w) = c$, $c \ne 0$. The function $\Psi(w) - \frac{c}{w^{\varkappa}} P_{\varkappa}(w)$ vanishes at infinity, and therefore it can be represented by the Cauchy type integral

$$\Psi(w) - \frac{cP_{\varkappa}(w)}{w^{\varkappa}} = \frac{1}{2\pi}\int\limits_{\gamma} \frac{[\Psi^+ - \frac{c}{\tau^{\varkappa}}P_{\varkappa}] - [\Psi^- - \frac{c}{\tau^{\varkappa}}P_{\varkappa}]}{\tau - w} d\tau =$$
$$= \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\Psi^+(\tau) - \Psi^-(\tau)}{\tau - w} d\tau.$$

If $P_{\varkappa}(w) = A_0 w^{\varkappa} + A_1 w^{\varkappa-1} + \cdots + A_{\varkappa}$, then we find that

$$\Psi(w) = \frac{1}{2\pi}\int\limits_{\gamma} \frac{\Psi^+(\tau) - \Psi^-(\tau)}{\tau - w} d\tau + cA_0 - \frac{c}{2\pi i}\int\limits_{\gamma} \Big(\frac{A_1}{\tau} + \cdots + \frac{A_{\varkappa}}{\tau^{\varkappa}}\Big)\frac{d\tau}{\tau - w} =$$
$$= \frac{1}{2\pi i}\int\limits_{\gamma} \frac{[\Psi^+(\tau) - \Psi^-(\tau)] + R(\tau)}{\tau - w} d\tau + cA_0,$$

where

$$R(\tau) = -c\Big(\frac{A_1}{\tau} + \cdots + \frac{A_{\varkappa}}{\tau^{\varkappa}}\Big),$$

and

$$\Psi^+ - \Psi^- = \lambda^+\Big(g_2 + X^+ S_\gamma \frac{g_2}{X^+} + X^+ P_{\varkappa}\Big) - \lambda^-\Big(\frac{g_2}{G_2} + X^- S_\gamma \frac{g_2}{X^+} +$$
$$+ X^- P_{\varkappa}\Big) = \Big[\lambda^+\Big(g_2 + X^+ S_\gamma \frac{g_2}{X^+}\Big) - \lambda^-\Big(\frac{g_2}{G_2} + X^- S_\gamma \frac{g_2}{X^+}\Big)\Big] +$$
$$+ X^+\lambda^+ P_{\varkappa} - X^-\lambda^- P_{\varkappa} = \Psi_0 + X^-\lambda^-\Big(\frac{X^+\lambda^+}{X^-\lambda^-} - 1\Big)P_{\varkappa} =$$
$$= \Psi_0 + \Psi_1 P_{\varkappa};\quad \Psi_0 = \Big[\lambda^+\Big(g_2 + (g_2) + X^+ S_\gamma \frac{g_2}{X^+}\Big) - \lambda^-\Big(\frac{g_2}{G_2} - X^- S_\gamma \frac{g_2}{X^+}\Big)\Big].$$

For the inclusion $\Psi \in m_p(\gamma)$ it is necessary and sufficient that the condition

$$cA_0 = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\Psi^+(\tau) - \Psi^-(\tau) + R(\tau)}{1 - \tau} d\tau,$$

that is,

$$cA_0 = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\Psi_0(\tau) + \Psi_1(\tau)P_{\varkappa}}{1 - \tau} d\tau + c(A_1 + \cdots + A_{\varkappa})$$

be fulfilled.

Hence we have

$$c(A_0 - A_1 - \cdots - A_\varkappa) = a + A_0 b_1 + \cdots + A_\varkappa b_\varkappa, \tag{6.12}$$

where

$$a = \frac{1}{2\pi i}\int_\gamma \frac{\Psi_0(\tau)d\tau}{1-\tau}, \quad b_j = \frac{1}{2\pi i}\int_\Gamma \frac{\lambda^-(\tau)X^-(\tau)(G_1-1)\tau^{\varkappa-j-1}}{1-\tau}d\tau.$$

With respect to the unknown numbers $A_0, \ldots, A_\varkappa$ and $\varkappa > 0$, the equation (6.12) is solvable. Indeed, this is evident for $\varkappa(G_2) > 0$. Substituting the above-found values in (6.11), we obtain the general solution of the problem. Note that this solution contains $\varkappa(G_2)$ arbitrary constants. For $\varkappa = 0$, we will proceed from the representations (6.3) and (6.11). According to the former, $\Psi(\infty) = \frac{1}{2\pi i}\int_\gamma \frac{\Psi_0(\tau)d\tau}{1-\tau}$, while to the latter, $\Psi(\infty) = A_0$. Whence

$$A_0 = \frac{1}{2\pi i}\int_\gamma \frac{\Psi_0 d\tau}{1-\tau}.$$

Thus, if $\operatorname{ind}_p G_2 > -1$, the coefficients of the polynomial $P_\varkappa$ satisfy (6.12) and the function Ψ is given by (6.11), then the function $\phi(z) = \Psi\left(\frac{z-i}{z+i}\right)$ is a solution of the problem (6.4).

II. $\varkappa(G_2) \leq -1$ (that is, $\varkappa(G) \leq 0$). In this case the problem (6.5) may have perhaps only one solution given by equation (6.11) with $P_\varkappa = 0$. For its solvability, the following conditions are to be fulfilled:

$$\int_\gamma g_2(\tau)\frac{\tau^k}{X^+(\tau)}d\tau = 0, \quad k = 0, 1, \ldots, |\varkappa| - 1. \tag{6.13}$$

Show that the function Ψ belongs to $m_p(\gamma)$. Since the conditions (6.13) are fulfilled, there exists $\lim_{w\to\infty} F(w) = c$, i.e., $\Psi(\infty) = c$. Then

$$\Psi(w) = \frac{1}{2\pi i}\int_\gamma \frac{\Psi^+(\tau) - \Psi^-(\tau)}{\tau - w}d\tau + c$$

and for the inclusion $\Psi \in m_p(\gamma)$ we must have

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{\Psi^+(\tau) - \Psi^-(\tau)}{1-\tau}d\tau. \tag{6.14}$$

The function $\frac{\Psi^+(w)}{1-w} = \frac{(1-w)^{\frac{2}{p}}F(w)}{1-w}$ belongs to H^1, and therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{\Psi^+(\tau)d\tau}{1-\tau} = 0.$$

Further, (6.3) implies that $\Psi(\infty) = c = -\frac{1}{2\pi i}\int_\gamma \frac{\Psi^-(\tau)d\tau}{1-\tau}$, and hence (6.14) is fulfilled.

Thus, problem (6.5) for $\varkappa(G_2) \le -1$ possesses the solution in the class $m_p(\gamma)$ if and only if are fulfilled conditions (6.13). In this case the solution is unique and given by formula (6.11).

Transforming the variable in (6.11) and taking into account the fact that $\varkappa(G_2, \gamma) = \varkappa(G, D) - 1$, the solutions (if any) of problem (6.4) are given in all the cases by the equality

$$\phi(z) = \Big(\frac{2i}{z+i}\Big)^{\frac{2}{p}} X\Big(\frac{z-i}{z+i}\Big) \times$$
$$\times \Bigg[\frac{z+i}{2\pi i}\int_{-\infty}^{\infty} \frac{(t+i)^{\frac{2}{p}-1} g(t)dt}{X^+(\frac{t-i}{t+i})(t-z)} + P_{\varkappa(G)-1}\Big(\frac{z-i}{z+i}\Big)\Bigg]. \tag{6.15}$$

Note that if $\varkappa(G) \ge 0$, then the conditions (6.12) must be fulfilled ensuring $\varkappa(G) - 1$ arbitrary coefficients for the polynomial $P_{\varkappa(G)-1}$. If $\varkappa(G) < 0$, then for the problem to be solvable it is necessary and sufficient that the conditions

$$\int_{-\infty}^{\infty} g(t)\Big(\frac{t-i}{t+i}\Big)^k \frac{1}{X^+(\frac{t-i}{t+i})}\frac{dt}{(t+i)^2} = 0, \quad k = 0, \dots, |\varkappa_p(G)| - 1, \tag{6.16}$$

be fulfilled, where $X(z)$ is the factor-function for $G_2(\tau) = G\Big(i\frac{1+\tau}{1-\tau}\Big)\tau^{-\frac{2}{p}}$.

6.5. One theorem on weights in the case of a straight line. As a consequence of the result obtained in subsection 6.4 we present the following:

Proposition 6.1. *If* $-\frac{1}{p} < \alpha_k < \frac{1}{p'}$, $-\frac{1}{p} < \beta + \sum_{k=1}^n \alpha_k < \frac{1}{p}$ *and* X *is the factor-function of a function from* $\widetilde{A}(\rho)$ *on* γ, *then the functions*

$$w_1(t) = X^+\Big(\frac{t-i}{t+i}\Big)(t+i)^{1-\frac{2}{p}} \tag{6.17}$$

and

$$w(t) = (1 + |t|^\beta)\prod_{k=1}^n |t - t_k|^{\alpha_k}, \quad t_k \in D, \tag{6.18}$$

belong to the class $W_p(D)$.

Proof. The first assertion is a consequence of the fact that the function given by the equality (6.15) is a solution of the problem (6.4) in the class $\mathcal{K}^p(D)$.

To prove the second assertion, it suffices to show that the function $\rho(t) = (t+i)^\beta \prod\limits_{k=1}^{n} |t-t_k|^{\alpha_k}$ belongs to the class $W_p(D)$. Suppose

$$\nu_0 = 1 - \frac{2}{p} - \sum_{k=1}^{n} \alpha_k - \beta, \quad \nu_k = \alpha_k, \quad k = \overline{1,n}.$$

Then $\frac{1}{p} < \nu_0 < \frac{1}{p'}$. Indeed, by assumption we have $-\frac{1}{p'} - \sum_{k=1}^{n} \alpha_k < \beta < \sum_{k=1}^{n} \alpha_k + \frac{1}{p'}$. Therefore

$$\nu_0 < \left(1 - \frac{2}{p} - \sum_{k=1}^{n} \alpha_k\right) + \frac{1}{p} + \sum_{k=1}^{n} \alpha_k = \frac{1}{p'},$$
$$\nu_0 > \left(1 - \frac{2}{p} - \sum_{k=1}^{n} \alpha_k\right) + \sum_{k=1}^{n} \alpha_k - \frac{1}{p'} = -\frac{1}{p}.$$

Consequently, the function $\rho_1(\tau) = \prod_{k=1}^{n} |\tau - c_k|^{\nu_k}$, where $c_k = i\frac{1+t_k}{1-t_k}$, belongs to the class $W_p(\gamma)$. But $0 < m < \left|\frac{\rho_1(\tau)}{X^+(\tau)}\right| < M$, where X is the factor-function of the piecewise continuous on γ function of the class $\widetilde{A}(p)$. Therefore, by virtue of (6.17), we have $\rho_1\left(\frac{t-i}{t+i}\right)(t+i)^{1-\frac{2}{p}} \in W_p(D)$. THe transformation of the variable in $\rho_1(\tau)$ shows that the our assertion is valid. ■

6.6. On the problem of conjugation in the case of an infinite curve. Singular integrals on infinite lines or the problem of conjugation in domains with such boundaries have been investigated in [32], [168], etc. The problem in the class $\mathcal{K}^p(\Gamma)$ is considered in [111] for a class of curves. Following the reasoning of subsections 6.1–6.4 and using the results of [111] (such as the belonging of the curves under consideration to R, reduction of the problem to the case of bounded curve, and etc.) we can extend the result of the present section to the curves from the above-mentioned class. Despite the complete analogy, we do not dwell on the question.

§ 7. On the Riemann–Hilbert Problem in a Class of Cauchy Type Integrals

Let $\Gamma \in R$ be a closed Jordan curve bounding a finite domain D. Denote by $\mathcal{K}^p(D,\rho)$, $1 < p < \infty$ a set of analytic in D functions ϕ representable in terms of

$$\phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(t)dt}{t-z}, \quad z \in D, \quad \varphi \in L^p(\Gamma,\rho).$$

In other words, the class $\mathcal{K}^p(D,\rho)$ consists of the restrictions on D of the functions from the class $\mathcal{K}^p(\Gamma,\rho)$. Let a, b, c be real functions defined on Γ, and

$$a, b \in L^\infty(\Gamma), \quad c \in L^p(\Gamma, \rho), \quad \inf_{t\in\Gamma} |a^2 + b^2| > 0. \tag{7.1}$$

Consider the Riemann-Hilbert problem formulated as follows: define the function $\phi \in \mathcal{K}^p(D, \rho)$ whose angular values $\phi^+(t)$ satisfy almost everywhere on Γ the condition

$$\mathrm{Re}[(a(t) + ib(t))\phi^+(t)] = c(t). \tag{7.2}$$

This problem will be called homogeneous for $c = 0$.

In the sequel will be used Muskhelishvili method of reduction of Riemann-Hilbert problem to the problem of linear conjugation. For such a reduction it is necessary to know properties of the integrals $(\mathcal{K}_\Gamma f)(z(w))$ in a circle, where $z = z(w)$ is a conformal mapping of the circle onto D. The obtained linear conjugation problem has to be considered in the class $\widetilde{K}^p(\gamma, r)$, where r is independent not only of ρ but of the derivative $z'(w)$, namely $r(\zeta) = \rho(\zeta)\sqrt[p]{z'(\zeta)}$. In the general case r differs from the conventional weight- $w = \prod |\zeta - c_k|^{\alpha_k}$. Moreover, such a situation takes place even if $\rho = 1$ and Γ is a smooth curve.

7.1. Reduction of the problem (7.2) to the linear conjugation problem. The conditions

$$1)\ \Gamma \in R, \quad 2)\ \rho \in W_p(\Gamma), \quad 3)\ \rho(z(\zeta))\sqrt[p]{z'(\zeta)} \in W_p(\gamma) \tag{7.3}$$

will be assumed to be fulfilled. Note that the condition $\rho \in W_p(\Gamma)$ implies $\rho(z(\zeta))\sqrt[p]{z'(\zeta)} \in L^p(\gamma)$. The requirement 3) is more strict.

Let $\phi(z) = (\mathcal{K}_\Gamma f)(z)$ be a solution of problem (7.2) of the class $\mathcal{K}^p(D, \rho)$, $p > 1$. In the circle U, consider the function

$$\Psi(w) = \phi(z(w)) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t - z(w)},$$

which under the conditions (7.3) belongs to the class $\mathcal{K}^p(U, r)$ (see Chapter I, Proposition 4.1) and satisfies the boundary condition

$$\mathrm{Re}[(A(\zeta) + iB(\zeta))\Psi^+(\zeta)] = C(\zeta), \quad \zeta \in \gamma, \tag{7.4}$$

where $A(\zeta) = a(z(\zeta))$, $B(\zeta) = b(z(\zeta))$, $C(\zeta) = c(z(\zeta))$.

Conversely, if $\Psi \in \mathcal{K}^p(U, r)$ is a solution of the problem (7.4), then $\phi(z) = \Psi(w(z))$, where $w = w(z)$ is the inverse to $z = z(w)$ function, is a solution of the problem (7.2) belonging to the class $\mathcal{K}^p(D, \rho)$ (see Chapter I, Proposition 4.2).

This implies that if we possess all the required solutions of problem (7.4), then will have all reqired solutions for (7.2) as well.

If, however, there appear conditions for solvability of problem (7.4), then they can be interpreted as the conditions for solvability of problem (7.2). Moreover, since the systems of functions $\phi_1(z), \ldots, \phi_n(z)$ and $\phi_1(z(w))$,

$\phi_2(z(w)), \ldots, \phi_n(z(w))$ are simultaneously linearly independent in the domains D and U, respectively, the number of linearly independent solutions of the homogeneous problems (7.2) and (7.4) coincide.

Thus we have to solve the problem (7.4) in the class $\mathcal{K}^p(U,r)$, $r = \rho\sqrt[p]{z'}$ under the assumption that the conditions (7.3) are fulfilled.

In the same way as in [106], consider the function

$$\Omega(w) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi(\frac{1}{\overline{w}})}, & |w| > 1, \end{cases} \tag{7.5}$$

where $\Psi(w) = (\mathcal{K}_\Gamma f)(z(w))$, $f \in L^p(\gamma, r)$.

Following the proof of Theorem 1 of Chapter IV in [66] (in which $r = \prod |t - t_k|^{\alpha_k}$, $t_k \in \gamma$), we can easily verify the validity (see also [118], Lemma 4) of the following

Lemma 7.1. *Let $\psi \in L^p(\gamma, r)$, $r \in W_p(\gamma)$ and*

$$\Omega(w) = \begin{cases} \frac{1}{2\pi i}\int_\gamma \frac{\psi(\zeta)d\zeta}{\zeta - w}, & |w| < 1 \\ \overline{\frac{1}{2\pi i}\int_\gamma \frac{\psi(\zeta)d\zeta}{\zeta - \frac{1}{\overline{w}}}}, & |w| > 1. \end{cases}$$

Then the function $\Omega_1(w) = \Omega(w) - \Psi(0)$, $\Psi(0) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(\zeta)}{\zeta}d\zeta$ is representable by the Cauchy type integral with density from the class $L^p(\gamma, r)$, i.e., $\Omega \in \widetilde{\mathcal{K}}^p(\gamma, r)$.

Now we are on the point of reducing the problem (7.4) to the linear conjugation problem.

Rewrite (7.4) as follows:

$$(A + iB)\Psi^+ + \overline{(A + iB)\Psi^+} = 2C. \tag{7.6}$$

Define the funcyion Ω by (7.5). Taking into account that $\Omega^+ = \Psi^+$, $\Omega^- = \overline{\Psi^+}$, the equality (7.6) takes the form

$$(A + iB)\Omega + (A - iB)\Omega^- = 2C. \tag{7.7}$$

Thus, if Ψ is a solution of the Riemann-Hilbert problem (7.4) of the class $\mathcal{K}^p(U,r)$, then Ω is a solution of the problem (7.7) of the class $\widetilde{\mathcal{K}}^p(\gamma, r)$.

If Ω is a solution of the problem (7.7) of the class $\widetilde{\mathcal{K}}^p(\gamma, r)$, then its restriction on U fails, generally speaking, to be a solution of the problem (7.4). For this to be so, it is necessary that the equality

$$\Omega^+ = \overline{\Omega^-} \tag{7.8}$$

be fulfilled. If this equality does take place, then using the same equalities as in obtaining (7.7) but in reverse order, we can see that the restriction on U of the function Ω is a solution of the problem (7.4).

Assume that

$$\Omega_* = \overline{\Omega(\frac{1}{\overline{w}})}, \quad |w| \neq 1.$$

The equalities

$$[\Omega_*(w)]_* = \Omega(w), \quad \Omega_*^-(\zeta) = \overline{\Omega^+(\zeta)}, \quad \Omega_*^+(\zeta) = \overline{\Omega^-(\zeta)}$$

are obvious.

Passing in (7.7) to complex conjugate functions, we obtain

$$(A - iB)\overline{\Omega^+} + (A + iB)\overline{\Omega^-} = 2C,$$

or

$$(A - iB)\Omega_*^- + (A + iB)\Omega_*^+ = 2C, \tag{7.9}$$

It is not difficult to prove that if $\Omega \in \widetilde{\mathcal{K}}^p(\gamma, r)$, then $\Omega_* \in \widetilde{\mathcal{K}}^p(\gamma, r)$. Therefore, owing to (7.9) we conclude: if Ω is a solution of the problem (7.7) of the class $\widetilde{\mathcal{K}}^p(\gamma, r)$, then Ω^* is likewise a solution of the same problem. Then the function $\frac{1}{2}(\Omega + \Omega_*)$ which already satisfies the condition (7.8) is also the solution of the problem (7.7). Consequently, the restriction $\Psi(w) = \frac{1}{2}(\Omega(w) + \Omega_*(w))$, of this function on U will be a solution of the problem (7.4) of the class $K^p(U, r)$. However, if Ψ is a single solution of (7.4), then the equalities $\psi(w) = \Omega(w) = \Omega_*(w)$ in U are valid, and hence $\Psi(w) = \frac{1}{2}(\Omega(w) + \Omega_*(w))$. Thus we have proved the following

Lemma 7.2. *Every solution $\Omega \in \widetilde{\mathcal{K}}^p(\gamma, r)$ of problem (7.7) generates a solution of problem (7.4) of the class $\mathcal{K}^p(D, \rho)$ specified by the equality $\Psi(w) = \frac{1}{2}(\Omega(w) + \Omega_*(w))$, $|w| < 1$ and vice versa, every solution of problem (7.4) of the class $\mathcal{K}^p(D, \rho)$ has such a form.*

Write the boundary condition (7.7) as

$$\Omega^+ = G\Omega^- + g, \tag{7.10}$$

where

$$G = (iB - A)(iB + A)^{-1}, \quad g = 2C(iB + A)^{-1}.$$

By Lemma 7.2, it suffices to investigate the problem (7.10) in the class $\widetilde{\mathcal{K}}^p(\gamma, r)$, $r = \rho \sqrt[p]{z'}$.

Since $r(\zeta) \in W_p(\gamma)$ (see (7.3)), there exists by Theorem 4.8 a real function $\psi \in L^\infty(\gamma)$ such that $r(\zeta)$ is equivalent to the function $\exp \frac{i}{2} S_\Gamma \psi$, i.e., $0 < m \leq \left| \frac{\exp \frac{i}{2} S_\gamma \psi}{r} \right| \leq M$, and we may assume that

$$r(\zeta) = \exp \frac{i}{2}(S_\gamma \psi)(\zeta). \tag{7.11}$$

The assumptions (7.1), (7.3) and (7.11) imply that the conditions of Theorem 5.1 are fulfilled and we are able to reduce problem (7.10) of the class $\widetilde{\mathcal{K}}_p(\gamma, r)$ to the linear conjugation problem.

Theorem 7.1. *Let the conditions* (7.1), (7.3) *and* (7.11) *be fulfilled, and*

$$Y(w) = \exp\left[\frac{1}{2\pi}\int\limits_{\gamma}\frac{\psi(\zeta)d\zeta}{\zeta - w}\right], \quad |w| \neq 1.$$

Then if $F(w)$ is a solution of the problem

$$F^{+}(\zeta) = G(\zeta)[\exp(i\psi(\zeta))]F^{-}(\zeta) + g(\zeta) \tag{7.12}$$

of the class $\widetilde{\mathcal{K}}^{p}(\gamma, r)$, then the function

$$\Psi(w) = \frac{1}{2}[\Omega(w) + \Omega_{*}(w)], \ \ \Omega(w) = F(w)[Y(w)]^{-1}, \ \ |w| < 1, \tag{7.13}$$

is a solution of the problem (7.4) *of the class $\mathcal{K}^{p}(D, \rho)$, and the function*

$$\phi(z) = \Psi(w(z)), \quad z \in D,$$

is the solution of problem (7.2) *of the class $\mathcal{K}^{p}(D, \rho)$. All the solutions can be obtained in such a way.*

Let us give to the condition (7.12) a somewhat different form. To this end we assume that

$$\rho(t) = \rho(z(\zeta)) = \exp\frac{1}{2\pi}\int\limits_{\gamma}\frac{\mu(\tau)d\tau}{\tau - \zeta}, \tag{7.14}$$

$$\lim_{w\to\zeta}\arg z'(w) = \beta(\zeta), \tag{7.15}$$

$$G(\zeta) = |G(\zeta)|\exp i\nu(\zeta). \tag{7.16}$$

Then the following corollary is valid.

Corollary. *If the weight function $\rho \in W_{p}(\Gamma)$ is of the form* (7.14) *and the function $r(\zeta) = \exp\left\{\frac{1}{2\pi}\int_{gm}\frac{\beta(\tau)p^{-1}+\mu(\tau)}{\tau-\zeta}d\tau\right\}$ belongs to $W_{p}(\gamma)$, then the conclusion of Theorem* 7.1 *with F as a solution of the problem*

$$F^{+}(\zeta) = |G(\zeta)|\exp\Big[i\Big(\frac{\beta(\zeta)}{p} + \mu(\zeta) + \nu(\zeta)\Big)\Big]F^{-}(\zeta) + g \tag{7.17}$$

is valid.

The character of the solvability of the problem (7.17) depends on the coefficient

$$|G|\exp\Big[i\Big(\frac{\beta}{p} + \mu + \nu\Big)\Big].$$

Here $|G|$ and $\exp i\nu$ are defined by means of the coefficients a and b of problem (7.2); $\exp i\frac{\beta}{p}$ is defined by a curve and by a class of unknown functions, and $\exp i\mu$ by a weight function ρ. On the basis of the above-said, varying the sets of unknown functions (i.e., those of weight functions and of the exponent p), coefficients of the boundary condition and the domains of analyticity of unknown functions, the corollary of Theorem 7.1 allows one to deal with such of linear conjugation problems which have already been well studied and thus to obtain the appropriate results for the Riemann-Hilbert problem. Of course, we intend to apply for this purpose Theorem 1.1, but postpone this until Chapter IV bearing in mind an application of properties of the function $\beta(\zeta)$ which will be established in Chapter III.

§ 8. On the Inhomogeneous Linear Conjugation Problem in a Class of Cauchy Type $\widetilde{L}$-Integrals

In this section the use will be made of the results obtained in Chapter I, §6 to solve an inhomogeneous linear conjugation problem under the following assumptions.

Let Γ be a simple, rectifiable curve, belonging to the class R, G satisfy the Hölder condition on Γ and differ on Γ from zero, $g \in L(\Gamma)$.

A solution will be sought in the class of functions representable in the form

$$\phi(z) = \begin{cases} K_\Gamma(\varphi_1)(z) & \text{for} \quad z \in D^+, \\ K_\Gamma(\varphi_2)(z) + P(z) & \text{for} \quad z \in D^-, \end{cases} \tag{8.1}$$

where φ_1, $\varphi_2 \in L(\Gamma)$, P is a polynomial. The last class coincides the class of functions representable by the Cauchy type $\widetilde{L}$-integral. whith polynomial principal part at infinity. The above-formulated problem in the case under consideration has, in general, no solutions in the class of functions representable by the Cauchy-Lebesgue type integral even under the simplest assumptions with respect to Γ and G which can be illustrated in the following example.

Let $\Gamma = \gamma$ be a unit circumference, $G(t) \equiv -1$, g be a summable on γ function for which $S_\gamma(g)$ is not summable.

Suppose there exists a solution of the problem ϕ representable by the Cauchy-Lebesgue type integral $K_\gamma(\varphi)$, $\varphi \in L(\gamma)$. Then by the Sokhotskiĭ-Plemelj formulas,

$$\phi^+(t) + \phi^-(t) = S_\gamma(\varphi)(t).$$

On the other hand, by the assumption,

$$\phi^+(t) + \phi^-(t) = g(t).$$

Consequently, $S_\gamma(\varphi)(t) = g(t)$. Using the inversion formula of a singular Cauchy integral from Chapter I, §6, we obtain $\varphi(t) = S_\gamma(g)(t)$ which is impossible since $\varphi \in L(\gamma)$ and $S_\gamma(g) \notin L(\gamma)$.

The solution of the boundary value problem can be obtained as well under the above assumptions as is done in Chapter II, §3, by substituting the Lebesgue integral by the $\widetilde{L}$-integral. The general solution will have the form

$$\phi(z) = \frac{X(z)}{2\pi i} \int\limits_{\Gamma} \frac{g(t)dt}{X^+(t)(t-z)} + X(z)P(z), \tag{8.2}$$

where X is a canonical solution of the homogeneous problem representable by the Cauchy integral with density from the class $H(\Gamma)$, and P is a polynomial.

Using the results of subsection 6.4, Remark 1 to Theorem 6.5 from Chapter I, and the fact that $S_\Gamma \varphi \in H(\Gamma)$ for $\varphi \in H(\Gamma)$ and $\Gamma \in R$ (see [28], p. 253), it can be verified that the function ϕ is representable by the Cauchy type $\widetilde{L}$-integral with polynomial principal part at infinity (i.e., in the form of (8.1)).

It follows from formula (8.2) that:

(1) if g and $S_\Gamma(g)$ are summable on Γ, then any solution of the problem is representable by the Cauchy-Lebesgue type integral;

(2) if $g \in L_p(\Gamma)$, $p > 1$, then any solution of the problem is representable by the Cauchy-Lebesgue type integral with density from the class $L_p(\Gamma)$;

(3) if $g \in H(\Gamma)$, then any solution of the problem is piecewise holomorphic.

Thus, the extension in cases (1)–(3) of the class of unknown functions to the class of functions representable in the form of (8.1) fails to provide us with new solutions.

Notes and Comments to Chapter II

Theorem 2.1 can be considered as a generalization of V. Smirnov's theorem ([146], see also [43], p. 401) in which it is stated that if Γ is a circumference and φ is continuous, then X belongs to the Hardy class H^p for any $p > 0$. On the other hand, this theorem can also be assumed to be a generalization of A. Zygmund's theorem on the integrability of a function of the kind $\exp \lambda \widetilde{\varphi}$, ([169]). The analogues of Theorem 2.1 under different assumptions regarding Γ and φ can be found in I. Simonenko [141], I. Danilyuk [21], V. Shelepov [139].

Problem (I) was investigated in the classes $\mathcal{K}^p(\Gamma)$ by B. Khvedelidze [66] when $G \in H$ and Γ is a Lyapunov curve. G. Manjavidze and B. Khvedelidze [90] studied the mentioned problem in case $G \in C(\Gamma)$ under the same assumption on curves. The last authors have not focused their attention on extension of the class of boundary curves. Neverthless their method, in fact, works in a more general situation when $\Gamma \in R$. For more detail consideration see [44–45]. In the case of piecewise continuous coefficients E. Gordadze [44–45] studied Problem (I) under some restrictions at the points of discontinuity of G. In [7] it was shown that for a Noethericity of (II) it is necessary to bind the character of discontinuity of G and geometric properties of G in a neighborhood of discontinuity points (see also [138]).

In [78] it was derived that the well-known formulas for solution in the case of Lyapunov curves remain valid for $\Gamma \in R$ as well.

In I. Simonenko's paper [141], a class of functions $A(p)$ has been introduced and a complete investigation of the problem (I) with the coefficient $G \in A(p)$, when Γ is the Lyapounov curve, is presented. Analogous results have been obtained by I. Danilyuk [18] for coefficients admitting the representation $G = \exp(\varphi_0 + \varphi_1)$, where $\varphi_0 \in C(\Gamma)$ and φ_1 is a function with a finite variation. It is proved by V. Shelepov [139] that the above-mentioned results remain also valid for the curves with bounded rotation.

As for singular equations with oscillating coefficients, the work due to [7] is worth mentioning.

Problem (I) in the class $E^1(\Gamma)$ has been studied in the case of Lyapunov Γ-curves [25].

The main goal of investigations carried out in papers [78–79] and presented in §1–4 is simultaneous extension both of a class of boundary curves and of a class of coefficients. For example, the class $A(p)$ does not contain all admissible piecewise continuous coefficients. To make up a deficiency, the class $\widetilde{A}(p)$ has been introduced. Along with this, has established that boundary curves belong to that class of curves which contains, for example, any curves (without cusps) made up of smooth arcs and of those with bounded rotation.

Recently, in [46-47] has been considered the discontinuous boundary value problem in a domain with an arbitrary piecewise smooth boundary with coefficient $G \in A(p)$ which satisfies at the cusps some additional condition under oscillation.

Reduction of the problem of conjugation (I) in the class $\mathcal{K}^p(\Gamma, w)$ to the analogous weightless problem was first performed in [110] in the case where Γ is the Lyapunov curve. The general case of curves from the R class has been considered in [78]. The statement appearing in the corollary of Theorem 5.1 is due to David's theorem. Reduction of singular integral equation (II) in the class $L^p(\Gamma, w)$, $w = \prod |t - c_k|^\alpha$ to the equation in the class $L^p(\Gamma)$ is also available in [107] for the case of Lyapounov curves. The case $\Gamma \in R$, $w \in W_p(\Gamma)$ is considered in [121]. In [149] the criterion for equation (II) with piecewise continuous coefficients to be Noetherian in the classes $L^p(\Gamma, w)$ with weights w from the class A_p, is established.

The method of reducing the boundary problems to the problem with shifts is pointed out in [92].

The results obtained with respect to the boundary value problem on a straight line in that generality which is set forth here, are presented for the first time (see also [32], [168], [111]).

The technique permitting one to reduce the Riemann-Hilbert problem in the class $\mathcal{K}^p(D, w)$ to the linear conjugation problem with circular boundary contour γ, whose coefficient absorbs all singularities of the boundary, weight and initial coefficient, has been suggested in [118].

CHAPTER III
APPLICATION OF SOLUTION OF THE LINEAR CONJUGATION PROBLEM TO CONFORMAL MAPPINGS

§ 1. One Representation of a Derivative of Conformal Mapping of a Circle onto a Domain with a Piecewise Smooth Boundary and Its Consequences

Below the use will be made of the following fact (see, e.g., [43], p. 405]: if a function $z = z(w)$ maps conformally the unit circle U onto a finite domain D bounded by a closed, rectifiable curve Γ, then: (i) $z' \in H^1$; (ii) z is continuous on $\overline{U}$ and absolutely continuous on its boundary γ; (iii) for almost all $\theta \in [0, 2\pi]$ there exists an angular boundary limit of the functions $z'(w)$ and

$$\lim_{w \widehat{\to} \exp i\theta} z'(w) = -ie^{-i\theta}\frac{dz(e^{i\theta})}{d\theta}. \tag{1.1}$$

Since $t = z(e^{i\theta})$ is the equation of the curve Γ, we have $\frac{dz(e^{i\theta})}{d\theta} = \left|\frac{dz(e^{i\theta})}{d\theta}\right| e^{i\alpha(\theta)}$, where $\alpha(\theta)$ is the angle between the oriented tangent to Γ at the point $z(e^{i\theta})$ and the real axis.

Let $z'(0) > 0$ and consider an analytic in D function

$$\ln z'(w) = \ln|z'(w)| + i\arg z'(w), \quad \arg z'(0) = 0.$$

From the equality (1.1) we can conclude that there exists an angular limit

$$\lim_{w \widehat{\to} \exp i\theta} \arg z'(w) = \alpha(\theta) - \theta + \frac{\pi}{2} + 2k(\theta)\pi = \beta(\theta) + 2k(\theta)\pi, \tag{1.2}$$

where $k(\theta)$ is a function taking integer values.

If Γ is a smooth curve, then in the equality (1.2) as $\alpha(\theta)$ one can take a continuous on $[0, 2\pi]$ function with the condition

$$\alpha(2\pi) = \alpha(0) + 2\pi. \tag{1.3}$$

If Γ is a piecewise smooth curve with angular points t_k, $k = \overline{1, n}$, then as $\alpha(\theta)$ will be taken a piecewise continuous function which at the points c_k corresponding to the points t_k $(z(c_k) = t_k)$ has jump discontinuities $h_k = \alpha(c_k+) - \alpha(c_k-) = \pi - \pi\nu_k$, where $\nu_k\pi$, $0 \le \nu_k \le 2$ is the interior with respect to D angle at the vertex t_k. The function α will be called the tangential function of the curve Γ.

Theorem 1.1. *If $z = z(w)$ maps conformally a unit circle U onto the domain D bounded by a closed piecewise smooth Jordan curve Γ, then*

$$z'(w) = z'(0)\exp\left(\frac{1}{\pi}\int_\gamma \frac{\beta(\theta)d\tau}{\tau - w}\right), \quad \tau = e^{i\theta}, \tag{1.4}$$

where

$$\beta(\theta) = \alpha(\theta) - \theta - \frac{\pi}{2}. \tag{1.5}$$

Proof. Consider the function

$$\Omega(w) = \begin{cases} \sqrt{z'(w)}, & |w| < 1, \\ \overline{\sqrt{z'(\frac{1}{\overline{w}})}}, & |w| > 1. \end{cases} \tag{1.6}$$

This function belongs to the class $\widetilde{\mathcal{K}}^2(\gamma)$ (see Chapter II, Lemma 7.1), and owing to (1.2) we have

$$\Omega^{\pm}(\tau) = \sqrt{|z'(\tau)|}\exp[\pm\frac{i}{2}(\beta(\theta) + 2\pi k(\theta))], \quad \tau = e^{i\theta}.$$

Assume

$$G(\tau) = \Omega^{+}(\tau)[\Omega^{-}(\tau)]^{-1},$$

Then

$$G(e^{i\theta}) = \exp i\beta(\theta). \tag{1.7}$$

In the class $\widetilde{\mathcal{K}}^2(\gamma)$, consider the problem

$$\Phi^{+}(\tau) = G(\tau)\Phi^{-}(\tau). \tag{1.8}$$

Let

$$X(w) = \exp\left\{\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\ln G(\tau)d\tau}{\tau - w}\right\} = \exp\left\{\frac{1}{2\pi}\int\limits_{\gamma}\frac{\beta(\theta)d\tau}{\tau - w}\right\}. \tag{1.9}$$

Represent the function β in terms of $\beta = \beta_0 + \beta_1$, where β_0 is continuous on $[0, 2\pi]$ and $\beta_0(2\pi) = \beta_0(0) + 2\pi$, while β_1 is a piecewise linear function. Then it follows from (1.9) that in the neighbourhood of γ (see Chapter II, Lemma 3.6 and inequality (3.24)) we have

$$X(w) = O\Big(\prod_{k=1}^{n}(w - c_k)^{\frac{\nu_k - 1}{2}}\exp\Big(\frac{1}{2\pi}\int\limits_{\gamma}\frac{\beta_0(\theta)d\tau}{\tau - w}\Big)X_0(w)\Big), \tag{1.10}$$

where $0 < C_1 \leq |X_0(w)| \leq C_2 < \infty$.

Let first

$$\nu_k \in [0, 2), \quad k = \overline{1, n} \tag{1.11}$$

Then on the basis of (1.10) we conclude that $\frac{1}{X} \in \widetilde{K}^2(\gamma)$. Therefore $\Phi X^{-1} \in \widetilde{K}(\gamma)$ and we can easily verify that the solutions of the problem (1.8) are contained in the set of functions given by the equality

$$\Phi(w) = CX(w) = C \exp\left(\frac{1}{2\pi} \int\limits_{\gamma} \frac{\beta(\theta) d\tau}{\tau - w}\right), \tag{1.12}$$

where C is an arbitrary complex constant.

Since the function $\Omega(w)$ is one of the solutions of the class $\widetilde{\mathcal{K}}^2(\gamma)$ of the problem (1.8), this implies that

$$\Omega = \widetilde{C} X \tag{1.13}$$

and from (1.12) for $|w| < 1$ we have

$$\sqrt{z'(w)} = \widetilde{C} \exp\left\{\frac{1}{2\pi} \int\limits_{\gamma} \frac{\beta(\theta) d\tau}{\tau - w}\right\} = \widetilde{C} X(w), \tag{1.14}$$

where $\widetilde{C}$ is a constant. Find $\widetilde{C}$.

By virtue of (1.13)–(1.12), for $|w| > 1$ we have

$$\overline{\sqrt{z'(\frac{1}{\overline{w}})}} = \widetilde{C} \exp\left(\frac{1}{2\pi} \int\limits_{\gamma} \frac{\beta(\theta) d\tau}{\tau - w}\right).$$

Passing to limit as $w \to \infty$, we obtain $\widetilde{C} = \overline{\sqrt{z'(0)}}$ and as $z'(0) > 0$, we have

$$\widetilde{C} = \sqrt{z'(0)}. \tag{1.15}$$

From (1.14) and (1.15) we obtain the equality (1.4) with the additional, for the time being, assumption (1.11).

We will dwell on the case where $\nu_k = 2$ for some k. Without loss of generality we assume that there exists only one such point. Denote the corresponding angular point on Γ by C and let $z(c) = C$, $c = \exp i\theta_c$.

Consider on Γ two sequences of points τ_n and t_n converging to C, such that $\arg w(\tau_n) \uparrow \theta_c$, $\arg w(t_n) \downarrow \theta_c$ respectively. Let $\widetilde{\Gamma}_n$ be the part of Γ left after removing the arc $\tau_n C t_n$. Draw smooth Jordan arcs γ_n through the points τ_n, t_n and C, that is, construct on the segment $[\arg w(\tau_n), \arg w(\tau_n)]$ the functions $\mu_n = \mu_n(\theta)$ for which μ_n' are continuous and different from zero. Moreover, the tangents of γ_n and Γ are assumed to coincide at the points τ_n and t_n. Since $\mathbb{C}\backslash\widetilde{\Gamma}_n$ is a domain and two smooth arcs of the curve Γ meet at the point C, such arcs γ_n can be constructed. Indeed, if $\rho < \min(\rho_1, \rho_2)$, where ρ_i are standard radii of smooth arcs of the curve Γ meeting in C (for the definition and for the properties of such a radius see [106], p. 18), then the circumference with center in C and radius ρ intersects Γ in two points τ and t only, and therefore this neighbourhood does not contain the points from $\Gamma \backslash \tau C t$. Of course, we may assume that $|C-\tau_n| < \rho$, $|C-t_n| < \rho$ and draw arcs γ_n inside the standard circumference. Moreover, we also assume that the oriented tangents form at the points τ_n and t_n starting from n_0 an angle which is less than $\pi+\varepsilon$, say $\frac{3\pi}{2}$. Therefore one can construct the arcs γ_n in such a way that the tangent oscillation along them would not exceed $\frac{7\pi}{4}$.

Let $\Gamma_n = \widetilde{\Gamma}_n \cup \gamma_n$. Then Γ_n is a closed piecewise smooth Jordan curve with angular points for which the condition (1.11) is fulfilled. Denote by D_n a finite domain bounded by Γ_n. From the construction of the domains D_n it follows that they converge to the domain D as to the kernel (see [43], p. 56). Furthermore, we can easily verify that the mapping of Γ onto Γ_n given by the equality

$$M_n(z) = \begin{cases} z, & z \in \widetilde{\Gamma}_n, \\ \mu_n(e^{i\theta}), & z(e^{i\theta}) \in \gamma_n, \end{cases}$$

is one-to-one and continuous. Note that $\lim\limits_{n\to\infty} [M_n(z) - z] = 0$.

Thus all the assumptions of Rado's theorem are fulfilled (see, e.g., [43], p. 62–63), and if $z_n = z_n(w)$, $z_n(0) = z(0)$, $z_n'(0) = z'(0)$, then z_n converges uniformly on $\overline{U}$ to $z(w)$. Obviously,

$$\lim z_n'(w) = z'(w), w \in U. \tag{1.16}$$

Since Γ_n are piecewise smooth curves not containing angular points with the interior cusp, by virtue of the already proven we have that

$$z_n'(w) = z_n'(0) \exp\left(\frac{1}{2\pi} \int\limits_{\gamma} \frac{\beta_n(\theta) d\tau}{\tau - w}\right),$$

where $\beta_n(\theta) = \alpha_n(\theta) - \theta - \frac{\pi}{2}$, and α_n is the tangential function of the curve Γ_n. Because β_n is a sequence of uniformly bounded functions ($|\beta_n| < \frac{7\pi}{4}$) converging on $\gamma \backslash \{c\}$ to $\beta(\theta)$, then passing in the last equality to limit and taking into account (1.16), we arrive to (1.4) for the case under consideration as well. ■

Corollary 1. *If Γ is a smooth curve, then*

$$z', \frac{1}{z'} \in \bigcap_{p>0} H^p, \tag{1.17}$$

$$z'(\tau) \in \bigcap_{p>0} W_p(\gamma), \tag{1.18}$$

$$w', \frac{1}{w'} \in \bigcap_{p>0} E^p(D), \tag{1.19}$$

$$\ln z' \in \bigcap_{p>0} H^p. \tag{1.20}$$

Proof. Indeed, the assertions (1.17) and (1.18) follow from the properties of the factor function X for the continuous function G, since $z' = \widetilde{C}X^2$. The relation (1.19) is a consequence of the equality

$$\int\limits_{\Gamma} |w'(z)|^{\pm p} |dz| = \int\limits_{\gamma} |z'(w)|^{\pm p-1} |dw|$$

and of assertion (1.17). Finally, (1.20) follows from the equality

$$\ln \frac{z'(w)}{z'(0)} = K_\gamma(2i\beta)(w). \quad \blacksquare$$

Theorem 1.1 establishes the connection between z' and the Cauchy type integral of the function β which is defined by the geometry of the curve Γ, that is, by the angle of slope of its tangent to the real axis. Having known the character of the variation of the function β and proceeding from the equality (1.4), we can throw light on the properties of z'. Below we will adduce some results obtained in this way.

Theorem 1.2 (Lindelöf). *If a function $z = z(w)$, $z'(0) > 0$ maps conformally the circle U onto the domain D bounded by a closed smooth curve, and $w = w(z)$ is the inverse to it function, then* $\arg z'(w)$ *is a function continuous on $\overline{U}$, and* $\arg w'(z)$ *is continuous on $\overline{D}$. Moreover, for $\theta \in [0, 2\pi]$ we have*

$$\arg z'(e^{i\theta}) = \alpha(\theta) - \theta - \frac{\pi}{2} + 2m\pi, \tag{1.21}$$

where m is an integer.

Proof. In the case under consideration, z' is representable by the equality (1.4), where β is a continuous function. Suppose $w = re^{i\theta}$, $\Delta_r x = 1 - 2r\cos x + x^2$. Then

$$z'(w) = z'(0)\exp\left(\frac{1}{2\pi}\int\limits_{\otimes}^{2\pi} \frac{\beta(s) i e^{is}\, ds}{e^{is} - re^{i\theta}}\right) =$$

$$= z'(0)\exp\left\{\frac{1}{\pi}\int\limits_{0}^{2\pi} \beta(s)\Big[\frac{r\sin(s-\theta)}{\Delta_r(s-\theta)} + i\frac{1 - r\cos(s-\theta)}{\Delta_r(s-\theta)}\Big] ds\right\} =$$

$$= z'(0)\exp\left\{\frac{1}{\pi}\int\limits_{0}^{2\pi} \beta(s)\Big[\frac{r\sin(s-\theta)}{\Delta_r(s-\theta)} + i\frac{1-r}{\Delta_r(s-\theta)} + i\frac{r - r\cos(s-\theta)}{\Delta_r(s-\theta)}\Big] ds\right\} =$$

$$= z'(0)\exp\left(\frac{1}{\pi}\int\limits_{0}^{2\pi} \frac{r\sin(s-\theta)}{\Delta_r(s-\theta)}\beta(s)ds\right)\exp\left(\frac{i}{(1+r)\pi}\int\limits_{0}^{2\pi} \frac{1-r^2}{\Delta_r(s-\theta)}\beta(s)ds\right) \times$$

$$\times \exp\left(\frac{ir}{\pi}\int\limits_0^{2\pi}\frac{1-\cos(s-\theta)}{\Delta_r(s-\theta)}\beta(s)ds\right). \tag{1.22}$$

Thus in the representation of z' there appear the Poisson integral of a continuous function β, its conjugate function and also the function $k(r,x)=\frac{1-\cos x}{\Delta_r x}$, which is continuous on the rectangle $[0,1;-2\pi,2\pi]$. Taking into account the properties of these integrals, from (1.22) we obtain

$$\lim_{w\overset{\wedge}{\to}\exp i\theta} z'(w)=z'(0)\exp[\widetilde{\beta}(\theta)]\exp[i\beta(\theta)]\exp\left[\frac{i}{2\pi}\int\limits_0^{2\pi}\beta(s)ds\right], \tag{1.23}$$

for almost all $\theta\in[0,2\pi]$, where $\widetilde{\beta}$ is the conjugate to β function (see 0.10).

Prove that

$$\int\limits_0^{2\pi}\beta(s)ds=4k\pi^2, \tag{1.24}$$

where k is an integer. Indeed, from (1.14) we have

$$\widetilde{C}=\sqrt{z'(0)}\exp\left(\frac{i}{2\pi}\int\limits_0^{2\pi}\beta(s)ds\right),$$

which together with (1.15) yields

$$\sqrt{z'(0)}=\sqrt{z'(0)}\exp\left(\frac{i}{2\pi}\int\limits_0^{2\pi}\beta(s)ds\right),$$

whence it follows the validity of equality (1.24). The equality (1.23) with regard for (1.24) takes the form

$$\lim_{w\overset{\wedge}{\to}\exp i\theta} z'(w)=z'(0)[\exp\widetilde{\beta}(\theta)]\exp[i\beta(\theta)]. \tag{1.25}$$

By virtue of (1.4) and (1.22) we now have

$$\ln z'(w)=\ln z'(0)+\frac{1}{\pi}\int\limits_\gamma\frac{\beta(\theta)d\tau}{\tau-w}=\ln z'(0)+\frac{1}{2\pi}\int\limits_0^{2\pi}\frac{2r\sin(s-\theta)}{\Delta_r(s-\theta)}\beta(s)ds+$$

$$+\frac{i}{(1+r)\pi}\int\limits_0^{2\pi}\frac{1-r^2}{\Delta_r(s-\theta)}\beta(s)ds+\frac{ir}{\pi}\int\limits_0^{2\pi}\frac{1-\cos(s-\theta)}{\Delta_r(s-\theta)}\beta(s)ds+2\lambda\pi i,$$

where λ is an integer. The latter equality results in

$$\arg z'(w) = \frac{1}{(1+r)\pi}\int_0^{2\pi}\frac{1-r^2}{\Delta_r(s-\theta)}\beta(s)ds +$$

$$+\frac{r}{\pi}\int_0^{2\pi}\frac{1-\cos(s-\theta)}{\Delta_r(s-\theta)}\beta(s)ds + 2\lambda\pi. \tag{1.26}$$

The right-hand side of the above expression contains the sum of continuous in $\overline{U}$ functions. Bearing in mind that

$$\lim_{w \stackrel{\wedge}{\to} \exp i\theta} \frac{r}{\pi}\int_0^{2\pi}\frac{1-\cos(s-\theta)}{\Delta_r(s-\theta)}\beta(s)ds = \frac{1}{2\pi}\int_0^{2\pi}\beta(s)ds = 2k\pi,$$

from (1.26) we find that

$$\arg z'(e^{i\theta}) = \lim_{w \stackrel{\wedge}{\to} e^{i\theta}} \arg z'(w) = \beta(w) + 2(k+\lambda)\pi.$$

which is the very equality (1.21) with $m = k + \lambda$.

The continuity of the function $\arg w'(z)$ follows from the equality $w'(z) = \frac{1}{z'(w)}$. ■

Remark 1. It is obvious that the continuous function α in all the equalities can be replaced by the function $\alpha(\theta) + 2j\pi$ with an arbitrary integer j. Therefore, if we take $\alpha = \alpha_1 - 2\pi$, then the geometrical meaning of the function α_1 will remain the same as before, and the equality (1.21) will take the form

$$\arg z'(e^{i\theta}) = \alpha_1(\theta) - \theta - \frac{\pi}{2}.$$

Remark 2. From the equality (1.26) and from the property of the Poisson integral we obtain a more general assertion than Theorem 1.2, namely:

If Γ is a piecewise smooth curve, then $\arg z'(w)$ is continuously extendable at all different from c_k points of the circumference γ (while $\arg w'(z)$ is continuously extendable at all points of Γ different from angular points t_k), and equality (1.21) is valid everywhere except for the points c_k.

Theorem 1.3 (Kellog). *If the function $z = z(w)$ maps conformally the unit circle U onto a domain D bounded by a Lyapunov curve for which*

$$|\alpha(s_1) - \alpha(s_2)| \le M|s_1 - s_2|^\lambda, \quad 0 < \lambda < 1, \tag{1.27}$$

then the functions $z'(w)$ and $\ln z'(w)$ belong to the Hölder class $H(\lambda)$ in the closed circle $\overline{U}$.

Proof. We have

$$|\alpha(\theta_1)-\alpha(\theta_2)| = |\alpha(\theta(s_1))-\alpha(\theta(s_2))| \leq$$
$$\leq M|s_1-s_2|^\lambda = \Big(\int\limits_{\theta_1}^{\theta_2}|z'(e^{i\sigma})|d\sigma\Big)^\lambda. \qquad (1.27')$$

Since $z' \in \cap_{p>0}H^p$, this implies that on γ the function $\alpha(\tau)=\alpha(e^{i\theta})$ for any $p>1$ belongs to the Hölder class $H\left(\frac{\lambda}{p}\right)$. By virtue of (1.5) and (1.3), β belongs to the same class as well. But then $(\mathcal{K}_\Gamma\beta)(z)$ belongs to $H\left(\frac{\lambda}{p}\right)$ in $\overline{U}$ (see, e.g., [106], §21). From (1.27′) we can now conclude that α, and thereby β, belongs to $H(\lambda)$. As a result of the above reasoning, we obtain the both assertions of Theorem 1.3. ■

Remark. If $\lambda=1$ in (1.27), then $\beta \in H(1)$, and in addition to the above-said, from Privalov theorem on a singular integral with density from the class $H(1)$ we obtain

$$|z'(e^{i\theta_1})-z'(e^{i\theta_2})| \leq M|\theta_1-\theta_2|\ln\frac{1}{|\theta_1-\theta_2|}, \quad |\theta_1-\theta_2|<1.$$

Theorem 1.4. *Let Γ be a piecewise smooth, closed, oriented Jordan curve with angular points t_k, bounding the finite domain D, and let $\pi\nu_k$, $0\leq\nu_k\leq 2$, $k=\overline{1,n}$ be sizes of interior (with respect to D) angles at these points. Then, if the function $z=z(w)$ maps conformally the unit circle U onto D, and*

$$z'(0)>0, \quad z(c_k)=t_k, \quad c_k\in\gamma, \qquad (1.28)$$

then

$$z'(w)=\prod_{k=1}^n (w-c_k)^{\nu_k-1}\exp\Big(\frac{1}{\pi}\int\limits_\gamma\frac{\delta(\tau)d\tau}{\tau-w}\Big), \qquad (1.29)$$

where $\delta=\delta(t)$ is a continuous on γ function.

Proof. By Theorem 1.1 we have

$$z'(w)=z'(0)\exp\Big(\frac{1}{\pi}\int\limits_\gamma\frac{\beta(\theta)d\tau}{\tau-w}\Big), \quad \tau=e^{i\theta}, \quad \beta(\theta)=\alpha(\theta)-\theta-\frac{\pi}{2},$$

where α is a piecewise continuous on $[0,2\pi]$ function.

Let $c_k=\exp i\theta_k$, $0\leq\theta_1<\theta_2<\cdots<\theta_n<2\pi$ and let $\Gamma_{1,k}$ be an oriented subarc of γ with the ends 1 and c_k, and $\Gamma_{2,k}=\gamma\backslash\Gamma_{1,k}$. Consider piecewise linear functions

$$\beta_k(\theta)=\begin{cases}0, & e^{i\theta}\in\Gamma_{1,k} \quad (\text{i.e.,} \quad 0\leq\theta<\theta_k),\\ \frac{2\pi-\theta}{2\pi-\theta_k}h_k, & e^{i\theta}\in\Gamma_{2,k} \quad (\text{i.e.,} \quad \theta_k\leq\theta\leq 2\pi),\end{cases}$$

where $k = \overline{1,n}$, and

$$h_k = \beta(\theta_k+) - \beta(\theta_k-) = \alpha(\theta_k+) - \alpha(\theta_k-) = \pi - \pi\nu_k = \pi(1-\nu_k). \quad (1.30)$$

Suppose

$$\beta^*(\theta) = \sum_{k=1}^{n} \beta_k(\theta).$$

The function $\beta_0(\theta) = \beta(\theta) - \beta^*(\theta)$ is continuous on $[0, 2\pi]$ and $\beta_0(0) = \beta_0(2\pi)$, i.e., $\beta_0(\tau) \equiv \beta_0(\theta)$ is continuous on γ.

We now have

$$z'(w) = z'(0) \exp\left(\frac{1}{\pi}\int_\gamma \frac{\beta(\theta)d\tau}{\tau - w}\right) =$$

$$= z'(0) \exp\left(\frac{1}{\pi}\int_\gamma \frac{\beta_0(\theta)d\tau}{\tau - w}\right) \exp\left(\frac{1}{\pi}\int_\gamma \frac{\beta^*(\theta)d\tau}{\tau - w}\right). \quad (1.31)$$

To estimate the integral $\frac{1}{\pi}\int_\gamma \frac{\beta^*(\theta)d\tau}{\tau-w}$ in the neighbourhood of c_k, we will use the following result from [106] (§26):

Let nonintersecting smooth arcs L_k, $k = \overline{1,n}$ meet at the point c; the function φ belongs to the Hölder class on $\overline{L}_k$; and $L = \cup L_k$. Then

$$\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)d\tau}{\tau - w} = A\ln(w - c) + \phi_0(w),$$

where $\phi_0(w)$ is continuous in every closed sector into which L divides the neighbourhood of the point c, ϕ_0^+ belongs on L to the Hölder class, and

$$A = \sum_{k=1}^{n} \frac{\pm\varphi(c_k)}{2\pi i};$$

the plus sign is taken for the incoming in c arcs and the minus sing for the outgoing arcs, and $\varphi(c_k) = \lim_{t\to c_k, t\in L_k} \varphi(t)$.

In connection with this assertion, we have

$$\frac{1}{\pi}\int_\gamma \frac{\beta_k d\tau}{\tau - w} = \frac{1}{2\pi i}\int_\gamma \frac{2i\beta_k}{\tau - w}d\tau = \frac{1}{2\pi i}\int_{\Gamma_{1,k}} \frac{2i\beta_k d\tau}{\tau - w} + \frac{1}{2\pi i}\int_{\Gamma_{2,k}} \frac{2i\beta_k d\tau}{\tau - w} =$$

$$= -\frac{h_k}{\pi}\ln(w - c_k) + \phi_{ok} = (\nu_k - 1)\ln(w - c_k) + \phi_{ok}(w)$$

This and (1.31) imply that

$$z'(w) = \prod_{k=1}^{n} (w - c_k)^{\nu_k - 1} \exp\left(\frac{1}{\pi}\int_\gamma \frac{\beta_0(\theta)d\tau}{\tau - w}\right) \exp\sum_{k=1}^{n} \phi_{0,k}(w), \quad (1.32)$$

where $\phi_{0,k}$ are holomorphic in U and continuous in $\overline{U}$ functions, and $\phi_{0,k}(\tau)$ belongs on γ to the Hölder class. Let $\phi_0 = \sum_{k=1}^{n} \phi_{0,k}$. This function is representable in U by its Cauchy integral, that is, $\phi_0(w) = (\mathcal{K}_\gamma \phi_0^+)(w)$, where ϕ_0^+ is a Hölder continuous function. Now (1.32) yields

$$z'(w) = \prod_{k=1}^{n} (w - c_k)^{\nu_k - 1} \exp\Big(\frac{1}{\pi}\int_\gamma \frac{\delta(\tau)d\tau}{\tau - w}\Big),$$

where $\delta(\tau) = \beta_0(\tau) + \frac{1}{2i}\phi_0(\tau)$, and hence it is continuous on γ. ■

From Theorem 1.4 we get the following Warschawski

Theorem 1.5. *Let Γ be a piecewise Lyapunov, closed Jordan curve with angular points t_k, bounding the finite domain D, and let, $\nu_k\pi$, $0 < \nu_k \le 2$ be the interior angles at the point t_k. If $z = z(w)$, $z'(0) > 0$ is a function conformally mapping the circle U onto D, then*

$$z'(w) = \prod_{k=1}^{n} (w - c_k)^{\nu_k - 1} z_0(w), \tag{1.33}$$

where $z_0(w)$ is a Hölder class function on $\overline{U}$, different from zero.

Remark. It should be noted that in [164] was proved only the continuity of z_0. The Hölder continuity of this function was established by I.N. Vekua ([159], p. 38).

Proof of Theorem 1.5. We will proceed from the equality (1.32) from which it is seen that to prove the theorem, it suffices to state that the function $\beta_0(\theta) = \alpha(\theta) - \theta - \frac{\pi}{2} - \sum_{k=1}^{n} \beta_k(\theta)$ satisfies the Hölder condition.

Let $t = z(e^{i\theta})$ and $t = t(s)$, $0 \le s \le l$ be the equations of the curve Γ with respect to the arc abscissa. Then $\theta = \theta(s)$, and by assumption, $\beta(s) = \beta(\theta(s))$ is a piecewise Hölder function on $[0, l]$.

Since $\beta_0 \in C(\gamma)$, its Hölder continuity on γ will be proved if we show that β_0 satisfies the Hölder condition in the neighbourhood of the points c_k.

Suppose $c_k = \exp i\theta_k = t(s_k)$, $\sigma_1 < \theta_k < \sigma_2$, $z(e^{i\sigma_1}) = t(s^{(1)})$, $z(e^{i\sigma_2}) = t(s^{(2)})$, $|s^{(1)} - s^{(2)}| < \min_k |s_k - s_{k+1}|$, $s_{n+1} = s_1 + l$.

We have

$$\begin{aligned}|\beta_0(\sigma_1) - \beta_0(\sigma_2)| \le |\beta_0(\sigma_1) - \beta_0(\theta_k)| + |\beta_0(\theta_k) - \beta_0(\sigma_2)| = \\ = |\beta(t(s^{(1)})) - \beta(t(s_k))| + |\beta(t(s_k)) - \beta(t(s^{(2)}))| \le\end{aligned}$$

$$\le M\big(|s_k - s^{(1)}|^\lambda + |s_k - s^{(2)}|^\lambda\big) \le 2M|s^{(1)} - s^{(2)}|^\lambda = 2M\Big(\int_{\sigma_1}^{\sigma_2} |z'(w)||dw|\Big)^\lambda.$$

By Theorem 1.4 (equality (1.29)), we obtain

$$|\beta_0(\sigma_1) - \beta_0(\sigma_2)| \le$$

$$\leq 2M_1\Big(\int\limits_{\sigma_1}^{\sigma_2}|w-c_k|^{\nu_k-1}\Big|\exp\Big(\frac{1}{\pi}\int\limits_{\gamma}\frac{\delta(\tau)d\tau}{\tau-w}\Big)\Big||dw|\Big)^{\lambda}. \qquad (1.34)$$

Since δ is continuous, we have $\exp\left(\frac{1}{\pi}\int_\gamma\frac{\delta(\tau)d\tau}{\tau-w}\right)\in\bigcap\limits_{p>0}L^p(\gamma)$. Moreover, by assumption, $\nu_k>0$, and therefore $|w-c_k|^{(\nu_k-1)(1+\varepsilon)}$ is summable and hence

$$|\beta_0(\sigma_1)-\beta_0(\sigma_2)|\leq 2M_2|\sigma_1-\sigma_2|^{\frac{\varepsilon\lambda}{1+\varepsilon}}.$$

Consequently, β_0 and thus z_0, belong to the Holder class. ■

§ 2. On Derivatives of Conformal Mappings in Case of Oscillating Tangent of Boundaries

The method of the proof of Theorem 1.1 can also be applied to the domains with more complicated boundaries than piecewise smooth curves.

2.1. A class $T(\mu)$ of curves. Let $t=t(s)$, $0\leq s\leq l$ be the equation of a simple, rectifiable curve Γ, and let $\alpha(s)$ be the angle formed by a oriented tangent at the point $t(s)$ and the x-axis.

Definition. We say that Γ belongs to the class $T(\mu)$, $\mu\in(0,\pi]$ if for every point $t\in\Gamma$ there exists its arc neighbourhood on which the values of the function $e^{i\alpha(s)}$ lie in an angle of size μ and with the vertex at the origin.

Let $\Gamma\in T(\pi)$, and define specifically the tangential function $\alpha(\theta)\equiv\alpha(s(\theta))$. Here the use will be made of the rule of selection of an argument's branch of the function from the class $A(p)$ (see Chapter II, §1).

Since $\Gamma\in T(\pi)$, the curve Γ can be covered by a finite number of arcs $\Gamma_k=(t_{k,1},t_{k,2})=(t(s_{k,1});t(s_{k,2}))$, $k=\overline{1,n}$, $t_{k+1,1}\in\Gamma_k$ such that for $s\in(s_{k,1},s_{k,2})$ the values of $e^{i\alpha(s)}$ lie in an angle less than π with the vertex at the origin. Add to the points $t(s_k)$ the point $t_0=z(e^{i\theta_0})$ at which the equality (1.1) holds, and let

$$\alpha_0=\lim_{w\overset{\wedge}{\to}\exp i\theta_0}\arg z'(w)+\theta_0+\frac{\pi}{2}.$$

Without restriction of generality, we may assume that the curve Γ at the points $t_{k,1}$, $t_{k,2}$ has tangents. For the sake of definiteness, assume that t_0 lies on the arc Γ_1, and replace it by the arcs $(t_{1,1},t_0)$ and $(t_0,t_{1,2})$. Define on $(t_0,t_{1,2})$ a function $\alpha(t)=\alpha(t(s))=\alpha(s)$ such that $\alpha(s_0)=\alpha_0$ and $|\alpha(s)-\alpha_0|<\pi$ for all s for which $t(s)\in(t_0,t_{1,2})$. Thus we have defined the value $\alpha(s_{2,1})$. Next, on the arc Γ_2 we define the function $\alpha(s)$, keeping to the condition $|\alpha(s)-\alpha(s_{2,1})|<\pi$. Continuing this process, we define the function $\alpha(s)$ on $[0,l]$ (and hence the function $\alpha(s(\theta))$ on $[0,2\pi]$).

Consequently, we have defined the tangential function $\alpha(s)$ for the curves from $T(\pi)$.

2.2. A derivative of conformal mapping of a circle onto a domain with a boundary from $T(\frac{\pi}{\lambda})$, $\lambda > 1$. The following theorem is valid.

Theorem 2.1. *If a simply connected domain is bounded by a curve from the class $T(\frac{\pi}{\lambda})$, $\lambda > 1$, then $(z')^{\pm 1} \in H^\lambda$, and for any $p \in [\frac{1+\lambda}{\lambda}, +\infty]$ the function $[z'(e^{i\theta})]^{\pm p}$ belongs to $W_{p\lambda}(\gamma)$.*

Proof. Let

$$p \geq \frac{1+\lambda}{\lambda} \tag{2.1}$$

and

$$\Omega(w) = \begin{cases} \sqrt[p]{z'(w)}, & |w| < 1, \\ \overline{\sqrt[p]{z'(\frac{1}{\bar{w}})}}, & |w| > 1. \end{cases} \tag{2.2}$$

It can be easily verified that $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$ and

$$\phi^+(e^{i\theta}) = \exp\Big\{\frac{i2\beta(\theta)}{p}\Big\}\phi^-(e^{i\theta}), \tag{2.3}$$

where $\beta(\theta) = \alpha(\theta) - \theta - \frac{\pi}{2}$, and $\alpha(\theta) = \alpha(s(\theta))$ is the targential function defined in subsection 2.1.

Suppose

$$G(\tau) = G(e^{i\theta}) = \exp\Big\{\frac{i2\beta(\theta)}{p}\Big\}.$$

It follows from the condition $\Gamma \in T(\frac{\pi}{\lambda})$ and the fact that the function $G_1(\theta) = \exp[-i(\frac{\pi}{2} + \theta)]$ is continuous that every point $\tau \in \gamma$ possesses a neighbourhood on which the values of the function $G(\tau)$ lie in an angle of size $\frac{2\pi}{\lambda p}$. This means that $G \in A(p\lambda)$, and owing to (2.1), $p\lambda \geq 1 + \lambda \geq 2$. Calculate the index of the function G. First consider the function $\varphi(e^{i\theta}) = \exp i\beta(\theta) = \exp i\alpha(\theta)\exp(-i(\frac{\pi}{2}+\theta)) = \varphi_1(\theta)\varphi_2(\theta)$. Since angular degree of simple curve equal 1 ([100], p. 84), one can see that the index of the function $\varphi_1(e^{i\theta}) = \exp i\alpha(\theta)$ in $\widetilde{\mathcal{K}}^{p\lambda}(\gamma)$ is equal to 1. Obviously, $\operatorname{ind}\varphi_2 = -1$. But if $\varphi_1 \in A(p)$, and φ_2 is continuous, then $\operatorname{ind}(\varphi_1\varphi_2) = \operatorname{ind}\varphi_1 + \operatorname{ind}\varphi_2$. Therefore $\operatorname{ind}\varphi = \operatorname{ind}(\varphi_1\varphi_2) = 0$. Represent the function φ in terms of $\varphi = g_1 g_2$, where g_1 belongs to the Lipschitz class on γ,

$$\operatorname{ind} g_1 = 0, \quad g_2(\theta) = \exp i\mu(\theta), \quad |\mu(\theta)| < \frac{\pi}{2\lambda} \tag{2.4}$$

(see Chapter II, subsection 1.3). Since $G = \varphi^{2/p}$, this implies that

$$G = g_1^{\frac{2}{p}} g_2^{\frac{2}{p}} = g_1^{\frac{2}{p}} \exp\Big\{\frac{i2\mu}{p}\Big\}.$$

Denote $G_1 = g_1^{2/p}$, $G_2 = \exp\Big\{\frac{i2\mu}{p}\Big\} = \exp i\nu$. Then $\operatorname{ind} G = 0$, $\operatorname{ind} G_1 = 0$, $|\nu| < \frac{\pi}{\lambda p}$.

Let

$$X_1(w)=\exp\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\ln G_1 d\tau}{\tau-w}\right),\quad X_2(w)=\exp\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{i\nu d\tau}{\tau-w}\right).$$

Then $X(w)=X_1(w)X_2(w)$ is the factor function of G in the class $\widetilde{\mathcal{K}}^{p\lambda}(\gamma)$. Consequently, the function $\phi = CX$, where C is an arbitrary constant, is a solution of the problem (2.3). Since Ω is a solution of the problem (2.3) of the class $\widetilde{\mathcal{K}}^{p\lambda}(\gamma)$, we have $\Omega = C_0 X$, where $C_0 \neq 0$ is a constant. But for $|w|<1$ we have $\Omega = \sqrt[p]{z'(w)}$, and thus $z'(w) = C_0^p X^p(w)$, where X is the factor function of the function $G \in A(p\lambda)$ in the class $\mathcal{K}^{p\lambda}(\gamma)$. Since $p\lambda \geq 2$, both assertions of the theorem follow from the properties of the factor function. ■

2.3. Curves of the class $\widetilde{T}(\frac{\pi}{\lambda})$ and properties of conformal mapping of a circle U onto a domain with a boundary from $\widetilde{T}(\frac{\pi}{\lambda})$. As is seen from subsection 2.2, the class $T(\mu)$ has been introduced in order for the function Ω constructed according to the equality (2.2) by means of $z'(w)$ to be a solution of the class $\widetilde{\mathcal{K}}^{p\lambda}(\gamma)$ of the problem (2.3) whose coefficient belongs to the class $A(p\lambda)$. Bearing these arguments in mind and proceeding from the classes $\widetilde{A}(p)$, one can extend the class $T\left(\frac{\pi}{\lambda}\right)$ so that the results analogous to those of Theorems 1.1 and 2.1 will remain valid for domains bounded by curves from the extended class.

Definition. The curve Γ belongs to the class $\widetilde{T}(\mu)$, $\mu \in (0,\pi]$ if for every point $t \in \Gamma$, except possibly for the points $t_1, t_2, \dots, t_n$, there exists an arc neighbourhood on which the values of the function $t'(s) = \exp i\alpha(s)$ lie in an angle of size less than μ with the vertex at the origin, and in the points t_k there exist limits $t'(s_k\pm)$).

Let $\Gamma \in \widetilde{T}(\pi)$. Exactly as in subsection 2.1, following the rule for definition of an argument branch of the function from the class $\widetilde{A}(p)$, define on that curve a tangential function $\alpha(\theta)$. Then the function $G(e^{i\theta}) = \exp i\alpha(s(\theta))$ satisfies the following condition: every point from γ, except possibly the points c_j corresponding to the cusps, possesses a neighbourhood on which the values $G(e^{i\theta})$ lie in an angle less than π, while the function $\alpha(\theta)(\equiv \alpha(s(\theta))$ at the points $c_j = \exp i\theta_j$ has one-sided limits; moreover, $\alpha(\theta_j+) - \alpha(\theta_j-) = (\nu_j - 1)\pi$, $\nu_j \overline{\in} \{0, 2\}$.

Using the results of subsection 3.7 from Chapter II and arguing as when deducing Theorems 1.1 and 2.1, we conclude that the following theorem is valid.

Theorem 2.2. *If D is a simply connected domain bounded by a curve* $\Gamma \in$

$\widetilde{T}\left(\frac{\pi}{\lambda}\right)$, $\lambda > 1$, *then*

$$z'(w) = \prod_{k=1}^{n}(w - c_k)^{\nu_k - 1} z_0(w), \tag{2.5}$$

where z_0, $\frac{1}{z_0} \in H^\lambda$, *and for every* $p \in [\frac{1+\lambda}{\lambda}, +\infty)$, *the function* $[z_0(e^{i\theta})]^{\pm p} \in W_{p\lambda}(\gamma)$.

Corollary 1. *If* D *is a simply connected domain with the boundary from* $\widetilde{T}(\pi)$, *then for almost all* $\theta \in [0, 2\pi]$ *there exists an angular limit of the function* $\arg z'(w)$ *for* $w \to \exp i\theta$, *and the equality*

$$\lim_{w \widehat{\to} \exp i\theta} \arg z'(w) = \beta(\theta) = \alpha(\theta) - \theta - \frac{\pi}{2} + 2m\pi \tag{2.6}$$

holds, where $\alpha(\theta)$ *is the tangential function of the curve* Γ, *and* m *is an integer.*

Corollary 2. *If* Γ *is a curve with bounded rotation and* $\{c_k\} = \{z(t(s_k))\}$, *where* $\{s_k\}$ *is a set of points of discontinuity of* $t'(s)$, *then*

$$z'(w) = \prod_{k=1}^{\infty}(w - c_k)^{\nu_k - 1} z_0(w), \tag{2.7}$$

where

$$z_0^{\pm 1} \in \cap_{p>0} H^p. \tag{2.8}$$

In particular, $z' \in H^\delta$ *for any* $\delta < \inf_{\nu_k < 1}\left\{\frac{1}{|\nu_k - 1|}\right\}$ *and* $\frac{1}{z'} \in H^\mu$ *for any* $\mu < \inf\limits_{\nu_k > 1}\left\{\frac{1}{\nu_k - 1}\right\}$.

§ 3. Behaviour of Conformal Mapping in the Neighbourhood of Angular Points

The results obtained in allow one to describe for the domain under consideration the behaviour of the functions $z = z(w)$ and $z'(w)$ in the neighbourhood of the points c_k, of the inverse function $w = w(z)$ as well as of the function $w = w'(z)$ in the neighbourhood of the points t_k.

Lemma 3.1. *Let a simply connected domain* D *be bounded by a closed, piecewise smooth curve* Γ *with angular points* $t_k = z(c_k)$, *being sizes of interior angles* $\nu_k \pi$, $0 \le \nu_k \le 2$. *Then the function*

$$z_k(w) = [z(w) - z(c_k)](w - c_k)^{-\nu_k}$$

belongs to H^{p_0} *for some* $p_0 > 1$.

Proof. Consider the equality

$$z_k(w) = \frac{z(w) - z(c_k)}{(w - c_k)^{\nu_k}} = \frac{1}{(w - c_k)^{\nu_k}} \int\limits_{\gamma_{wc_k}} z'(\zeta) d\zeta, \tag{3.1}$$

where $\gamma_{wc_k} = \gamma_{ww_k} \cup \gamma_{w_k c_k}$, γ_{ww_k} is assumed to be the small arc of the circumference with center at the origin, passing through the points w and $w_k = |w| \exp i\theta_k$, $\theta_k = \arg c_k$ while $\gamma_{w_k c_k}$ is a rectilinear segment connecting the points w_k and c_k. Therefore

$$\int\limits_{\gamma_{wc_k}} |z'(\zeta)||d\zeta| \leq \int\limits_{\gamma_{ww_k}} |z'(\zeta)||d\zeta| + \int\limits_{\gamma_{w_k c_k}} |z'(\zeta)||d\zeta|. \tag{3.2}$$

Let now $\nu_k < 1$. Then it is clear from (3.1) that

$$|z_k(w)| \leq \frac{M}{|w - c_k|^{\nu_k}} \tag{3.3}$$

and therefore $z' \in H^{p_0}$, $p_0 \in (1, \frac{1}{\nu_k})$. However, if $\nu_k \in [1, 2]$, then by Theorem 1.4 we have $z'(w) = (w - c_k)^{\nu_k - 1} \widetilde{z}_0(w)$, where $\widetilde{z}_0(w) = \prod\limits_{j \neq k} (w - c_j)^{\nu_j - 1} z_0(w)$, $z_0 \in \bigcap\limits_{p>0} H^p$. Put $m = \min\limits_{1 \leq i,j \leq n, i \neq j} |c_i - c_j|$ and $|w - c_k| < \frac{1}{2} m$.

If we denote by $d(E, F)$ the distance between the sets E and F, then we obtain $d(\gamma_{wc_k}, \{c_j\}_{j \neq k}) \geq \frac{1}{2} m$. Moreover, when $\zeta \in \gamma_{wc_k}$, then $|\zeta - c_k| < |w - c_k|$.

Now for an arbitrary $\alpha > 1$ we have

$$\begin{aligned}
\int\limits_{\gamma_{w_k c_k}} |z'(\zeta)||d\zeta| &\leq \int\limits_{\gamma_{w_k c_k}} |\zeta - c_k|^{\nu_k - 1} \prod_{\nu_j < 1} |\zeta - c_j|^{\nu_j - 1} \prod_{\nu_j \geq 1} |\zeta - c_j|^{\nu_j - 1} |z_0(\zeta)||d\zeta| \leq \\
&\leq |w - c_k|^{\nu_k - 1} \left(\frac{m}{2}\right)^{\sum_{\nu_j < 1} (\nu_j - 1)} 2^{\sum_{\nu_j \geq 1} (\nu_j - 1)} \int\limits_{\gamma_{w_k c_k}} |z_0(\zeta)||d\zeta| \leq \\
&\leq |w - c_k|^{\nu_k - 1} M_1 \Bigg(\int\limits_{\gamma_{w_k c_k}} |z_0(\zeta)|^\alpha |d\zeta| \Bigg)^{\frac{1}{\alpha}} |w - c_k|^{\frac{\alpha - 1}{\alpha}} \leq \\
&\leq M_1 |w - c_k|^{\nu_k - \frac{1}{\alpha}} \Bigg(\int\limits_{-1}^{1} |z_0(x e^{i\theta_k})|^\alpha dx \Bigg)^{\frac{1}{\alpha}}.
\end{aligned} \tag{3.4}$$

But according to the Fejer-Riesz theorem (see [27], p. 46),

$$\int\limits_{-1}^{1} |z_0(x e^{i\theta_k})|^\alpha dx \leq \frac{1}{2} \int\limits_0^{2\pi} |z_0(e^{i\theta})|^\alpha d\theta = \frac{1}{4\pi} \int\limits_\gamma |z_0(e^{i\theta})||d\zeta|,$$

and therefore (3.4) yields

$$\int\limits_{\gamma_{w_k c_k}} |z'(\zeta)||d\zeta| \le M|w-c_k|^{\nu_k-\frac{1}{\alpha}}\Big(\int\limits_{\gamma} |z_0(\zeta)|^{\alpha}|d\zeta|\Big)^{1/\alpha}. \tag{3.5}$$

Further,

$$\int\limits_{\gamma_{ww_k}} |z'(\zeta)||d\zeta| \le$$

$$\le |w-c_k|^{\nu_k-1}\Big(\frac{1}{2}m\Big)^{\sum_{\nu_j<1}(\nu_j-1)} 2^{\sum_{\nu_j\ge 1}(\nu_j-1)} \int\limits_{\gamma_{ww_k}} |z_0(\zeta)||d\zeta| \le$$

$$\le M_1|w-c_k|^{\nu_k-1}\Big(\int\limits_{\gamma_{ww_k}} |z_0(\zeta)|^{\alpha}|d\zeta|\Big)^{\frac{1}{\alpha}}(l_{ww_k})^{\frac{\alpha-1}{\alpha}}. \tag{3.6}$$

But we have that $|c_k-w|>|w_k-w|\ge \frac{2}{\pi}|\gamma_{ww_k}|$, and the inequality (3.5) results in

$$\int\limits_{\gamma_{ww_k}} |z'(\zeta)||d\zeta| \le M_1|w-c_k|^{\nu_k-\frac{1}{\alpha}}\Big(\frac{\pi}{2}\Big)^{\frac{\alpha-1}{\alpha}}\Big(\int\limits_{|\zeta|=w} |z_0(\zeta)|^{\alpha}|d\zeta|\Big)^{1/\alpha} \le$$

$$\le M_2|w-c_k|^{\nu_k-\frac{1}{\alpha}}\Big(\int\limits_{\gamma} |z_0(\zeta)|^{\alpha}|d\zeta|\Big)^{1/\alpha}. \tag{3.7}$$

By virtue of (3.5) and (3.7), from (3.2) we obtain

$$\int\limits_{\gamma_{wc_k}} |z'(\zeta)||d\zeta| \le 2M|w-c_k|^{\nu_k-\frac{1}{\alpha}}\Big(\int\limits_{\gamma} |z_0(\zeta)|^{\alpha}|d\zeta|\Big)^{1/\alpha}. \tag{3.8}$$

Having the inequality (3.8) at hand, we can easily conclude from (3.1) that

$$|z_k(w)| \le \frac{2M}{|w-c_k|^{1/\alpha}}\Big(\int\limits_{\alpha} |z_0(\zeta)|^{\alpha}|d\zeta|\Big)^{1/\alpha}. \tag{3.9}$$

This implies that $z_k \in H^{p_0}$ for every $p_0 \in (1,\alpha)$. ■

Remark 1. Proceeding from the fact that α is any number greater than 1, we have proved much more than it has been stated in the lemma, namely, that $z' \in \bigcap\limits_{1<p<p_0} H^p$, $p_0 = \min\limits_{\nu_k<1}\{\frac{1}{\nu_k}\}$.

Remark 2. As is seen from the proof, the statement of Lemma 3.1 remains valid for domains which are bounded by the curves from $\widetilde{T}(\frac{\pi}{\lambda})$, $\lambda > 1$; note that as p_0 one can take any number from the interval $(1, \lambda)$ (thus in the inequality (3.4) one can take $\alpha = \lambda$).

Theorem 3.1. *Let the domain D be bounded by a simple, piecewise smooth curve Γ with angular points t_k, $k = \overline{1,n}$ and with sizes of interior with respect to D angles $\pi\nu_k$. If $z = z(w)$ is the function mapping conformally the unit circle U onto D, $w = w(z)$ is an inverse to it function, and $z(c_k) = t_k$, then*

$$z(w) = z(c_k) + (w - c_k)^{\nu_k} z_k(w), \quad \nu_k \in [0,2], \tag{3.10}$$

$$w(z) = w(t_k) + (z - t_k)^{1/\nu_k} w_k(z), \quad \nu_k \in (0,2], \tag{3.11}$$

$$z'(w) = \prod_{k=1}^{n} (w - c_k)^{\nu_k - 1} z_0(w), \quad \nu_k \in [0,2], \tag{3.12}$$

$$w'(z) = \prod_{k=1}^{n} (z - t_k)^{\frac{1}{\nu_k} - 1} w_0(z), \quad \nu_k \in (0,2], \tag{3.13}$$

where the functions w_k, $z_k(w)$ and $z_k(e^{i\theta})$, $k = \overline{1,n}$, satisfy the conditions

$$(z_k)^{\pm 1} \in \cap_{p>1} H^p, \quad (w_k)^{\pm 1} \in \cap_{p>1} E^p(D), \quad [z_k(e^{i\theta})]^{\pm 1} \in \bigcap_{p>1} W_p(\gamma). \tag{3.14}$$

Proof. First of all it should be noted that the assertion (3.12) follows immediately from Theorem 1.4 if we take into account the properties of the function $\exp\left(\frac{1}{\pi}\int_\gamma \frac{\beta d\tau}{\tau - w}\right)$, when β is a continuous function.

Alongside with the function $z_k(w)$ we consider the function $\overline{z_k(\frac{1}{\overline{w}})}$ $|w| > 1$ and put $G_k(\tau) = z_k(\tau)/\overline{z_k(1/\overline{\tau})}$, $|\tau| = 1$. It can be easily verified that $G_k(\tau)$ is different from zero, continuous everywhere on γ function. Really,

$$G_k(\tau) = \frac{z(\tau) - z(c_k)}{\overline{z(\tau) - z(c_k)}} \frac{\overline{(\tau - c_k)^{\nu_k}}}{(\tau - c_k)^{\nu_k}} = G_{1k}(\tau) G_{2k}(\tau),$$

and since $|G_k(\tau)| = 1$, it suffices to show that $\alpha_\tau(\tau) = \arg G_k(\tau)$ is continuous at the point c_k. Indeed the argument of the function $G_{1k}(\tau)$ at the point c_k has a jump equal to $2\pi\nu_k$, while the function $G_{2k}(\tau)$ has the jump equal to $(-2\pi\nu_k)$. Hence $\alpha_k(\tau)$ is continuous.

The above-mentioned jumps are simultaneously equal to the increments of the corresponding functions as the result of a circuit of the point τ around γ. Therefore the increment of the function $\arg G_k(\tau)$ equals zero, i.e., $\varkappa(G_k) = [\arg G_k]_\gamma = 0$.

Consider the boundary value problem of linear conjugation

$$\phi^+(\tau) = G_k(\tau)\phi^-(\tau). \tag{3.15}$$

As we know, the problem (3.15) for continuous G_k, $\varkappa(G_k)=0$, is solvable in any class $\widetilde{\mathcal{K}}^p(\gamma)$, $p>1$, and all its solutions (the same for any p) are given by the formula

$$\phi_k(w) = CX_k(w), \tag{3.16}$$

where C is an arbitrary constant and X_k satisfies the conditions

$$(X_k)^{\pm 1} \in \cap_{p>1}\widetilde{\mathcal{K}}^p(\gamma) \quad \text{and} \quad [X_k^+(e^{i\theta})]^{\pm 1} \in \cap_{p>1}W_p(\gamma). \tag{3.17}$$

Let us now show that the function z_k for some $p_0 \geq 1$ is the restriction on U of the solution of the problem (3.15) of the class $\widetilde{\mathcal{K}}^{p_0}(\gamma)$. Such a solution is presented by the function

$$\Omega_k(w) = \begin{cases} z_k(w), & |w|<1, \\ \overline{z_k(1/\overline{w})}, & |w|>1. \end{cases} \tag{3.18}$$

By lemma 3.1 we have $z_k \in H^{p_0}$, $p_0>1$. Owing to this fact, we state that $\Omega_k \in \widetilde{\mathcal{K}}^{p_0}(\gamma)$ (see Chapter II, §7). Consequently, z_k is the required solution of the problem (3.15). But then according to (3.16), there exists a constant $C_k \neq 0$ such that $\Omega_k(w) = C_kX_k(w)$. Therefore the functions $(z_k)^{\pm 1}$ coincide in U with $X_k(w)$. However, by (3.17), $X_k \in \widetilde{K}^p(\gamma)$ $(\forall p>1)$ and as is known (Chapter I, §3), the restrictions of such functions on U belong to the class H^p. This implies that $(z_k)^{\pm 1} \in \bigcap\limits_{p>1} H^p$ and $[z_k(e^{i\theta})]^{\pm 1} \in \bigcap\limits_{p>1} W_p(\gamma)$, taking into account (3.17).

Consider now the function w_k. Substituting the values $w=w(\tau)$ and $c_k = w(t_k)$ into the equality $z(w)-z_k(\tau) = (w-c_k)^{\nu_k}z_0(w)$, we obtain

$$z - t_k = (w(z)-w(t_k))^{\nu_k} z_0(w(z)).$$

Hence

$$\begin{aligned} w(z)-w(t_k) &= (z-t_k)^{\frac{1}{\nu_k}}[z_0(w(z))]^{-\frac{1}{\nu_k}} = \\ &= (z-t_k)^{\frac{1}{\nu_k}} w_k(z), \quad w_k(z) = [z_0(w(z))]^{-\frac{1}{\nu_k}}. \end{aligned} \tag{3.19}$$

Show that $(w_k)^{\pm 1} \in \bigcap\limits_{p>1} E^p(D)$. Indeed, for any $p>0$, $\varepsilon>0$ we have

$$\begin{aligned} \int\limits_{\Gamma_r} |w_k(z)|^{\pm p}|dz| &= \int\limits_{|w|=r} \frac{1}{|z_0(w)|^{\pm\frac{p}{\nu_k}}}|z'(w)||dw| \leq \\ &\leq \Bigg(\int\limits_{|w|=r} \frac{|dw|}{|z_0(w)|^{\pm\frac{p}{\nu_k}\frac{1+\varepsilon}{\varepsilon}}}\Bigg)^{\frac{\varepsilon}{1+\varepsilon}} \Bigg(\int\limits_{|w|=r}|z'(w)|^{1+\varepsilon}|dw|\Bigg)^{\frac{1}{1+\varepsilon}}. \end{aligned} \tag{3.20}$$

By Theorem 1.4 and condition $\nu_k>0$, we can choose $\varepsilon>0$ so small that the right-hand side of inequality (3.20) becomes finite. Thus $(w_k)^{\pm 1} \in E^p(D)$ for any $p>0$. Consequently, $(w_k)^{\pm 1} \in \bigcap\limits_{p>1} E^p(D)$.

There remains to prove the relation (3.13) and the inclusion $(w_0)^{\pm 1} \in \bigcap\limits_{p>0} E^p(D)$.

We have

$$w'(z) = \frac{1}{z'(w(z))} = \frac{1}{\prod_{k=1}^{n}(w(z)-w(t_k))^{\nu_k - 1} z_0(w(z))} =$$
$$= \prod_{k=1}^{n}(w(z) - w(t_k))^{-\nu_k+1}\widetilde{w}_1(z), \quad \widetilde{w}_1(z) = \frac{1}{z_0(w(z))}. \tag{3.21}$$

Substituting in it the values of the difference $w(z) - w(t_k)$ from the equality (3.19), we arrive at

$$w'(z) = \prod_{k=1}^{n}(z - t_k)^{\frac{1}{\nu_k}-1} w_k^{1-\nu_k}(z)\widetilde{w}_1(z), \quad w_k(z) = [z_0(w(z))]^{-\frac{1}{\nu_k}}. \tag{3.22}$$

Therefore, $w_0(z) = [z_0(w(z))]^{\sum_{k=1}^{n}\frac{1-\nu_k}{\nu_k}-1}$ and $w_0^{\pm 1} \in \bigcap\limits_{p>1} E^p(D)$. ■

Corollary. *If* Γ *is a piecewise Lyapunov curve with angular points* t_k *by conditions* $0 < \nu_k \leq 2$ $(k = 1, 2, \ldots, n)$, *then the functions* z_k, w_k, $k = \overline{0, n}$, *in the representations* (3.10)–(3.13) *are Hölder class functions different from zero.*

This follows from the fact that the function G_k (the coefficient of the problem (3.15)) in the case under consideration belongs to the Hölder class.

Notes and Comments to Chapter III

The assertions that the derivative of a function mapping conformally the unit circle onto a simply connected domain satisfy the inclusions $z' \in \bigcap\limits_{p>0} H^p$ and $\ln z' \in H^1$, which are the particular cases of Corollary 1 of Theorem 1.1, are well known (see [43], pp. 410–411).

Formulation of Lindelöf's theorem (Theorem 1.2) is taken from [43], p. 409. For more general Lindelöf's result see [89], [167], [86].

The proof of Theorem 1.4, different from that cited in the text, can be found in [65].

Warschawski's theorem (Theorem 1.5) is given in the form as it is in [159].

Classes of the curves $T(\mu)$ and $\widetilde{T}(\mu)$ have been introduced in [117]. The proofs of Theorems 2.1 and 2.2 (without corollaries) based on the generalizations of Lindelöf's theorem for domains with a boundary from the class $T(\pi)$ and also the particular cases of Theorem 1.4 are given therein.

Corollary 2 of Theorem 2.2 is due to Warschawski [166].

CHAPTER IV
BOUNDARY VALUE PROBLEMS IN THE DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES

The Riemann-Hilbert problem will be investigated on the basis of the results obtained in Chapter II, §7 and in Chapter III. We begin with the consideration of particular cases of the problem, that is, with the Dirichlet and Neumann problems for harmonic functions.

§ 1. The Dirichlet Problem in $e^p(D)$ in Domains with a Piecewise Smooth Boundary

1.1. Statement of the problem and its reduction to the problem of linear conjugation. Let D be a simply connected domain bounded by a simple, piecewise smooth curve Γ. Denote by $e^p(D)$ the set of harmonic functions presenting a real part of functions from the class $E^p(D)$, i.e.,

$$e^p(D) = \{u : u = \operatorname{Re}\Phi(z), \quad \Phi \in E^p(D)\}. \tag{1.1}$$

The functions of this class possess almost everywhere on Γ angular boundary values forming a function of the class $L^p(\Gamma)$.

Consider the following Dirichlet problem:

Find a harmonic in D function $u(t)$, $p > 1$ from the class $e^p(D)$ whose angular boundary values coincide almost everywhere on the boundary Γ of the domain D with the given on it real function f from the class $L^p(\Gamma)$.

Thus we have to determine the function u for which

$$\left.\begin{array}{c} \Delta u = 0, \quad u \in e^p(D), \quad p > 1, \\ u(t) = f(t), \quad t \in \Gamma, \quad f \in L^p(\Gamma). \end{array}\right\} \tag{1.2}$$

Let $u = \operatorname{Re}\Phi$, and assume

$$\Psi(w) = \sqrt[p]{z'(w)}\Phi(z(w)), \quad |w| < 1, \tag{1.3}$$

where $z = z(w)$ is a function mapping conformally the unit circle U onto D. Following [102], [105] suppose

$$\Omega(w) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi(\frac{1}{\overline{w}})}, & |w| > 1, \end{cases} \tag{1.4}$$

$$\Omega_*(w) = \overline{\Omega}\Big(\frac{1}{\overline{w}}\Big), \quad |w| \neq 1. \tag{1.5}$$

The problem (1.2) is equivalent to the problem in the following statement (see Chapter II, §7)

$$\left.\begin{array}{c} \Omega^+(\tau) = -\dfrac{\sqrt[p]{z'(\tau)}}{\overline{\sqrt[p]{z'(\tau)}}}\Omega^-(\tau) + g(\tau), \; g(\tau) = 2f(z(\tau))\sqrt[p]{z'(\tau)}, \\ \Omega \in \widetilde{\mathcal{K}}^p(\gamma), \quad \Omega(w) = \Omega_*(w) \end{array}\right\} \tag{1.6}$$

in the sense that any solution u $(= \operatorname{Re}\phi)$ of (1.2) generates by means of equalities (1.3)–(1.4) the function Ω which satisfies the conditions (1.6), and vice versa, if Ω satisfies (1.6), and Ψ is its restriction on U, then

$$u = \operatorname{Re}\Big[\frac{\Psi(w)}{\sqrt[p]{z'(w)}}\Big]$$

is a solution of (1.2).

The problem

$$\left.\begin{aligned} \Delta u = 0, \quad u \in e^p(D), \quad p > 1, \\ u(t) = 0, \quad t \in \Gamma \end{aligned}\right\} \tag{1.2_0}$$

will be called the homogeneous problem corresponding to (1.2), or simply, the homogeneous problem.

We quote here a lemma which will be used below.

Lemma 1.1. *If $X \in H^p$, $g \in L(\Gamma)$ then the function $X\mathcal{K}_\gamma g$ belongs to the Hardy class H^δ for some $\delta > 0$.*

To prove this, it suffices to notice that the Cauchy type integral $K_\gamma g$ belongs to $\cap_{\delta<1} H^\delta$ ([133], p. 116).

1.2. The solution of problem (1.2) in case of one angular point. We assume here that Γ contains only one angular point C with the angle size $\nu\pi$, $0 \le \nu \le 2$ and consider separately the cases (i) $0 < \nu < p$; (ii) $p < \nu$; (iii) $\nu = p$; (iv) $\nu = 0$.

(i) $0 < \nu < p$. Let

$$X(w) = \begin{cases} -\sqrt[p]{z'(w)}, & |w| < 1, \\ \overline{\sqrt[p]{z'(\frac{1}{\overline{w}})}}, & |w| > 1. \end{cases} \tag{1.7}$$

Since $z' \in H^1$, then $X \in H^p$ in U. Moreover, by Theorem 1.4 of Chapter III, we have

$$\begin{gathered} X^+(\tau) = (\tau - c)^{\frac{\nu-1}{p}} z_0^{\frac{1}{p}}(\tau), \quad c = w(C), \\ z_0^{\pm 1} \in \cap_{\delta>1} H^\delta, \quad (z_0^+)^{\pm 1} \in \cap_{\delta>1} W_\delta. \end{gathered} \tag{1.8}$$

In the case under consideration, $\frac{\nu-1}{p} \in (-\frac{1}{p}, \frac{1}{p'})$. Therefore, using Lemma 7.1 of Chapter II, we can easily conclude that $X^{-1} \in \widetilde{\mathcal{K}}^{p'}(\gamma)$. Consequently,

$$\Omega_0(w) = \alpha X(w),$$

where α is an arbitrary complex constant, is the general solution of the homogeneous problem given by (1.6). Since

$$(\alpha X)_* = \begin{cases} \overline{\alpha}\sqrt[p]{z'(w)}, & |w| < 1, \\ -\overline{\alpha}\overline{\sqrt{z'(\frac{1}{\overline{w}})}}, & |w| > 1, \end{cases}$$

the condition $\Omega_0 = (\Omega_0)_*$ yields $\overline{\alpha} = -\alpha$. Thus $\operatorname{Re}\alpha = 0$, and hence $u_0 = \operatorname{Re}\left[\frac{\alpha\Omega_0(w)}{\sqrt[p]{z'(w)}}\right] = -\operatorname{Re}\alpha = 0$.

Further, since $\frac{\Omega}{X} \in \widetilde{\mathcal{K}}^1(\gamma)$, $\frac{g}{X^+} \in L(\gamma)$ and

$$\left(\frac{\Omega}{X}\right)^+ = \left(\frac{\Omega}{X}\right)^- + \frac{g}{X^+},$$

the function

$$\Omega(w) = \frac{X(w)}{2\pi i}\int\limits_{\gamma} \frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - w} \tag{1.9}$$

is the only one possible solution of the problem (1.6). By Lemma 1.1, $\Omega \in H^\delta$ for some $\delta > 0$. On the other hand, if in the representation (1.8)

$$\rho_1(\tau) = (\tau - c)^{\frac{\nu-1}{p}}, \quad \rho_2(\tau) = z_0^{\frac{1}{p}}(\tau),$$

then $\rho_1 \in W_{p+\varepsilon}(\gamma)$, $\rho_2 \in \bigcup\limits_{\delta>1} W_\delta(\gamma)$ and applying theorem from (0.20) to the product $\rho_1\rho_2$, we obtain $X^+ \in W_p(\gamma)$. This, by virtue of Sokhotskiĭ-Plemelj's formulas applied to Ω, implies that $\Omega^+ \in L^p(\gamma)$ from which by means of Smirnov's theorem we conclude that $\Omega \in H^p$. But then $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$, by Lemma 7.1 of Chapter II.

Thus the problem of linear conjugation (1.6) has a solution given by (1.9). Consequently,

$$\Omega(w) = \frac{1}{2}\left[\frac{X(w)}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)d\zeta}{X^+(\zeta)(\zeta - w)} + \overline{\left(\frac{X(\frac{1}{\overline{w}})}{2\pi i}\right)}\int\limits_{\gamma} \overline{\frac{g(\zeta)d\zeta}{X^+(\zeta)(\zeta - \frac{1}{\overline{w}})}}\right]$$

is a solution satisfying $\Omega(w) = \Omega_*(w)$. Since $X(w(z)) = -\frac{1}{\sqrt[p]{w'(z)}}$, $X(\frac{1}{\overline{w(z)}}) = \frac{1}{\sqrt[p]{w'(z)}}$ it follows from (1.3) and the above equality that

$$u(z) = \operatorname{Re}\left[\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(z(\zeta))d\zeta}{\zeta - w(z)} + \frac{w(z)}{2\pi i}\int\limits_{\gamma} \frac{\overline{f(z(\zeta))}}{\zeta}\frac{d\zeta}{\zeta - w(z)}\right].$$

Taking into account that f is a real function, we finally obtain

$$u(z) = \operatorname{Re}\left[\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(z(\zeta))}{\zeta}\frac{\zeta + w(z)}{\zeta - w(z)}d\zeta\right]. \tag{1.10}$$

(ii) $p < \nu$. Suppose

$$X_1(w) = X(w)(w - c)^{-1}, \tag{1.7$_1$}$$

where X is a function given by (1.7). Using the representation (1.29) of Chapter III, we conclude that

$$X_1(w) = O\Big((w-c)^{\frac{\nu-p-1}{p}} z_1(w)\Big), \quad z_1(w) = z_0^{\frac{1}{p}}(w), \tag{1.11}$$

in the neighbourhood of the point c. Since $-\frac{1}{p} < \frac{\nu-p-1}{p} < \frac{1}{p'}$ (this is equivalent to the inequality $p < \nu < 2p$ which is valid for the assumptions $p < \nu$, $p > 1$, $\nu \leq 2$), all possible solutions of the conjugation problem from (1.6) are contained in the set of functions

$$\Omega(w) = \frac{X_1(w)}{2\pi i} \int\limits_{\gamma} \frac{g(\tau)}{X_1^+(\tau)} \frac{d\tau}{\tau - w} + (\alpha w + \beta) X_1(w), \tag{1.12}$$

where α and β are arbitrary complex constants. The general solution of the corresponding homogeneous problem will be the function $\Omega_0(w) = (\alpha w + \beta)X_1$.

In order for the second condition from (1.6) to be fulfilled, it is necessary to have

$$-\Big(\overline{\alpha}\frac{1}{w} + \overline{\beta}\Big) \frac{\sqrt[p]{z'(w)}}{\frac{1}{w} - \overline{c}} = \frac{(\alpha w + \beta)\sqrt[p]{z'(w)}}{w - c}.$$

Since $\overline{c} = c^{-1}$,

$$-\Big(\frac{\overline{\alpha}}{w} + \overline{\beta}\Big) \frac{wc}{c - w} = \frac{\alpha w + \beta}{w - c}.$$

Consequently, $\overline{\beta}c - \alpha = 0$, $\overline{\alpha}c - \beta = 0$ and we find that $\beta = \overline{\alpha}c$. That is, if β is assumed to be arbitrary, we must have $\alpha = \overline{\beta}c$. Thus, the function

$$u_0(z(w)) = \operatorname{Re}\Big[\frac{1}{\sqrt[p]{z'(w)}} \frac{w\overline{\beta}c + \beta}{w - c} \sqrt[p]{z'(w)}\Big] = \operatorname{Re}\Big[\frac{w\overline{\beta}c + \beta}{w - c}\Big]$$

is a solution of the homogeneous Dirichlet problem (1.2_0).

But if $w = re^{i\theta}$, $c = c_1 + ic_2 = e^{i\theta_c}$, $\beta = \lambda + i\mu$ then

$$\frac{\overline{\beta}cw + \beta}{w - c} = \frac{(\overline{\beta}cw + \beta)(\overline{w} - \overline{c})}{|w - c|^2} = \frac{(\overline{\beta}cr^2 - \beta\overline{c}) - (\overline{\beta}w - \beta\overline{w})}{|w - c|^2}.$$

Taking into account that $\operatorname{Re}[\overline{\beta}w - \beta\overline{w}] = 0$ and supposing $\overline{\beta}c = d + ie$, we obtain

$$\operatorname{Re}\frac{\overline{\beta}cw + \beta}{w - c} = \operatorname{Re}\frac{\overline{\beta}cr^2 - \beta\overline{c}}{|w - c|^2} = \operatorname{Re}\frac{(d + ie)r^2 - (d - ie)}{|w - c|^2} = \frac{d(r^2 - 1)}{|w - c|^2},$$

where $d = \operatorname{Re}\overline{\beta}c = \operatorname{Re}[(\lambda - i\mu)(c_1 + ic_2)] = \lambda c_1 + \mu c_2$.

Hence

$$u_0(z(re^{i\theta})) = \frac{-(\lambda c_1 + \mu c_2)(1 - r^2)}{1 + r^2 - 2r\cos(\theta - \theta_c)},$$

where λ and μ are arbitrary real constants. Obviously, $-(\lambda c_1+\mu c_2)$ passes through all real numbers, so that

$$u_0(z(re^{i\theta}))=M\operatorname{Re}\frac{c+w}{c-w},$$

where M is an arbitrary real constant.

Thus

$$u_0(z)=M\operatorname{Re}\frac{c+w(z)}{c-w(z)} \tag{1.13}$$

is the general solution of problem (1.2_0).

One can obtain a particular solution of the problem (1.2) by using again the equality $u_f=\frac{1}{2}[\Omega(w(z))+\Omega_*(w(z))]$, where Ω is the function defined by (1.12) for $\alpha=\beta=0$.

As a result, we obtain the particular solution having the form

$$u_f(z)=$$
$$=\operatorname{Re}\Big[\frac{1}{w(z)}\Big(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(z(\zeta))(\zeta-c)}{\zeta-w(z)}d\zeta-\frac{cw^2(z)}{2\pi i}\int\limits_{\gamma}\frac{f(z)(\overline{\zeta}-\overline{c})}{\zeta(\zeta-w(z))}d\zeta\Big)\Big]. \tag{1.14}$$

Consequently, for $p<\nu$, the Dirichlet problem (1.2) is solvable for any $f\in L^p(\Gamma)$ and has the set of solutions given by the equality

$$u(z)=u_0(z)+u_f(z), \tag{1.15}$$

where u_0 and u_f are defined by the formulas (1.13) and (1.14), respectively.

(iii) $\nu=p$. Consider first the homogeneous problem.

If X is given by (1.7), then $X(w)=O((w-c)^{\frac{1}{p'}}z_0(w))$, $z_0\in\bigcap\limits_{\delta>1}\widetilde{\mathcal{K}}^\delta(\Gamma)$ (including the case $\nu=2=p$). The function $F(w)=\Omega(w)[X(w)]^{-1}$ satisfies the condition $F^+=F^-$, belongs to the class $\bigcap\limits_{\delta<1}H^\delta$ in U and to the class $\bigcap\limits_{\delta<1}\widetilde{H}^\delta$ in $\mathbb{C}\backslash\overline{U}$. Let us show that the function F is regular at all, different from c, points of the plane. Let ζ be an arbitrary on γ point, different from c. Choose on γ a pair of points ζ_1 and ζ_2 on either side from ζ so close to ζ that the circumference arc with these ends $\gamma(\zeta_1,\zeta_2)$ does not contain c. Consider the domain S_ζ^+ (a part of the circle U) bounded by the radii passing through the points ζ_1 and ζ_2 and by the arc $\gamma(\zeta_1,\zeta_2)$. Since the function $\Omega\in H^\delta$, according to the Fejer-Riesz theorem ([27], p. 46) we have

$$\int\limits_0^1|\Omega(re^{i\theta_0})|^\delta dr\le M\int\limits_0^{2\pi}|\Omega(e^{i\theta})|^\delta d\theta,\quad \theta_0\in[0,\pi],\quad \delta\in(0,+\infty),$$

and thus we can easily state that $\Omega\in E^p(S_\zeta^+)$. Analogously we show that $\Omega\in E^p(S_\zeta^-)$, where S_ζ^- is the domain bounded by the arc, extension of radii passing through the points ζ_1 and ζ_2 and by the arc of circumference $|w| = 1+\eta$, $\eta > 0$. Since $\zeta \overline{\in} \gamma(\zeta_1, \zeta_2)$, $[X(w)]^{-1}$ is bounded in domains $S_\zeta^\pm$; hence $F \in E^p(S_\zeta^\pm)$, $p > 1$ and therefore it is representable in these domains by the Cauchy integral

$$F(w) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_1} \dfrac{F(\zeta)d\zeta}{\zeta - w}, & w \in S_\zeta^+, \\ 0, & w \in S_\zeta^-, \end{cases}$$
$$F(w) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_2} \dfrac{F(\zeta)d\zeta}{\zeta - w}, & w \in S_\zeta^-, \\ 0, & w \in S_\zeta^+, \end{cases} \tag{1.16}$$

where γ_1 and γ_2 are the boundaries of the domains S_ζ^+ and S_ζ^-, respectively.

Let $\gamma_3 = (\gamma_1 \cup \gamma_2) \backslash \gamma(\zeta_1, \zeta_2)$. Since on $\gamma(\zeta_1, \zeta_2)$ we have $F^+ = F^-$, the function

$$F_1(w) = \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2} \frac{F(\zeta)d\zeta}{\zeta - w} = \frac{1}{2\pi i} \int_{\gamma_3} \frac{F(\zeta)d\zeta}{\zeta - w}$$

is regular at the points of the arc $\gamma(\zeta_1, \zeta_2)$. On the other hand, the function F inside of γ_3 on account of (1.16) coincides with F.

Thus, the function F is regular everywhere with the exclusion of the point c at which it may possibly have a pole of the first order, since it fails to belong to the class $\bigcup\limits_{\delta<1} H^\delta$ otherwise. Consequently, $F(w) = \alpha + \frac{\beta}{w-c}$, and all possible solutions of the homogeneous problem from (1.6) are contained in the set of functions given by

$$\Omega_0(w) = \alpha X(w) + \beta(w-c)^{-1} X(w) = \alpha X(w) + \beta X_1(w), \tag{1.17}$$

where α and β are arbitrary constants.

Let us now find conditions for Ω_0 to belong to $\widetilde{\mathcal{K}}^p(\gamma)$. Since $X \in \widetilde{\mathcal{K}}^p(\gamma)$, for the inclusion $\Omega_0 \in \widetilde{\mathcal{K}}^p(\gamma)$ the condition $X_1 \in \widetilde{\mathcal{K}}^p(\gamma)$ must be satisfied. With this end in view, it is necessary and sufficient for X_1 to be the function H^p in U. In this connection, there may appear two possible cases.

I. $X_1 \in H^p$ in U. As in case (ii), we obtain that Ω_0 is a solution of (1.6) if β is arbitrary and $\alpha = \overline{\beta} c$. Then the general solution of the homogeneous problem (1.2_0) is again given by equality (1.13).

II. $X_1 \overline{\in} H^p$. Then $\Omega_0 \in \widetilde{\mathcal{K}}^p(\Gamma)$ if and only if $\beta = 0$. Moreover, by virtue of the second condition from (1.6) we again arrive at $\operatorname{Re}\alpha = 0$, and therefore the problem (1.2_0) has only zero solution.

Consider now the question whether the inhomogeneous problem has a solution.

Let X_1 be given by the equality (1.7_1). Then in the case under consideration, owing to (1.29) of Chapter III, in the neighbourhood of the point

c we have $X_1 = O((w-c)^{-\frac{1}{p}} z_0^{\frac{1}{p}}(w))$; moreover, $(z_1^+)^{\pm 1} \in \underset{\delta>1}{W}{}^{\delta}(\gamma)$, where $z_1 = z_0^{\frac{1}{p}}$. Since $X_1^{-1} \in \widetilde{\mathcal{K}}^{p'}(\gamma)$, we can easily find that a possible solution of the problem (1.6) is contained in the set

$$\Omega(w) = \frac{X_1(w)}{2\pi i} \int\limits_{\gamma} \frac{g(\zeta)}{X_1^+(\zeta)} \frac{d\zeta}{\zeta - w} + (\alpha w + \beta) X_1(w).$$

Consider again two cases.

I. $X_1 \in H^p$. In this case $(\alpha w + \beta)X_1 \in \widetilde{\mathcal{K}}^p(\gamma)$ and $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$, if and only if

$$\Omega_g = \frac{X_1(w)}{2\pi i} \int\limits_{\gamma} \frac{g(\zeta)}{X_1^+(\zeta)} \frac{d\zeta}{\zeta - w} \tag{1.18}$$

belongs to the class $\widetilde{\mathcal{K}}^p(\gamma)$. Obviously, this equality is equivalent to the condition $\Omega_g \in H^p$ in U. However, Ω_g does not belong to H^p for some $g \in L^p(\gamma)$. Indeed, assume Ω_g to belong to H^p for any $g \in L^p(\gamma)$. Then, by Lemma 1.1 and Smirnov's theorem from (0.19), we have $\Omega_g^+ \in L^p(\gamma)$ which, owing to Sokhotskiĭ-Plemelj's formulas, is equivalent to the condition that the function

$$T_1 g = \frac{X_1^+(\zeta_0)}{2\pi i} \int\limits_{\gamma} \frac{g(\zeta)}{X_1^+(\zeta)} \frac{d\zeta}{\zeta - \zeta_0}$$

belongs to the class $L^p(\gamma)$ for all $g \in L^p(\gamma)$.

But

$$\begin{aligned} T_1 g(\zeta_0) &= X_1^+(\zeta_0) \int\limits_{\gamma} \frac{g(\zeta)}{X_1^+(\zeta)} \frac{d\zeta}{\zeta - \zeta_0} = \\ &= (\zeta_0 - c)^{-\frac{1}{p}} z_1(\zeta_0) \int\limits_{\gamma} \frac{g(\zeta)}{(\zeta - c)^{-\frac{1}{p}} z_1(\zeta)} \frac{d\zeta}{\zeta - \zeta_0}, \end{aligned} \tag{1.19}$$

and if the function $T_1 g$ belongs to the class $L^p(\gamma)$ for any $g \in L^p(\gamma)$, then by Theorem 2.2 of Chapter I, T_1 will be an operator, continuous on $L^p(\gamma)$. However, T_1 is not such. Indeed, the function $\rho(\zeta_0) = (\zeta_0 - c)^{-\frac{1}{p}} z_0(\zeta)$ under such an assumption belongs to $W_p(\gamma)$ and hence to $W_{p+\varepsilon}(\gamma)$ as well (see Chapter I, §4). Therefore there must be $\rho \in L^{p+\varepsilon}(\gamma)$. But this is impossible, since the condition $z_0^{\pm 1} \in \bigcap\limits_{\delta>1} L^\delta(\gamma)$ implies that $\rho \overline{\in} \bigcap\limits_{\delta>p} L^\delta(\gamma)$.

Thus, if $X_1 \in H^p$, there exist functions g_0 of the class $L^p(\gamma)$ for which the problem (1.6) is unsolvable. Consequently, the problem (1.6) is unsolvable for the functions $f_0(t) = \widetilde{g}_0(w(t)) \left[\sqrt[p]{w'(t)} \right]^{-1}$, $f_0 \in L^p(\Gamma)$.

II. $X_1 \overline{\in} H^p$. In the case of piecewise smooth Lyapunov curves with angle sizes $\nu\pi$, $\nu \neq 0$, it follows from Warschawski's theorem (Theorem 1.5, Chapter III) that $X_1(w) \overline{\in} H^p$. However, as it will be shown in subsection 1.3, for any such curve with one angular point c, of angular size $p\pi$, $1 < p \le 2$ there exists a function $f_0 \in L^p(\Gamma)$ for which the problem (1.2) is unsolvable.

Thus when the curve has an angular point c of the angular size $p\pi$, the problem (1.2) is, generally speaking, unsolvable if the given function f is required to satisfy the condition $f \in L^p(\Gamma)$ only.

On the basis of Theorem 4.7 of Chapter I, one can point out a rather wide set of functions f from the class $L^p(\Gamma)$ for which the problem (1.2) becomes solvable.

Suppose

$$f(t)\ln|w(t) - C| \in L^p(\Gamma). \tag{1.20}$$

Then

$$g(\zeta)\ln|\zeta - c| \in L^p(\gamma), \tag{1.20$_1$}$$

and thus $g(\zeta)\ln(\zeta - c) \in L^p(\gamma)$.

Let X be the function defined by equality (1.7). Then $\frac{g(\zeta)\ln(\zeta-c)}{X^+(\zeta)} \in L(\gamma)$.

Consider the function

$$\Omega_g(w) = \frac{X(w)}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta - w} \tag{1.21}$$

and show that this function is a particular solution of the problem (1.6). For this purpose, it is seen to be sufficient to show (observing that $\Omega_g \in H^\delta$ for some $\delta > 0$ in domains U and $\mathbb{C}\backslash\overline{U}$) that $\Omega_g^+ \in L^p(\gamma)$. But

$$\Omega_g^+(\zeta_0) = \frac{1}{2}g(\zeta_0) + \frac{X^+(\zeta_0)}{2\pi i}\int\limits_{\gamma} \frac{g(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta - \zeta_0}, \tag{1.22}$$

where $X^+(\zeta_0) = O((\zeta_0 - c)^{\frac{1}{p'}} z_0^{\frac{1}{p}}(\zeta_0))$ and $z_0(\zeta_0) = \exp(\mathcal{K}_\Gamma\beta)^+(\zeta_0)$ with the function β continuous on Γ. By Corollary of Theorem 1.1 of Chapter III, $z_0^q \in \bigcap\limits_{\delta>1} W^\delta$ for any $q \in \mathbb{R}$. From this and the expression (1.20$_1$) it follows immediately that to the second summand in (1.22) we have applied Theorem 4.7 of Chapter I on the basis of which we can conclude that this summand, and thus $\Omega_g^+ \in L^p(\gamma)$. Consequently, $\Omega_g \in H^p$ in U and by Lemma 7.1 of Chapter I we find that $\Omega_g \in \widetilde{\mathcal{K}}^p(\gamma)$ as well. Again, a particular solution of the problem (1.2) is given by (1.10).

Summarizing the above results, we conclude that: if $\nu = p$, then the problem (1.2) has only zero solution if in the circle U the function $X_1(w) = (w - c)^{-\frac{1}{p}} z_0^{\frac{1}{p}}(w) \overline{\in} H^p$, or a set of solutions given by (1.13), when $X_1 \in H^p$. The inhomogeneous problem is, generally speaking, unsolvable. If the condition (1.20) is fulfilled, then the problem (1.2) is solvable, and its general solution is given by the equality

$$u(z) = u_f(z) + u_0(z), \tag{1.23}$$

where u_f is the function given by equality (1.10), and

$$u_0(z) = \begin{cases} 0, & \text{for,} \quad X_1 \overline{\in} H^p, \\ M \operatorname{Re} \dfrac{w(z)+c}{w(z)-c}, & \text{for} \quad X_1 \in H^p. \end{cases} \tag{1.24}$$

Remark 1. If Γ is a piecewise Lyapunov curve, then $0 < m < |z_0| \leq M$. Therefore $X_1 \overline{\in} H^p$, and hence the problem (1.2_0) for $\nu = p$ has only the zero solution. Moreover, using Corollary 1 of Theorem 2.2 from Chapter III, condition (1.20) can be written in the form

$$g(t) \ln |t-c| \in L^p(\Gamma). \tag{1.20$_2$}$$

Remark 2. On the basis of the above-considered cases, we have the following picture for the solvability of the problem (1.2) in domains with one interior angle of size $\nu = 2$:

for $p < 2$, the problem (1.2) *is solvable not uniquely, and all its solutions are given by* (1.15), (1.13) *and* (1.14). *For $p > 2$, it is solvable uniquely, and its unique solution is given by* (1.10). *If $p = 2$, then the problem* (1.2_0) *has only the zero solution for $X_1 \overline{\in} H^p$, and the set of solutions is given by* (1.13) *for $X_1 \in H^p$. The inhomogeneous problem* (1.2) *is, generally speaking, unsolvable. If, however, $f(t) \ln |w(t) - C| \in L^2(\Gamma)$, then this problem is solvable, and its general solution is given by* (1.23) *and* (1.24).

(iv) $\nu = 0$. If X is specified by (1.7), then in the neighbourhood of the point c we have $X = O((w-c)^{-\frac{1}{p}} z_0^{\frac{1}{p}})$ so $\frac{1}{X} \in \widetilde{\mathcal{K}}^{p'}(\gamma)$. Therefore $\Omega_0 = \alpha X$, and the homogeneous conjugation problem (1.6) is solvable, and its general solution is given by the equality $\Omega_0 = \alpha X$. As in case (i), we conclude that $\operatorname{Re} \alpha = 0$. Thus the problem (1.2_0) has only the zero solution. All the solutions of the inhomogeneous problem are contained in the set of functions

$$\Omega(w) = \frac{X(w)}{2\pi i} \int_\gamma \frac{g(\zeta)}{X^+(\zeta)} \frac{d\zeta}{\zeta - w} + \alpha X(w), \tag{1.25}$$

where $X^+(\zeta) = (\zeta - c)^{-\frac{1}{p}} z_0^{\frac{1}{p}}$.

Obvious $X \in \widetilde{\mathcal{K}}^p(\gamma)$ as $X \in H^p$ in U. Therefore $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$ if and only if $X \mathcal{K}_\Gamma \left(\frac{g}{X^+}\right) \in \widetilde{\mathcal{K}}^p(\gamma)$. However, if this inclusion is fulfilled for any $g \in L^p(\gamma)$, then the operator T_1, defined by (1.19), is continuous in $L^p(\gamma)$ (by Theorem 2.2 of Chapter II). But as is shown in the case (iii), the operator T_1 is not such. Hence, there exists a function $g_0 \in L(\gamma)$ for which the problem (1.6) is unsolvable. This implies the existence of such $f_0 \in L^p(\Gamma)$ for which the problem (1.2) is unsolvable.

Assume again that the condition (1.20) is fulfilled and let us show that the function Ω given by (1.25) for $\alpha = 0$ provides a particular solution of the problem (1.6), i.e., $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$. It suffices to show that $\Omega \in H^p$ in U. For this purpose, as it follows from Lemma 1.1 and Smirnov's theorem (0.19), it suffices in its turn to show that $\Omega^+ \in L^p(\gamma)$.

Suppose $g_1(\zeta) = g(\zeta)\ln|\zeta - c|$, $\rho(\zeta) = X^+(\zeta)(\zeta - c)^{\frac{1}{p}} z_0^{\frac{1}{p}}(\zeta)$. Then $g_1 \in L^p(\gamma)$ (because of (1.20)), and $\rho^q \in W_p(\gamma)$ for every $q \in \mathbb{R}$ (see Theorem 2.2 of Chapter II).

We now have

$$2\pi i\Omega^+(\zeta_0) = g(\zeta_0) + X^+(\zeta_0)\int_\gamma \frac{g_1(\zeta)(\zeta - c)}{(\zeta - c)^{\frac{1}{p'}}\rho(\zeta)\ln|\zeta - c|}\frac{d\zeta}{\zeta - \zeta_0} =$$

$$= g(\zeta_0) + X^+(\zeta_0)\int_\gamma \frac{g_1(\zeta)[(\zeta - \zeta_0) + (\zeta_0 - c)]}{(\zeta - c)^{\frac{1}{p'}}\rho(\zeta)\ln|\zeta - c|}\frac{d\zeta}{\zeta - \zeta_0} =$$

$$= g(\zeta_0) + AX^+(\zeta_0) + \rho(\zeta_0)(\zeta_0 - c)^{\frac{1}{p'}}\int_\gamma \frac{g_1(\zeta)}{(\zeta - c)^{\frac{1}{p'}}\rho(\zeta)\ln|\zeta - c|}\frac{d\zeta}{\zeta - \zeta_0} =$$

$$= g(\zeta_0) + AX^+(\zeta_0) + (Tg_1)(\zeta_0),\ A = \int_\gamma \frac{g_1(\zeta)d\zeta}{(\zeta - c)^{\frac{1}{p}}\rho(\zeta)\ln|\zeta - c|}. \quad (1.26)$$

Here $X^+ \in L^p(\gamma)$, since $X^+(\zeta) = \sqrt[p]{z'(\zeta)}$, and it follows from the above-mentioned properties of ρ (by Theorem 4.7 of Chapter I) that $Tg_1 \in L^p(\gamma)$. Thus, (1.26) implies that $\Omega^+ \in L^p(\gamma)$. Consequently, $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$, and hence this is a unique solution of the problem (1.6). The solution of (1.2) is given by the equality (1.10).

From the above considered cases (i)–(iv) we obtain

Theorem 1.1. *Let Γ be a simple, closed, piecewise smooth curve containing one angular point C with interior angle of size $\nu\pi$, $0 \le \nu \le 2$ and $X_1(w) = (w - c)^{-1}X(w)$, where X is given by* (1.7), *$c = w(C)$. Then the Dirichlet problem* (1.2)*:*

– is uniquely solvable for $o < \nu < p$;

– has for $p < \nu$ a set of solutions depending on only one parameter;

– is, in general, unsolvable for $p = \nu$ and becomes solvable when the condition (1.20) *is fulfilled. Moreover, it has a unique solution, if $X_1 \overline{\in} H^p$, and a set of solutions depending on only one parameter for $X_1 \in H^p$;*

– is, in general, unsolvable for $\nu = 0$. If condition the (1.20) *is fulfilled, it has a unique solution.*

When the solution exists, it is given by (1.10) *for $0 \le \nu < p$, by* (1.15), (1.14) *and* (1.13) *for $p < \nu$ and by* (1.23) *and* (1.24) *for $p = \nu$.*

1.3. An example of the function $f_0 \in L^p(\Gamma)$ for which the problem (1.2) has no solution if Γ contains an angular point with an angle of size $p\pi$. Let Γ be an arbitrary, closed, piecewise Lyapunov curve containing only one angular point with an angle (interior with respect to a finite domain bounded by this curve) $p\pi$, $1 < p \le 2$. To construct the function f_0 for which the problem

(1.2) is unsolvable, we will first show that upon its solvability the solution is obtained from that solution of problem (1.6) which has a specific form.

Write the boundary condition from (1.6) in the following form:

$$(\zeta - c)\Omega^+(\zeta) = -\frac{(\zeta - c)\sqrt[p]{z'(\zeta)}}{\overline{\sqrt[p]{z'(\zeta)}}}\Omega^-(\zeta) + g(\zeta)(\zeta - c).$$

It is easily verified that if X is the function given by (1.7), then the function $F(w) = (w - c)\Omega(w)X^{-1}(w)$ will, after subtraction from it of some linear function, belong to the class $\widetilde{\mathcal{K}}^p(\gamma)$. Therefore all possible solutions of the problem (1.6) lie in the set of functions

$$\widetilde{\Omega}(w) = \frac{X(w)}{w - c}\frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)(\zeta - c)}{X^+(\zeta)(\zeta - w)}d\zeta + \frac{aw + b}{w - c}X(w), \tag{1.27}$$

where a and b are arbitrary constants. Since $X \in \widetilde{K}^p(\gamma)$, $\widetilde{\Omega}$ will be a solution of the class $\widetilde{\mathcal{K}}^p(\gamma)$, if one can choose a constant B such that the function

$$\Omega_0(w) = \frac{X(w)}{w - c}\frac{1}{2\pi i}\int_\gamma \frac{g(\zeta)(\zeta - c)}{X^+(\zeta)}\frac{d\zeta}{\zeta - w} + \frac{BX(w)}{w - c} \tag{1.28}$$

would belong to the class $\widetilde{\mathcal{K}}^p(\gamma)$.

We can easily show that in the unit circle and in its complement $\Omega_0 \in \bigcap\limits_{\gamma<1} H^\delta$, and therefore $\Omega_0 \in \widetilde{\mathcal{K}}^p(\gamma)$ if and only if $\Omega_0^+ L^p(\gamma)$. For this it is necessary and sufficient that the function

$$h_{g,B} = \frac{X^+(\zeta_0)}{\zeta_0 - c}\int_\gamma \frac{g(\zeta)(\zeta - c)d\zeta}{X^+(\zeta)(\zeta - \zeta_0)} + \frac{BX^+(\zeta_0)}{\zeta_0 - c} \tag{1.29}$$

would belong to the class $L^p(\gamma)$.

Construct now a function $g_0 \in L^p(\gamma)$ such that $h_{g_0,B}\overline{\in}L^p(\gamma)$ for any values of B.

Suppose $c = 1$ and let

$$g_0(\zeta) = g_0(e^{i\theta}) = \begin{cases} \dfrac{m_n X^+(\zeta)}{\zeta - 1}, & \theta_n \le \theta < \theta_{n+1}, \\ 0, & \theta \in (1, 2\pi), \end{cases} \tag{1.30}$$

where $\theta_n = \frac{1}{n}$ and $m_n = \frac{1}{\ln(n+1)}$.

Since Γ is a piecewise Lyapunov curve, near $c = 1$ (see Warschawski's theorem in Chapter III) we have $X^+ = O((\zeta - 1)^{\frac{1}{p'}} z_0(\zeta))$, where $z_0 \in H(\gamma)$,

$z_0(1) \neq 0$. Therefore $\left|\frac{X^+(\zeta)}{\zeta-1}\right| \leq \frac{M}{|\zeta-1|}$, and

$$\int_0^{2\pi} |g_0|^p d\theta \leq \sum_{n=1} m_n^p \int_{\theta_n}^{\theta_n+1} \frac{d\theta}{|\zeta-1|} \leq$$

$$\leq \sum_{n=1}^{\infty} \frac{m_n^p}{\sqrt{(1-\cos\frac{1}{n})^2+\sin^2\frac{1}{n}}}\Big(\frac{1}{n}-\frac{1}{n+1}\Big) \leq \text{const}\sum_{n=1}^{\infty}\frac{1}{n\ln^p(n+1)} < \infty.$$

Thus $g_0 \in L^p(\gamma)$. Show that $h_{g_0,B}\overline{\in}L^p(\gamma)$. We have

$$h_{g_0,B} = \frac{X^+(\zeta_0)}{\zeta_0-1}\Big[\int_\gamma \frac{g_0(\zeta)(\zeta-1)}{X^+(\zeta)(\zeta-\zeta_0)}d\zeta + B\Big],$$

where $\frac{X^+(\zeta_0)}{\zeta_0-1}\overline{\in}L^p(\gamma)$. Therefore if we prove that

$$\lim_{\zeta_0\to 1}\int_\gamma \frac{g_0(\zeta)}{X^+(\zeta)}\frac{\zeta-1}{\zeta-\zeta_0}d\zeta = \infty, \tag{1.31}$$

then this will imply that the assertion regarding $h_{g_0,B}$ is valid.

Given $k > 0$ and N such that $\sum_{n=1}^{N}\frac{1}{n\ln(n+1)} > k$. Suppose $\zeta_0 = e^{i\theta}$, $\theta \in \Big(0, \frac{1}{N+1}\Big)$, $\zeta_n = e^{i\theta_n}$, $\theta_n = \frac{1}{n}$. We have

$$\Big|\int_\gamma \frac{g_0(\zeta)(\zeta-1)d\zeta}{X^+(\zeta)(\zeta-\zeta_0)}\Big| \geq \Big|\sum_{n=1}^{\infty} m_n \ln\Big|\frac{\zeta_n-\zeta_0}{\zeta_{n+1}-\zeta_0}\Big|\Big|, \tag{1.32}$$

$$\ln\Big|\frac{\zeta_n-\zeta_0}{\zeta_{n+1}-\zeta_0}\Big| = \ln\Big|\frac{\sin(\frac{\theta}{2}+\frac{1}{2n})}{\sin(\frac{\theta}{2}+\frac{1}{2(n+1)})}\Big| > 0, \quad \theta\in\Big(0,\frac{1}{N+1}\Big).$$

Therefore from (1.32) we find that

$$\Big|\int_\gamma \frac{g_0(\zeta)(\zeta-1)}{X^+(\zeta)(\zeta-\zeta_0)}d\zeta\Big| \geq \sum_{n=1}^{\infty} m_n \ln\Big|\frac{\sin(\frac{\theta}{2}+\frac{1}{2n})}{\sin(\frac{\theta}{2}+\frac{1}{2(n+1)})}\Big| =$$

$$= \sum_{n=1}^{\infty} m_n \ln\left(1+\frac{2\cos(\frac{\theta}{2}+\frac{1}{4n(n+1)})\sin\frac{1}{4n(n+1)}}{\sin(\frac{\theta}{2}+\frac{1}{2(n+1)})}\right) \geq$$

$$\geq \frac{1}{2}\sum_{n=1}^{N} m_n \frac{2\cos(\frac{\theta}{2}+\frac{1}{4n(n+1)})\sin\frac{1}{4n(n+1)}}{\sin(\frac{\theta}{2}+\frac{1}{2(n+1)})} \geq$$

$$\geq \frac{1}{2}\sum_{n=1}^{N} m_n \frac{2\cos(\frac{\theta}{2}+\frac{1}{4n(n+1)})\sin\frac{1}{4n(n+1)}}{\sin(\frac{1}{2(N+1)}+\frac{1}{2(n+1)})} \geq m_0 k, \tag{1.33}$$

$$\text{where} \quad m_0 = \frac{1}{2\pi}\inf_N \cos\frac{2N+3}{4N(N+1)} > 0.$$

which proves the relation (1.31).

If $\frac{g_0}{\sqrt[p]{z'(\zeta)}} = g_1 + ig_2$, then it is obvious that the problem (1.2) is unsolvable for one of the functions $g_1(w(t))$ or $g_2(w(t))$.

1.4. Problem (1.2) in domains bounded by arbitrary piecewise smooth boundaries. From the results obtained in sections 1.2 and 1.3 it follows

Theorem 1.2. *Let Γ be a closed, piecewise smooth curve bounding the finite domain D, and let C_k, $k = \overline{1,n}$, be its angular points with interior angles $\nu_k\pi$, $0 \le \nu_k \le 2$. Denote by n_1 the number of angular points with the values n_1 from the interval $(p, 2]$ (assuming $(2,2] = \varnothing$). Then all the solutions of the homogeneous problem (1.2_0) are given by the equality*

$$u_0(z) = \sum_{\nu_k \in (p,2]} N_k(p) \operatorname{Re} \frac{w(C_k)+w(z)}{w(C_k)-w(z)} + \sum_{\nu_k = p} M_k(p) \operatorname{Re} \frac{w(C_k)+w(z)}{w(C_k)+w(z)}, \quad (1.34)$$

where N_k are arbitrary constants, $w = w(z)$ is a function mapping conformally the domain D onto U, and

$$M_k(p) = \begin{cases} 0, & \text{if } \ X_k \overline{\in} H^p, \quad X_k(w) = (w - c_k)^{-\frac{1}{p}} z_0^{\frac{1}{p}}, \\ M_k & \text{is an arbitrary constant if } \ X_k \in H^p, \end{cases} \quad (1.35)$$

z_0 is the function defined from equality (2.7) of Chapter III.

The inhomogeneous problem is, in general, unsolvable if there exist angular points with the values ν_k from the set $\{0, p\}$. If f satisfies the condition

$$f(t) \ln \Big| \prod_{\substack{k \\ \nu_k \in \{0,p\}}} (w(t) - w(C_k)) \Big| \in L^p(\Gamma), \quad (1.36)$$

then the problem is solvable.

In all the cases in which a solution exists, it is given by the equality

$$u(z) = u_0(z) + u_f(z), \quad (1.37)$$

where u_0 is defined by (1.34), and

$$u_f(z) = \operatorname{Re} \Bigg[\Bigg(\frac{1}{2\pi i} \frac{f(z(\zeta))\rho(\zeta)}{\zeta - w(z)} d\zeta + + \frac{(-1)^{n_1}}{2\pi i} w(z)^{n_1+1} \prod_{\nu_k \in (p,2]} c_k \int_\gamma \frac{f(z(\zeta))\overline{\rho}(\zeta)}{\zeta(\zeta - w(z))} \Bigg) \frac{1}{\rho(w(z))} \Bigg], \quad (1.38)$$

$$\rho(w(z)) = \prod_{\nu_k \in (p,2]} (w - c_k) \quad \text{and} \quad \rho \equiv 1, \quad \text{if} \quad \{\nu_k : \nu_k \in (p, 2]\} = \varnothing. \quad (1.39)$$

§ 2. The Neumann Problem in the Class $e'_p(D)$ in Domains with Piecewise Smooth Boundary

Suppose

$$e'_p(D)=\Big\{u:\Delta u=0,\ \sup_{0\le r<1}\int_{\Gamma_r}\Big(\Big|\frac{\partial u}{\partial x}\Big|^p+\Big|\frac{\partial u}{\partial y}\Big|^p\Big)|dz|\Big\},\ p>1,\quad (2.1)$$

where Γ_r is as usual the image of the circumference $|w|=r$ for the conformal mapping of U onto D.

Let $u\in e'_p(D)$, v be a function conjugate harmonically to it, and $\phi=u+iv$. Since $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$, and $\phi(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial y}$, $z=x+iy$, it follows from (2.1) that $\phi'\in E^p(D)$. Thus $e'_p(D)=\operatorname{Re}E'_p(D)$, where $E'_p(D)=\{\phi:\phi'\in E^p(D)\}$. This implies that the functions u from $e'_p(D)$ are continuous in $\overline{D}$, absolutely continuous on Γ (see, e.g., [133], p. 208), and $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ have on Γ angular boundary values $\left(\frac{\partial u}{\partial x}\right)^+$ and $\left(\frac{\partial u}{\partial y}\right)^+$ summable on γ with degree p.

Let

$$\Big(\frac{\partial u}{\partial n}\Big)_\Gamma=\Big(\frac{\partial u}{\partial x}\Big)^+\cos(n,x)+\Big(\frac{\partial u}{\partial y}\Big)^+\cos(n,y)=$$
$$=\Big(\frac{\partial u}{\partial x}\Big)^+(-\sin\alpha(t))+\Big(\frac{\partial u}{\partial y}\Big)^+\cos(\alpha(t)),$$

where (n,x) and (n,y) denote the angles formed by the normal at the point t and the coordinate axes and $\alpha(t)$ is the angle lying between the oriented tangent at the point t and the x-axis.

Consider the Neumann problem formulated as follows: define the function u for which

$$\left.\begin{array}{c}\Delta u=0,\quad u\in e'_p(D),\quad p>1,\\ \left(\frac{\partial u}{\partial n}\right)_\Gamma=f,\quad f\in L^p(\Gamma).\end{array}\right\}\qquad (2.2)$$

Let $u=\operatorname{Re}\phi$ be a solution of the problem (2.2). Then $\phi'\in E^p(D)$. Taking ϕ' in terms of $\phi'=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$ and assuming $a(t)=\cos\alpha(t)$, $b(t)=\sin\alpha(t)$, we can write the boundary condition from (2.2) in the following form:

$$\operatorname{Re}[i(a(t)+ib(t))\phi'(t)]=f(t)$$

([106], §74–75) or, what comes to the same thing, in the form

$$\operatorname{Re}[it'(s)\phi'(t(s))]=f(t(s)).\qquad (2.3)$$

Let

$$\Psi(w)=\sqrt[p]{z'(w)}\phi'(z(w)),\quad f(z(\tau))=g_1(\tau).\qquad (2.4)$$

Taking into account the fact that $t'(s) = \exp i\alpha(t(s))$, we write (2.3) as

$$\operatorname{Re}\Big[\frac{i\exp i\alpha(z(\tau))}{\sqrt[p]{z'(\tau)}}\Psi^+(\tau)\Big] = g_1(\tau).$$

If

$$\Omega(w) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi\Big(\frac{1}{\overline{w}}\Big)}, & |w| > 1, \end{cases} \tag{2.5}$$

then Ω satisfies the conditions

$$\left.\begin{aligned} \Omega^+(\tau) = \exp(-2i\alpha(z(\tau)))\frac{\overline{\sqrt[p]{z'(\tau)}}}{\sqrt[p]{z'(\tau)}}\Omega^-(\tau) + g(\tau), \\ \Omega \in \widetilde{\mathcal{K}}^p(\gamma), \quad \Omega(w) = \Omega_*(w), \end{aligned}\right\} \tag{2.6}$$

where

$$g(\tau) = -i\sqrt[p]{z'(\tau)}\exp(-i\alpha(z(\tau))2g_1(\tau), \quad g \in L^p(\gamma). \tag{2.7}$$

Let $\tau = \exp i\theta$. Assume $\alpha(z(\tau)) \equiv \alpha(\theta)$. Since

$$\lim_{w \overset{\wedge}{\to} \exp i\theta} \arg z'(w) = \alpha(\theta) - \theta - \frac{\pi}{2}$$

(see Corollary 1 of Theorem 2.2 in Chapter III), the conjugation problem from (2.6) will take the form

$$\Omega^-(\tau) = M_0\exp\Big[-2i\Big(\frac{\theta}{p} + \frac{\alpha(\theta)}{p'}\Big)\Big]\Omega^-(\tau) + g(\tau), \quad M_0 = \exp\Big(-\frac{\pi i}{p}\Big). \tag{2.8}$$

The coefficient $G(\tau) = M_0\exp\Big[-2i\Big(\frac{\theta}{p} + \frac{\alpha(\theta)}{p'}\Big]$ of the problem (2.8) has a jump discontinuity at the point $c = w(C)$. Moreover, when the point τ moves along the unit circumference in the positive direction, the $\arg G(\tau) = -2\Big(\frac{\alpha(\theta)}{p'} + \frac{\theta}{p}\Big)$ possesses the increment -4π. Choosing on γ a point $\tau_0 \neq C$ and assuming it to be the initial point of the going around γ, then $\arg G(\tau)$ will have a discontinuity equal to (-4π). Let $c = \exp i\theta_c$. Then $(\alpha(\theta_c+) - \alpha(\theta_c-))$ is equal to the angle between the right and the left tangents at the point C, that is, $\pi - \nu\pi$. Therefore the $\arg G(\tau)$ has at the point c a discontinuity equal to $\beta = \frac{2\pi(\nu-1)}{p'}$. Let $\varphi_1(\tau) = \frac{\beta}{2\pi}\theta_c(\tau)$ and $\varphi_2(\tau) = 2\theta_{\tau_0}(\tau)$ where θ_c, θ_{τ_0} are the continuous, respectively on $\gamma\backslash\{c\}$ and $\gamma\backslash\{\tau_0\}$, branches of the function $\arg\tau$. Then the function $\varphi_0(\tau) = \arg G(\tau) - \varphi_1(\tau) - \varphi_2(\tau)$ is continuous on γ.

Let now

$$X_c(w) = (w - \tau_0)^2\rho_c(w)\prod_{k=0}^{2}\exp\Big(\frac{1}{2\pi i}\int_\gamma\frac{\varphi_k(\tau)d\tau}{\tau - w}\Big) =$$

$$= (w-\tau_0)^2 \rho_c(w) \prod_{k=0}^{2} X_k(w), \tag{2.9}$$

where

$$\rho_c(w) = \begin{cases} 1 & \text{for } \nu < p' \\ w-c & \text{for } \nu \geq p', \end{cases} \quad X_k(w) = \exp\left(\frac{1}{2\pi i}\int_\gamma \frac{\varphi_k(\tau)d\tau}{\tau - w}\right). \tag{2.10}$$

The function $X_c(w)$ is a solution of the homogeneous boundary value problem (2.8) representable by the Cauchy type integral with density from the class $L^p(\gamma)$ and its principal part at infinity of order $\varkappa_c + 2$ (that is, $X_c(w) = \int_\gamma \frac{x(\tau)d\tau}{\tau - w} + P_{\varkappa_c+2}(w)$, $x \in L^p(\gamma)$ and $P_{\varkappa_c+2}$ is a polynomial of order $\varkappa_c + 2$), where $\varkappa_c = 0$ for $\nu < p'$ and $\varkappa_c = 1$ for $\nu \geq p'$. The function $[X_c(w)]^{-1}$ is representable by the Cauchy type integral with density from $\bigcap\limits_{\varepsilon>0} L^{p'-\varepsilon}(\gamma)$, while the functions $[X_c^\pm(\tau)]^{-1}$ are integrable, with degree p', on any closed portion of γ not containing the point c (see subsection 1.2). Further, $X_c^+[X_c^-]^{-1} = G$, and we can easily verify that near the point c,

$$\left.\begin{aligned} X_c(w) &= O((w-c)^{\frac{1-\nu}{p'}} X_0(w)) \quad \text{for} \quad \nu < p', \\ X_c(w) &= O((w-c)^{\frac{1-\nu}{p'}+1} X_0(w)) \quad \text{for} \quad \nu \geq p', \end{aligned}\right\} \tag{2.11}$$

where $X_0(w) = \exp\left(\frac{1}{2\pi}\int_\gamma \frac{\psi_0(\tau)d\tau}{\tau - w}\right)$ with the continuous on γ function ψ_0, and therefore $X_0^{\pm 1} \in \bigcap\limits_{\delta>1} H^\delta$, $X^+ \in \bigcap\limits_{\delta>1} W_\delta(\gamma)$.

If Ω_0 is a solution of the problem (2.6), and $F(w) = \Omega_0(w)[X_c(w)]^{-1}$, then: first, almost everywhere on γ we have $F^+(\tau) = F(\tau)$, and $F(w)$ has at infinity zero of order $\varkappa_c + 2$; secondly, F is regular at all points of γ when

$$\nu \overline{\in} \{0, 2, p'\} \quad \text{or} \quad \nu = 2 \quad \text{and} \quad p < 2. \tag{2.12}$$

If, however,

$$\nu \in \{0, p'\} \quad \text{or} \quad \nu = 2 \quad \text{and} \quad p \geq 2, \tag{2.13}$$

then the function F is regular on $\gamma \backslash \{c\}$, and at the point c it may perhaps have a pole of the first order. (This can be justified in the same manner as it was done in subsection 1.2.)

On this basis we can conclude that F, and hence Ω_0, is everywhere equal to zero. But then only the functions $u_0 \equiv M$, where M is an arbitrary real constant, will be solutions of the problem (2.2) for $f = 0$.

Construct now a particular solution of the inhomogeneous problem (2.2). For this purpose we notice that under the assumption (2.12) and owing to (2.11), we have $g[X_c^+]^{-1} \in L(\gamma)$, and consider the function

$$\widetilde{\Omega}(w) = \frac{X_c(w)}{2\pi i}\int_\gamma \frac{g(\tau)d\tau}{X_c^+(\tau)(\tau - w)}. \tag{2.14}$$

It is not difficult to see that $\widetilde{\Omega}$ is the only possible function providing us with the solution. It is also easy to show that $\widetilde{\Omega} \in H^p$ in U. In order for Ω to belong to the class $\widetilde{\mathcal{K}}^p(\gamma)$, it is necessary and sufficient that the conditions

$$\int_\gamma g(\tau)[X_c^+(\tau)]^{-1}\tau^k d\tau = 0, \quad k = 0, \varkappa_c, \tag{2.15}$$

$$x_c = \begin{cases} 0, & \text{if } \nu < p', \\ 1, & \text{if } \nu \geq p' \end{cases}$$

or, what is the same, the conditions

$$\int_\Gamma f(t)Z_c(t)w^k(t)dt = 0, \quad k = 0, \varkappa_c, \tag{2.16}$$

be fulfilled, where

$$Z_c(t) = \exp\left(\frac{1}{2\pi}\int_\gamma \frac{\alpha(z(\tau))d\tau}{\tau - w(t)}\right)(w(t) - w(C))^{-\mu_p}(w(t) - w(t_0))^2 w'(t), \tag{2.17}$$

$$t_0 = z(\tau_0), \quad \mu_p = \begin{cases} 0, & \text{if } \nu < p' \\ 1, & \text{if } \nu \geq p'. \end{cases}$$

If (2.13) takes place, and if Γ is a piecewise Lyapunov curve, then for any $p > 1$ we can, as in subsection 1.3, construct an example of the function $f_0 \in L^p(\Gamma)$ for which the problem (2.2) is unsolvable. Note that the condition (2.16) can be fulfilled for the function f_0.

This implies that if $f \in L^p(\Gamma)$ then the problem (2.2) under conditions (2.13), (2.16), is, in general, unsolvable. Therefore we assume that the condition (1.20), appearing in the previous section, is fullfilled, i.e.,

$$f(t)\ln|w(t) - C| \in L^p(\Gamma). \tag{2.18}$$

In this case we can show, as in the above-mentioned section (see the case (ii) of subsection 1.2), that the function $\widetilde{\Omega}$ given in U and in its complement by (2.14) belongs to some Hardy class. According to Smirnov's theorem, for the belonging to the class H^p of the function $\widetilde{\Omega}$, it is sufficient to have $\widetilde{\Omega}^+ \in L^p(\gamma)$. By (2.18) and (2.11), on the basis of Theorem 4.7 of Chapter I, we can conclude that this inclusion is fulfilled. In order for the function $\widetilde{\Omega}$ to be the desired solution (i.e., for the existence of the finite limit $\lim\limits_{z\to\infty}\widetilde{\Omega}(z)$), it is necessary and sufficient that the conditions (2.16) be fulfilled. If these conditions are fulfilled, then the solution of the problem (2.2) is given by the equality

$$u(z) = \mathrm{Re}\left[\int_0^{w(z)} \frac{\widetilde{\Omega}(\zeta)d\zeta}{X_c^+(\zeta)}\right] + M, \tag{2.19}$$

where X_c and $\widetilde{\Omega}$ are functions given by the equalities (2.9), (2.10) and (2.14), and the integral in (2.19) is taken over any path in U from the point 0 to $w(z)$, and M is an arbitrary real constant.

As a result of the above reasoning we have the following

Theorem 2.1. *Let Γ be a simple, closed, piecewise smooth curve containing one angular point C with the angle $0 \le \nu \le 2$, and let X_c be defined by* (2.9) *and* (2.10)*. Then for the solvability of the Neumann problem in the class $e'_p(D)$ it is necessary to fulfil the conditions* (2.16)*. When these conditions are fulfilled, the problem becomes solvable for $\nu \in \overline{\in}\{0,2,p'\}$ or for $\nu = 2$ and $p < 2$. However, if $\nu \in \{0, p'\}$ or $\nu = 2$ and $p \ge 2$, then the problem is, in general, unsolvable. In these cases, if along with conditions* (2.16) *for f the condition* (2.18) *is fulfilled, then the problem is solvable.*

In all the cases in which the solution exists, it is given by (2.19)*, where $\widetilde{\Omega}$ is defined by* (2.14)*.*

The case with a general piecewise smooth curve can be considered by means of the function which is the product of the functions X_{c_k}, where X_{c_k} are constructed by (2.9) with C replaced by C_k. In particular, the number of conditions for solvability is equal to $1 + n_p$, where n_p is the number of angular points C_k at which $\nu_k \ge p'$. If on the boundary there exist points ν_k from the set $\{0, p'\}$, or $\nu_k = 2$ and $p \ge 2$, then the fulfilment of the condition of orthogonality guarantees the solvability of the problem if

$$f(t)\ln \prod_{\nu_k \in \{p',0\}} |w(t) - C_k| \prod_{\nu_k = 2} |w(t) - C_k|^{\lambda(p)} \in L^p(\Gamma),$$

$$\lambda(p) = \begin{cases} 0, & \text{if } \ p < 2, \\ 1, & \text{if } \ p \ge 2. \end{cases}$$

We will dwell in detail on the case where the curve Γ contains only one angular point: a cusp C.

If $\nu = 2$ and $p < 2$, then the problem (2.2) is solvable for any $f \in L^p(\Gamma)$ under the condition that it must be orthogonal to the function $Z_c(t)$ defined by (2.17). If $p \ge 2$ and $f(t)\ln|w(t) - C| \in L^p(\Gamma)$, then for the problem to be solvable, it is necessary (and sufficient) for f to be orthogonal to the function $Z_c(t)w(t)$ as well.

If $\nu = 0$, then there exists in any $L^p(\Gamma)$, $p > 1$, the function $f_{0,p}$ for which the Neumann problem is unsolvable in the class $e'_p(D)$. If along with (2.16) condition (2.18) is fulfilled, then the problem is solvable.

Finally we will remark that the condition (2.16) with $\varkappa_c = 0$ takes also place in the case, where Γ is a smooth curve. In this case the condition (2.16) can be written in a more simple form.

In the case of domains bounded by smooth curves, $\ln z'(w) \in H^1$ (see, e.g., (1.20) from Chapter III), and therefore the equality

$$i \ln z'(w) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\mathrm{Re}[i \ln z']}{\tau} \frac{\tau + w}{\tau - w} d\tau \tag{2.20}$$

is valid. Moreover, it is not difficult to verify that

$$\int\limits_{\gamma}^{2\pi} \frac{\theta d\zeta}{\zeta - \tau} = \int\limits_{0}^{2\pi} \frac{\theta i e^{i\theta} d\theta}{e^{i\theta} - \tau} = \ln(1-\tau) - i\theta_0 - i\pi, \quad \tau = e^{i\theta_0}. \tag{2.21}$$

The condition (2.16) with regard for (2.20) and (2.21) takes the form

$$\int\limits_{\Gamma} f(t) ds = 0.$$

Consequently, the condition for solvability of the Neumann problem in the class $e'_p(D)$ in the case under consideration has the same form as the condition for its solvability in different classes of smooth functions (see, e.g., [106], §75).

§ 3. On the Asymptotics of the Solutions in the Neighbourhood of Angular Points

As we have seen from the foregoing sections, the solutions of the Dirichlet and Neumann problems, as they were formulated above, can be written out in quadratures by means of the Cauchy type integrals and conformal mapping of D onto U. These integrals and mappings are studied well enough. This circumstance allows one to obtain, under some additional assumptions regarding the given functions, the asymptotics of the solutions in the neighbourhood of angular points of the boundary. As an example, consider one case of the Dirichlet problem.

Let Γ be a piecewise Lyapunov curve containing one angular point with the angle $\pi\nu$, $0 < \nu < p$ and let $f(t) = |t - C|^{-\alpha}\varphi(t)$, $0 < \alpha < \frac{1}{p}$, $\varphi \in H(\Gamma)$, $\varphi(C) \neq 0$.

Since $t - C = w(\tau) - w(c) = (\tau - c)^{\nu} w_0(\tau)$, where $w_0 \in H(\gamma)$ and $w_0 \neq 0$ (see, e.g., Corollary of Theorem 3.4 from Chapter III),

$$f(z(\tau)) = |\tau - c|^{-\alpha\nu}\psi(\tau), \quad \psi(\tau) \neq 0, \quad \psi \in H(\gamma).$$

The solution of the Dirichlet problem in $e^p(D)$ is given by the formula

$$u(z) = \mathrm{Re}\left[\frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(z(\tau))}{\tau} \frac{\tau + w}{\tau - w} d\tau\right] + M \,\mathrm{Re}\, \frac{w(C) + w(z)}{w(C) - w(z)} = u_1(z) + u_2(z).$$

Consider the function $u_1(z)$,

$$u_1(z) = \operatorname{Re}\Omega(w(z)), \quad \Omega(w) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(z(\tau))}{\tau}\frac{\tau + w}{\tau - w}d\tau.$$

We write the function $\Omega(z)$ as follows:

$$\Omega(w) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{|\tau - c|^{-\alpha\nu}\psi(\tau)d\tau}{\tau - w} +$$

$$+\frac{w}{2\pi i}\int\limits_{\gamma} \frac{|\tau - c|^{-\alpha\nu}\psi(\tau)}{\tau(\tau - w)}d\tau = \Omega_1(w) + \Omega_2(w).$$

The estimates for the Ω_1 and Ω_2 are well known when $0 < \alpha\nu < 1$ ([106], §22). Consequently, if $\nu\alpha \in (0,1)$, then applying the appropriate results from [106] and separating the real part of Ω, we obtain the estimate of the function $u_1(z(w))$ in the neighbourhood of the point c, and thus the estimate of $u_1(z)$ in the neighbourhood of the point C. The estimate for the summand u_2 can be obtained easily from Theorem 3.1 of Chapter III.

§ 4. The Riemann–Hilbert Problem in Domains with Piecewise Smooth Boundaries

Let D be a simply connected domain bounded by a simple, closed, piecewise smooth curve Γ containing one angular point C with the angle $\nu\pi$, $0 \le \nu \le 2$. We will consider the Riemann-Hilbert problem and formulate it as follows: define the function Φ of the class $E^p(D)$ whose angular boundary values $\Phi^+(t)$ satisfy almost everywhere on Γ the condition

$$\operatorname{Re}[(a(t) + ib(t))\Phi^+(t)] = c(t), \tag{4.1}$$

where a, b, c are the given on Γ real functions, and $c \in L^p(\Gamma)$.

As for the coefficients a and b, they will be assumed to be measurable on Γ functions such that if $G(t) = [a(t) - ib(t)][a(t) + ib(t)]^{-1}$, then

$$G(t) \in \widetilde{A}(p), \quad \text{and} \quad G \quad \text{is continuous in the neighbourhood of} \quad C. \tag{4.2}$$

We denote the class of such functions G by $\widetilde{A}_C(p)$.

Let $\varphi_p(t) = \arg G(t)$ and $\varkappa = \varkappa(p) = \varkappa(p, G)$ (the index of the function G be defined as in subsection 1.2 of Chapter II).

Passing to the circle U as in §1–2 of the present chapter, we arrive at the problem of defining the function $\Omega \in \widetilde{\mathcal{K}}^p(\gamma)$ by the following conditions:

$$\Omega^+(\zeta) = -\frac{\sqrt[p]{z'(\zeta)}}{\overline{\sqrt[p]{z'(\zeta)}}}\frac{A(\zeta) - iB(\zeta)}{A(\zeta) + iB(\zeta)}\Omega^-(\zeta) + g(\zeta), \quad z \in \Gamma, \tag{4.3}$$

$$\Omega_*(w) = \Omega(w), \quad |w| \ne 1, \tag{4.4}$$

where Ω is defined by (1.4), $A(\zeta) = a(z(\zeta))$, $B(\zeta) = b(z(\zeta))$, $g(\zeta) = 2c(z(\zeta))[A(\zeta) + iB(\zeta)]^{-1}$.

The function

$$G_\gamma(\zeta) = [A(\zeta) - iB(\zeta)][A(\zeta) + iB(\zeta)]^{-1}$$

on the circumference belongs to the class $\widetilde{A}_c(p)$.

Owing to Theorem 3.5 of Chapter II, G_γ is factorizable in the class $\widetilde{\mathcal{K}}^p(\gamma)$, and its factor function has the form

$$Y(w) = \begin{cases} \exp\left(\frac{1}{2\pi}\int_\gamma \frac{\varphi_p(\zeta)d\zeta}{\zeta - w}\right), & |w| < 1, \\ w^{-\varkappa}\exp\left(\frac{1}{2\pi}\int_\gamma \frac{\varphi_p(\zeta)d\zeta}{\zeta - w}\right), & |w| > 1, \end{cases} \tag{4.5}$$

where $\varphi_p(\zeta) = \arg_p G_\gamma(\zeta)$.

Before we proceed to investigating the problem (4.1), we will assume that $C \neq t_k$ and consider the following cases separately: (i) $0 < \nu < p$; (ii) $p < \nu$; (iii) $\nu = p$; (iv) $\nu = 2$; (v) $\nu = 0$.

(i) If $0 < \nu < p$, and the function X is given by (1.7), then

$$T(w) = AY(w)X(w), \tag{4.6}$$

will be a factor function for $\widetilde{G}_\gamma = -\frac{\sqrt[p]{z'(\zeta)}}{\overline{\sqrt[p]{z'(\zeta)}}}G_\gamma$ of order $-\varkappa(p)$ at infinity. Here A is an arbitrary constsnt. Choosing A as in §41 of [106], we achieve the fulfilment of the equality $T_*(w) = w^{\varkappa(p)}T(w)$, and hence conclude: if $\varkappa = \varkappa(p) \geq 0$, then the homogeneous problem corresponding (4.1), has an infinite number of solutions given by the equality

$$\Phi(z) = T(w(z))P_\varkappa(w(z))[\sqrt[p]{w'(z)}]^{-1},$$

where $P_\varkappa(w) = a_0 + a_1 w + \cdots + a_x w^\varkappa$ is an arbitrary polynomial whose coefficients satisfy the condition

$$a_i = \overline{a_{\varkappa - i}}, \quad i = \overline{1, \varkappa}. \tag{4.7}$$

The inhomogeneous problem (4.1) is, unconditionally, solvable. However, if $\varkappa < 0$, then the homogeneous problem has only zero solution, and in order for the inhomogeneous problem to be solvable, it is necessary and sufficient that the conditions

$$\int_\Gamma w^k(t)\frac{c(t)w'(t)}{T^+(w(t))}dt = 0, \quad k = 0, 1, \ldots, |\varkappa| - 2, \tag{4.8}$$

be fulfilled (If $\varkappa = -1$, there are no conditions for the solvability).

(ii) $p < \nu < 2$. Suppose $T_1(w) = AY(w)X_1(w)$, where Y is defined by (4.5), and $X_1(w) = X(w)(w - c)^{-1}$.

By a suitable choice of A, we can arrive at the fulfilment of the equality

$$(T_1)_*(w) = w^{\varkappa(p)+1}T_1(w), \quad |w| \neq 1,$$

and therefore the result of the previous section with substitution of $\varkappa(p)$ by $\varkappa(p)+1$ holds valid .

Write out the conditions of solvability for $\varkappa(p)+1 < 0$. They have the form

$$\int\limits_{\Gamma} \frac{w^k(t)c(t)w'(t)dt}{T_1^+(w(t))} = 0, \quad k = 0, 1, \dots, |\varkappa + 1| - 2. \tag{4.9}$$

(iii) $\nu = p$. By analogy with subsection 1.2, we find that only the function

$$\Omega_0(w) = Y(w)\Big[P_\varkappa(w) + \frac{M}{w-c}\Big]X(w). \tag{4.10}$$

can be a solution of the homogeneous problem (4.3).

Let

$$\varkappa \geq 0, \quad T_1 \in H^p. \tag{4.11}$$

Then all solutions of the homogeneous problem are given by (4.10), where M is an arbitrary constant and $P_\varkappa$ is an arbitrary polynomial of order $\varkappa$ whose coefficients a_i satisfy (4.7).

However, if

$$\varkappa \geq 0, \quad T_1 \overline{\in} H^p, \tag{4.12}$$

then the general solution is again given by (4.10) with $M = 0$.

For $\varkappa < 0$, the homogeneous problem has only zero solution. The inhomogeneous problem is, generally speaking, unsolvable.

Let the condition (1.20) be fulfilled. Since G_γ at the point c is continuous, and in the neighbourhood of this point $(Y^+)^a \in \bigcup\limits_{\delta>1} W_\delta$ for any $a \in \mathbb{R}$, using Theorem 4.7 of Chapter I, we find that the function

$$\Omega(w) = \frac{T_1(w)}{2\pi i} \int\limits_{\gamma} \frac{g(\zeta)}{T_1^+(\zeta)} \frac{d\zeta}{\zeta - w} \tag{4.13}$$

for $\varkappa \geq 0$ is a solution of the inhomogeneous problem (4.3).

However, if $\varkappa < 0$, then in order for the problem to be solvable, it is necessary and sufficient that the conditions (4.8) be fulfilled.

(iv) $\nu = 2$. For $p < 2$ and $p \geq 2$ we have a diverse picture. If $p < 2$, we can prove as in §1 that the function $[Y(w)X(w)]^{-1}$ is analytically extendable everywhere on γ, with the exception of the point c. This implies that the general solution of the homogeneous problem (4.3) is given by the equality

$$\Omega_0(w) = Y(w)X(w)(w-c)^{-1}P_{\varkappa+1}(w), \tag{4.14}$$

where $P_{\varkappa+1} \equiv 0$ for $\varkappa < -1$, while for $\varkappa > -1$ the coefficients a_i of the polynomial $P_{\varkappa+1}$ are connected by the relation $a_i = \overline{a_{\varkappa+1-i}}$. The inhomogeneous problem is, undoubtedly, solvable for $\varkappa \geq -1$. However, if $\varkappa \leq -2$, then the problem is solvable if and only if the conditions (4.9) are fulfilled. In this case the solution can be written out easily.

If $p > 2$, then $2 = \nu < p$ (this case has been considered in (i)). If $p = 2$, then $\nu = 2 = p$ (this case has been considered in (iii).

(v) $\nu = 0$. For $\varkappa > 0$, the homogeneous problem (4.1) has an infinite number of solutions given by the equality (4.10) and the condition (4.7). However, if $\varkappa \leq 0$, then it has only the zero solution.

The inhomogeneous problem is, generally speaking, unsolvable.

If the condition (1.20) is fulfilled and $\varkappa \geq 0$, and if for $\varkappa < 0$, along with (1.20), there take place condition (4.8), then the inhomogeneous problem is solvable.

Finally we summarize the above-obtained results for the general case.

Theorem 4.1. *Let the Riemann-Hilbert problem be considered in the class $E^p(D)$, where D is bounded by the curve Γ, and $0 \in D$. Assume that:*

(i) *Γ is a simple, piecewise smooth curve containing angular points C_k, $k = \overline{1, n}$, with the angles $\nu_k \pi$, $0 \leq \nu_k \leq 2$;*

(ii) *$G(t) = (a(t) - ib(t))(a(t) + ib(t))^{-1}$ belongs to the class $\widetilde{A}(p)$, $C_k \neq t_i$ where t_i are the p-points of discontinuity of G, $G(t)$ being continuous in small neighbourhoods of the points C_k; $\varkappa(p) = \varkappa(p; G)$ is the index of G and $\varphi_p(\zeta) = \arg G(z(\zeta))$;*

$$Y(w) = \begin{cases} \exp\left(\dfrac{1}{2\pi}\displaystyle\int_\gamma \dfrac{\varphi_p(\zeta)d\zeta}{\zeta - w}\right), & |w| < 1, \\ w^{-\varkappa(p)} \exp\left(\dfrac{1}{2\pi}\displaystyle\int_\gamma \dfrac{\varphi_p(\zeta)d\zeta}{\zeta - w}\right), & |w| > 1; \end{cases} \tag{4.15}$$

(iii) *Let*

$$c_k = w(C_k), \quad h_p = \{c_k : \nu_k \geq p\}, \quad h_0 = \{c_k : \nu_k = 0\},$$
$$h_{p,1} = \left\{c_k : \nu_k = p, \;\; Z(w) = Y(w)(w - c_k)^{-\frac{1}{p}} z_0^{\frac{1}{p}} \overline{\in} H^p\right\},$$
$$h_{0,1} = \left\{c_k : \nu_k = 0, \;\; Z(w) = Y(w)(w - c_k)^{-\frac{1}{p}} z_0^{\frac{1}{p}} \overline{\in} H^p\right\}$$

and let $n_p, n_{p,1}, n_{0,1}$ be numbers of points of the sets h_p, $h_{p,1}$ and $h_{0,1}$, respectively. Here z_0 is the function defined by (2.5) *from Chapter* III.

Further, put

$$T(w) = Y(w)\rho(w), \tag{4.16}$$

where

$$\rho(w) = \begin{cases} \prod\limits_{c_k \in h_p} (w - c_k)^{-1} & \text{if } h_p \neq \varnothing, \\ 1 & \text{if } h_p = \varnothing \end{cases} \tag{4.17}$$

and

$$\varkappa = \varkappa(p; G) + n_p - n_{p,1} - n_{0,1}. \tag{4.18}$$

Then

(1) *All the solutions of the homogeneous problem are given by the equality*

$$\Phi_0(z) = AY(w(z))P_\varkappa(w(z)) \prod_{c_k \in h_{p,1} \cup h_{0,1}} (w(z) - c_k), \tag{4.19}$$

where for $\varkappa \geq 0$, $P_\varkappa(w) = \sum\limits_{k=0}^{\varkappa} a_k w^k$ *is an arbitrary polynomial with the condition*

$$a_j(-1)^\varkappa \prod_{c_k \in h_p} c_k \prod_{c_k \in h_{p,1} \cup h_{0,1}} c_k^{-1} = \overline{a_{\varkappa - j}}, \quad j = \overline{0, \varkappa}, \tag{4.20}$$

and $P_\varkappa(w) = 0$ *if* $\varkappa < 0$. *The constant* A *in* (4.19) *is defined from the condition*

$$(AY)_*(w) = w^\varkappa AY(w). \tag{4.21}$$

(2) *For the inhomogeneous problem we can conclude that:*

If $\varkappa \geq 0$ *and*

$$c(t) \ln \left| \prod_{\nu_k \in \{0,p\}} (w(t) - c_k) \right| \in L^p(\Gamma), \tag{4.22}$$

then the problem is solvable.

If $\varkappa < 0$ *and* (4.22) *holds, then in order for the problem to be solvable, it is necessary and sufficient that*

$$\int_\Gamma \frac{w^k(t)c(t)w'(t)}{T^+(w(t))} dt = 0, \quad k = 0, 1, \ldots, |\varkappa| - 2. \tag{4.23}$$

In all the cases, in which the solution exists, it is given by the equality

$$\Phi(z) = \Phi_c(z) + \Phi_0(z),$$

where Φ_0 *is defined by* (4.19), *and*

$$\Phi_c(z) = \frac{T(w(z))}{2\pi i} \int_\gamma \frac{c(z(\zeta))}{T^+(\zeta)} \frac{\sqrt[p]{z'(\zeta)} d\zeta}{\zeta - w(z)} +$$

$$+w(z)\overline{\Big(\frac{T(\frac{1}{\overline{w(z)}})}{2\pi i}\Big)}\int\limits_{\gamma}\frac{c(z(\zeta))}{T^{+}(\zeta)}\frac{\overline{\sqrt[p]{z'(\zeta)}}d\zeta}{\zeta(\zeta-w(z))}. \tag{4.24}$$

Remark. If Γ is a piecewise Lyapunov curve and $a(t)$, $b(t)$ belong to the Hölder class, then $h_{p,1} = \{c_k : \nu_k = p\}$ and $h_{0,1} = \varnothing$.

Notes and Comments to Chapter IV

The Dirichlet and Neumann problems, as well as Riemann-Hilbert problem for harmonic and analytic functions from Smirnov classes in domains with piecewise Lyapunov curves, containing no cusps with the zero angle have been investigated earlier by V. Kokilashvili and V. Paatashvili [80], [81]. Subsequently, generalization of Warschawski's theorem to the case of non-smooth boundaries, considered in §1 of Chapter III and new two-weight inequalities for singular integrals allowed us to extend the class of boundaries to the problems mentioned above. The results of Chapter IV regarding boundary value problems in domains with piecewise smooth boundaries (containing, generally speaking, cusps of any kind) have been announced earlier in [82] and [83].

The Dirichlet and Neumann problems for domains with boundaries admitting cusps in different functional classes are considered by A. Soloviev and V. Maz'ya and A. Soloviev [93–95].

A vast number of works is available which are devoted to the investigation of these problems in multi-dimensional domains (involving sometimes plane cases) under different assumptions for unknown functions to be harmonic in domains of harmonicity. For the cases with non-regular boundaries the reader can be referred to the papers [85], [96], [11] and etc.

General singular integral equations in a class of curves containing cusps of special type have been studied by R. Duduchava, T. Latsabidze and A. Saginashvili [26].

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