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ASYMPTOTIC DISTRIBUTION OF EIGENFUNCTIONS AND EIGENVALUES OF BASIC BOUNDARY-CONTACT OSCILLATION PROBLEMS OF THE COUPLE-STRESS ELASTICITY THEORY

Abstract. The basic three-dimensional boundary-contact oscillation eigenvalue problems of the couple-stress elasticity theory are considered for a piecewise homogeneous medium. Formulas for asymptotic distribution of eigenfunctions and eigenvalues are derived.

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რეზიუმე. ნაშრომში განხილულია მომენტური დრეკადობის თეორიის რხევის ძირითადი სამგანზომილებიანი სასაზღვრო-საკონტაქტო ამოცანები საკუთრივ მნიშვნელობებზე უბნობრივ-ერთგვაროვანი გარემოსათვის. დადგენილია აღნიშნული ამოცანების საკუთრივი რიცხვებისა და საკუთრივი ვექტორ-ფუნქციების ასიმპტოტური განაწილების ფორმულები.

1. In the present paper the use will be made of the following notation: $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$ are points in R^3; $|x-y|=(\sum\limits_{k=1}^{3}(x_k-y_k)^2)^{1/2}$ is the distance between the points x and y; $D_0\subset R^3$ is a finite domain bounded by closed surfaces $S_0,S_1,\dots,S_m$ of the class $\mathcal{L}_2(\alpha)$, $0<\alpha\leq 1$ [1] (note that S_0 encloses all the remaining domains S_k, while the latter do not enclose each other, $S_i\cap S_k=\varnothing$ for $i\neq k$, $i,k=\overline{0,m}$); the finite domain bounded by the surface $S_k (k=\overline{1,m})$ will be denoted by D_k; $\overline{D}_0=D_0\cup(\bigcup\limits_{k=0}^{m}S_k)$, $\overline{D}_k=D_k\cup S_k$, $k=\overline{1,m}$.

If $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_n)$ are real vectors, then uv is their scalar product: $uv=\sum\limits_{k=1}^{n}u_kv_k$; $|u|=(\sum\limits_{k=1}^{n}u_k^2)^{1/2}$. Multiplication of matrices is performed according to the rule: a row by a column; if $A=\|A_{ij}\|_{n\times n}$ is a matrix of dimension $n\times n$, then $|A|^2=\sum\limits_{i,j=1}A_{ij}^2$. Every vector $u=(u_1,\dots,u_n)$ is considered to be a one-column $n\times 1$-matrix: $u=\|u_i\|_{n\times 1}$; by $A_k=\|A_{jk}\|_{j=1}^{n}$ it will be denoted the k-th column vector of the matrix A.

The vector $u(x)=(u_1,(x),\dots,u_n(x))$ is said to be regular in D_k, if $u_i\in C^1(\overline{D}_k)\cap C^2(D_k)$, $\quad i=\overline{1,n}$.

The system of homogeneous oscillatory differential equations of the couple-stress elasticity theory for a homogeneous isotropic centrally symmetric medium is of the form [1]

$$\begin{cases}(\mu+\alpha)\Delta u+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}u+2\alpha\operatorname{rot}\omega+\rho\sigma^2u=0,\\(\nu+\beta)\Delta\omega+(\varepsilon+\nu-\beta)\operatorname{grad}\operatorname{div}\omega+2\alpha\operatorname{rot}u-4\alpha\omega+I\sigma^2\omega=0,\end{cases}\tag{1}$$

where Δ is the three-dimensional Laplace operator, $u(x)=(u_1,u_2,u_3)$ is the displacement vector, $w(x)=(w_1,w_2,w_3)$ is the rotation vector, $\rho=\text{const}>0$ is the density of the medium, $I=\text{const}>0$ is the moment of inertia, σ is the oscillation frequency, and λ, μ, α, ν, β, ε are elastic constants satisfying $\mu>0$, $3\lambda+2\mu>0$, $\alpha>0$, $\nu>0$, $3\varepsilon+2\nu>0$, $\beta>0$.

Introduce the matrix differential operator

$$M\equiv M(\partial x)=\big\|M_{ij}(\partial x)\big\|_{6\times 6}=\begin{Vmatrix}M^{(1)}(\partial x) & M^{(2)}(\partial x)\\ M^{(3)}(\partial x) & M^{(4)}(\partial x)\end{Vmatrix},$$

$$M^{(k)}(\partial x)=\big\|M_{ij}^{(k)}(\partial x)\big\|_{3\times 3},\quad k=\overline{1,4};$$

$$M^{(1)}(\partial x)=(\mu+\alpha)\delta_{ij}\Delta+(\lambda+\mu-\alpha)\frac{\partial^2}{\partial x_i\partial x_j},$$

$$M_{ij}^{(2)}(\partial x)=M_{ij}^{(3)}(\partial x)=-2\alpha\sum_{k=1}^{3}\varepsilon_{ijk}\frac{\partial}{\partial x_k},$$

$$M_{ij}^{(4)}(\partial x)=\delta_{ij}[(\nu+\beta)\Delta-4\alpha]+(\varepsilon+\nu-\beta)\frac{\partial^2}{\partial x_i\partial x_j},$$

where δ_{ij} and ε_{ijk} are respectively the Kronecker and Levi-Civita symbols. Then the system (1) written in the matrix vector form looks as follows:

$$M(\partial x)v(x) + r\sigma^2 v(x) = 0, \tag{2}$$

where $v(x) = (u(x), w(x)) = (u_1, u_2, u_3, w_1, w_2, w_3) = (v_1, v_2, \dots, v_6)$; r is a diagonal 6×6-matrix: $r = \|r_{ij}\|_{6\times 6}$; note that $r_{ij} = 0$ for $i \neq j$, $r_{ii} = \rho$ for $i = 1, 2, 3$ and $r_{ii} = I$ for $i = 4, 5, 6$.

Rewrite the equation (2) as

$$\widetilde{M}(\partial x)\widetilde{v}(x) + \sigma^2 \widetilde{v}(x) = 0, \tag{3}$$

where $\widetilde{M} = \widetilde{r}^{-1} M \widetilde{r}^{-1}$, $\widetilde{v} = \widetilde{r}v$, $\widetilde{r} = \|\sqrt{r_{ij}}\|_{6\times 6}$.

Denote by $T(\partial x, n(x))$ the stress operator of the couple-stress elasticity which is a matrix differential operator of dimension 6×6:

$$T \equiv T(\partial x, n(x)) = \big\|T_{ij}(\partial x, n(x))\big\|_{6\times 6} = \begin{Vmatrix} T^{(1)}(\partial x, n(x)) & T^{(2)}(\partial x, n(x)) \\ T^{(3)}(\partial x, n(x)) & T^{(4)}(\partial x, n(x)) \end{Vmatrix},$$

$$T^{(k)}(\partial x, n(x)) = \big\|T_{ij}^{(k)}(\partial x, n(x))\big\|_{3\times 3}, \quad k = \overline{1,4};$$

$$T_{ij}^{(1)}(\partial x, n(x)) = \lambda n_i(x)\,\frac{\partial}{\partial x_j} + (\mu - \alpha)n_j(x)\,\frac{\partial}{\partial x_i} + (\mu + \alpha)\delta_{ij}\,\frac{\partial}{\partial n(x)},$$

$$T_{ij}^{(2)}(\partial x, n(x)) = -2\alpha \sum_{k=1}^{3} \varepsilon_{ijk} n_k(x), \quad T_{ij}^{(3)}(\partial x, n(x)) = 0,$$

$$T_{ij}^{(4)}(\partial x, n(x)) = \varepsilon n_i(x)\,\frac{\partial}{\partial x_j} + (\nu - \beta)n_j(x)\,\frac{\partial}{\partial x_i} + (\nu + \beta)\delta_{ij}\,\frac{\partial}{\partial n(x)},$$

where $n(x)$ is an arbitrary unit vector at the point x (if $x \in S_k$, $k = \overline{0,m}$, then $n(x)$ is the outer (with respect to D_0) normal to S_k at x).

The domains $D_k (k = \overline{0, m_0})$ are assumed to be filled with homogeneous isotropic elastic media with constants λ_k, μ_k, α_k, ν_k, β, ε, and the remaining domains $D_k (k = \overline{m_0 + 1, m})$ are assumed to be empty inclusions. If in the operators M and T instead of λ, μ, α, ν, β, ε there appear λ_k, μ_k, α_k, ν_k, β_k, ε_k then we will write $\overset{k}{M}$ and $\overset{k}{T}$, respectively.

Introduce the notation

$$v^+(z) = \lim_{D_0 \ni x \to z \in S_k} v(x), \quad k = \overline{0,m}, \quad v^-(z) = \lim_{D_k \ni x \to z \in S_k} v(x), \quad k = \overline{1, m_0}.$$

The notation $(T(\partial z, n(z))v(z))^{\pm}$ has a similar meaning.

2. The matrix of fundamental solutions of the homogeneous oscillation equation (2) has the form [1]

$$\Gamma(x - y, \sigma^2) = \big\|\Gamma_{kj}(x - y, \sigma^2)\big\|_{6\times 6} = \begin{Vmatrix} \Gamma^{(1)}(x - y, \sigma^2) & \Gamma^{(2)}(x - y, \sigma^2) \\ \Gamma^{(3)}(x - y, \sigma^2) & \Gamma^{(4)}(x - y, \sigma^2) \end{Vmatrix},$$

where

$$\Gamma^{(e)}(x-y,\sigma^2)=\big\|\Gamma^{(e)}_{kj}(x-y,\sigma^2)\big\|_{3\times 3},\quad e=\overline{1,4};$$

$$\Gamma^{(1)}_{kj}(x-y,\sigma^2)=\sum_{e=1}^{4}\big(\delta_{kj}\alpha_e+\beta_e\,\frac{\partial^2}{\partial x_k\partial x_j}\big)\frac{e^{ik_er}}{r}, \tag{4}$$

$$\Gamma^{(2)}_{kj}(x-y,\sigma^2)=\Gamma^{(3)}_{kj}(x-y,\sigma^2)=\ \frac{2\alpha}{\mu+\alpha}\sum_{e=1}^{4}\sum_{p=1}^{3}\varepsilon_e\varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{e^{ik_er}}{r},$$

$$\Gamma^{(4)}_{kj}(x-y,\sigma^2)=\sum_{e=1}^{4}(\delta_{kj}\gamma_e+\delta_e\frac{\partial^2}{\partial x_k\partial x_j})\frac{e^{ik_er}}{r},$$

$$\alpha_e=\frac{(-1)^e(\sigma_1^2-k_e^2)(\delta_{3e}+\delta_{4e})}{4\pi(\mu+\alpha)(k_3^2-k_4^2)},\quad \beta_e=-\frac{\delta_{1e}}{4\pi\rho\sigma^2}+\frac{\alpha_e}{k_e^2},\ \sum_{e=1}^{4}\beta_e=0,$$

$$\gamma_e=\frac{(-1)^e(\sigma_1^2-k_e^2)(\delta_{3e}+\delta_{4e})}{4\pi(\beta+\nu)(k_3^2-k_4^2)},\quad \delta_e=-\frac{\delta_{2e}}{4\pi(I\sigma^2-4\alpha)}+\frac{\gamma_e}{k_e^2},\ \sum_{e=1}^{4}\delta_e=0,$$

$$\varepsilon_e=\frac{(-1)^e(\delta_{3e}+\delta_{4e})}{4\pi(\beta+\nu)(k_3^2-k_4^2)},\quad \sum_{e=1}^{4}\varepsilon_e=0,\quad r=|x-y|;$$

here

$$\sigma_1^2=\frac{\rho\sigma^2}{\mu+\alpha},\ \ \sigma_2^2=\frac{I\sigma^2-4\alpha}{\nu+\beta},\ \ k_1^2=\frac{\rho\sigma^2}{\lambda+2\mu},\ \ k_2^2=\frac{I\sigma^2-4\alpha}{\varepsilon+2\nu},$$

k_3^2 and k_4^2 satisfy

$$k_3^2+k_4^2=\sigma_1^2+\sigma_2^2+\frac{4\alpha^2}{(\mu+\alpha)(\nu+\beta)},\quad k_3^2\cdot k_4^2=\sigma_1^2\cdot\sigma_2^2,$$

and for large frequencies they assume the following asymptotic values [6]:

$$k_3^2=\frac{\rho\sigma^2}{\mu+\alpha},\quad k_4^2=\frac{I\sigma^2}{\nu+\beta}. \tag{5}$$

Since in the sequel we will be concerned with the asymptotics under large frequencies σ^2, we assume that k_3^2 and k_4^2 are defined by means of (5) and, moreover,

$$\sigma_2^2=\frac{I\sigma^2}{\nu+\beta},\quad k_2^2=\frac{I\sigma^2}{\varepsilon+2\nu}. \tag{6}$$

Let $\varkappa_0$ be an arbitraryly fixed real positive number and $\varkappa>\varkappa_0$ be an arbitrary number. If in (4) we substitute $\sigma=i\varkappa$ and take into account (5) and (6), then we get

$$
\begin{aligned}
&\Gamma^{(1)}_{kj}(x-y,-\varkappa^2)=\frac{\delta_{kj}}{4\pi(\mu+\alpha)}\,\frac{e^{-\frac{\varkappa r}{c^2}}}{r}+\frac{1}{4\pi\rho\varkappa^2}\,\frac{\partial^2}{\partial x_k\partial x_j}\frac{e^{-\frac{\varkappa r}{c_1}}-e^{-\frac{\varkappa r}{c_2}}}{r},\\
&\Gamma^{(2)}_{kj}(x-y,-\varkappa^2)=\Gamma^{(3)}_{kj}(x-y,-\varkappa^2)=\frac{2\alpha}{4\pi\varkappa^2(\rho(\nu+\beta)-I(\mu+\alpha))}\times\\
&\qquad\qquad\times\sum_{p=1}^{3}\varepsilon_{kjp}\frac{\partial}{\partial x_p}\,\frac{e^{-\frac{\varkappa r}{c_2}}-e^{-\frac{\varkappa r}{c_4}}}{r},\\
&\Gamma^{(4)}_{kj}(x-y,-\varkappa^2)=\frac{\delta_{kj}}{4\pi(\beta+\nu)}\,\frac{e^{-\frac{\varkappa r}{c_4}}}{r}+\frac{1}{4\pi I\varkappa^2}\,\frac{\partial^2}{\partial x_k\partial x_j}\frac{e^{-\frac{\varkappa r}{c_3}}-e^{-\frac{\varkappa r}{c_4}}}{r},
\end{aligned}
\tag{7}
$$

where $c_1^2=(\lambda+2\mu)\rho^{-1}$, $c_2^2=(\mu+\alpha)\rho^{-1}$, $c_3^2=(\varepsilon+2\nu)I^{-1}$, $c_4^2=(\nu+\beta)I^{-1}$.

For the sake of convenience, in what follows the use will be made of the equation (3). The matrix of fundamental solutions for the equation (3) is $\widetilde{\Gamma}(x-y,\sigma^2)=\widetilde{r}\Gamma(x-y,\sigma^2)\widetilde{r}$, and under the substitution $\sigma=i\varkappa$ it follows from (7) that

$$
\begin{aligned}
&\widetilde{\Gamma}^{(1)}_{kj}(x-y,-\varkappa^2)=\frac{\rho\sigma_{kj}}{4\pi(\mu+\alpha)}\,\frac{e^{-\frac{\varkappa r}{c_2}}}{r}+\frac{1}{4\pi\varkappa^2}\,\frac{\partial^2}{\partial x_k\partial x_j}\frac{e^{-\frac{\varkappa r}{c_1}}-e^{-\frac{\varkappa r}{c_2}}}{r},\\
&\widetilde{\Gamma}^{(2)}_{kj}(x-y,-\varkappa^2)=\widetilde{\Gamma}^{(3)}_{kj}(x-y,-\varkappa^2)=\frac{2\alpha\sqrt{\rho I}}{4\pi\varkappa^2(\rho(\nu+\beta)-I(\mu+\alpha))}\times\\
&\qquad\qquad\times\sum_{p=1}^{3}\varepsilon_{kjp}\frac{\partial}{\partial x_p}\,\frac{e^{-\frac{\varkappa r}{c_2}}-e^{-\frac{\varkappa r}{c_4}}}{r},\\
&\widetilde{\Gamma}^{(4)}_{kj}(x-y,-\varkappa^2)=\frac{I\sigma_{kj}}{4\pi(\beta+\nu)}\,\frac{e^{-\frac{\varkappa r}{c_4}}}{r}+\frac{1}{4\pi\varkappa^2}\,\frac{\partial^2}{\partial x_k\partial x_j}\frac{e^{-\frac{\varkappa r}{c_3}}-e^{-\frac{\varkappa r}{c_4}}}{r}.
\end{aligned}
\tag{8}
$$

Denote by c_0 the largest number among c_1, c_2, c_3, c_4. Let $\delta<\frac{1}{2c_0}$ be an arbitrary positive number, and let $a=\frac{1}{c_0}-\delta>0$. Then we have

$$
\frac{e^{-\frac{\varkappa r}{c_k}}}{r}=\frac{e^{-a\varkappa r}}{r}e^{-(\frac{1}{c_k}-a)\varkappa r},\quad \frac{1}{c_k}-a>0;\quad k=\overline{1,4}.
$$

Moreover,

1) $\displaystyle\frac{\partial}{\partial x_j}\Big(\frac{e^{-a\varkappa r}}{r}e^{-(\frac{1}{c_k}-a)\varkappa r}\Big)=\frac{e^{-a\varkappa r}}{r^2}\cdot e^{-(\frac{1}{c_k}-a)\varkappa r}\Big(-1-\frac{1}{c_k}\varkappa r\Big)\frac{\partial r}{\partial x_j}$;
2) $\displaystyle\frac{\partial^2}{\partial x_k\partial x_j}\Big(\frac{e^{-a\varkappa r}}{r}e^{-(\frac{1}{c_k}-a)\varkappa r}\Big)=\frac{e^{-a\varkappa r}}{r^3}\cdot e^{-(\frac{1}{c_k}-a)\varkappa r}\Big[\Big(3+\frac{3}{c_k}\varkappa r+ +\frac{1}{c_k^2}\varkappa^2r^2\Big)\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}+\Big(-1-\frac{1}{c_k}\varkappa r\Big)\delta_{kj}\frac{\partial r}{\partial x_k}\Big]$;
3) the functions $(\varkappa r)^n e^{-(\frac{1}{c_k}-a)^{\varkappa r}}$, $0,1,2,\dots$ are bounded in the interval $\varkappa\in[0,+\infty)$.

From this and the equation (8), we obtain the estimates

$$\Big|\frac{\partial^n \widetilde{\Gamma}^{(e)}_{pq}(x-y,-\varkappa^2)}{\partial x_1^i \partial x_2^j \partial x_3^k}\Big| \leq \frac{\text{const}}{r^{n+1}} e^{-a\varkappa r}, \quad i+j+k=n; \quad n=0,1,2,\dots \tag{9}$$
$$p,q=\overline{1,3}; \quad e=\overline{1,4}.$$

3. Let $x, y \in D_k$, $k=\overline{0,m_0}$ and let l_y be the distance from the point y to the boundary of D_k. Denote $\rho_y(x)=\max\{r,l_y\}$, and introduce an auxiliary matrix

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)=\Big[1-\Big(1-\frac{r^m}{\rho_y^m(x)}\Big)^n\Big]\overset{k}{\widetilde{\Gamma}}(x-y,-\varkappa^2). \tag{10}$$

Denote by $\mathbb{B}(y,l_y)$ the sphere of radius l_y with the center at the point y and let $c(y,l_y)$ be its boundary. It is easy to see that $(1-\frac{r^m}{\rho_y^m(x)})^n$ vanishes together with its derivatives up to the $(n-1)$-th order inclusive, when the point $x\in\mathbb{B}(y,l_y)$ tends to a point of the boundary $c(y,l_y)$. For $x\in D_k\backslash\overline{\mathbb{B}(y,l_y)}$, we have

$$1-\Big(1-\frac{r^m}{\rho_y^m(x)}\Big)^n=1 \quad \text{and} \quad \lim_{\mathbb{B}(y,l_y)\ni x\to z\in c(y,l_y)}\Big[1-\Big(1-\frac{r^m}{\rho_y^m(x)}\Big)^n\Big]=1.$$

Thus $\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)=\overset{k}{\widetilde{\Gamma}}(x-y,-\varkappa^2)$ for $x\in D_k\backslash\mathbb{B}(y,l_y)$, while the function $\overset{\widehat{k}}{\Gamma}$ and its derivatives up to the $(n-1)$-th order inclusive remain continuous when crossing the boundary $c(y,l_y)$.

Represent the function $\overset{\widehat{k}}{\Gamma}$ as follows:

$$\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)=\overset{k}{\widetilde{\Gamma}}(x-y,-\varkappa^2)(nr^m/\rho_y^m(x)+\cdots).$$

It can be easily seen that for $x=y$ the function $\overset{\widehat{k}}{\Gamma}$ and its derivatives up to the $(m-2)$-th order inclusive are continuous, and for $x\in\mathbb{B}(y,l_y)$ we have by virtue of (9) the estimates

$$\Big|\frac{\partial^s \overset{\widehat{k}}{\Gamma}_{pq}(x-y,-\varkappa^2)}{\partial x_1^i \partial x_2^j \partial x_3^k}\Big| \leq \frac{\text{const}\cdot e^{-a\varkappa r}}{l_y^m}\, r^{m-s-1}, \tag{11}$$
$$p,g=\overline{1,6}, \quad i+j+k=s, \quad m\geq s+1.$$

4. Calculate the limit

$$\lim_{x\to y}[\overset{k}{\widetilde{\Gamma}}(x-y,-\varkappa^2)-\overset{k}{\Gamma}(x-y,-\varkappa^2)], \quad x,y\in D_k, \quad k=\overline{0,m_0}.$$

Taking into account the expansion

$$\frac{e^{-\frac{\varkappa r}{c_1}}}{r} = \frac{1}{r} - \frac{\varkappa}{c_1} + \frac{\varkappa^2}{2!c_1^2}\,r - \frac{\varkappa^3}{3!c_1^3}\,r^2 + \cdots,$$

we obtain

1) $\dfrac{e^{-\frac{\varkappa r}{c_2}}}{r} - \dfrac{e^{-\frac{\varkappa_0 r}{c_2}}}{r} = \dfrac{\varkappa_0 - \varkappa}{c_2} + \dfrac{\varkappa^2 - \varkappa_0^2}{2!c_2^2}\,r + \cdots$

2) $\dfrac{e^{-\frac{\varkappa r}{c_2}}}{r} - \dfrac{e^{-\frac{\varkappa r}{c_2}}}{r} = \varkappa\Big(\dfrac{1}{c_1} + \dfrac{1}{c_2}\Big) + \varkappa^2\Big(\dfrac{1}{2!c_1^2} - \dfrac{1}{2!c_2^2}\Big)r +$
$\varkappa^3\Big(-\dfrac{1}{3!c_1^3} + \dfrac{1}{3!c_2^3}\Big)r^2 + \cdots$

3) $\dfrac{\partial^2}{\partial x_k \partial x_j}\dfrac{e^{-\frac{\varkappa r}{c_1}} - e^{-\frac{\varkappa r}{c_2}}}{r} = \varkappa^2\Big(\dfrac{1}{2!c_1^2} - \dfrac{1}{2!c_2^2}\Big)\dfrac{\partial^2 r}{\partial x_k \partial x_j} +$
$+\varkappa^3\Big(\dfrac{1}{6c_2^3} - \dfrac{1}{6c_1^3}\Big)\dfrac{\partial^2 r^2}{\partial x_k \partial x_j},$

4) $\dfrac{\partial^2 r^2}{\partial x_k \partial x_j} = 2\delta_{kj},$

5) $\lim\limits_{x\to y}\big[\overset{k}{\widetilde{\Gamma}}{}^{(2)}(x-y,-\varkappa^2) - \overset{k}{\widetilde{\Gamma}}{}^{(2)}(x-y,-\varkappa_0^2)\big] = 0,$

6) $\lim\limits_{x\to y}\big[\overset{k}{\widetilde{\Gamma}}{}^{(3)}(x-y,-\varkappa^2) - \overset{k}{\widetilde{\Gamma}}{}^{(3)}(x-y,-\varkappa_0^2)\big] = 0.$

The above relations result in

$$\lim_{x\to y}\big[\overset{k}{\widetilde{\Gamma}}{}^{(1)}_{pq}(x-y,-\varkappa^2) - \overset{k}{\widetilde{\Gamma}}{}^{(1)}_{pq}(x-y,-\varkappa_0^2)\big] =$$
$$= \frac{(\varkappa_0 - \varkappa)\rho_k^{3/2}}{4\pi(\mu_k+\alpha_k)^{3/2}}\delta_{pq} + \frac{(\varkappa_0 - \varkappa)\rho_k^{3/2}}{12\pi}\Big(\frac{1}{(\lambda_k+2\mu_k)^{3/2}} - \frac{1}{(\mu_k+\alpha_k)^{3/2}}\Big)\delta_{pq} =$$
$$= \frac{(\varkappa_0 - \varkappa)\rho_k^{3/2}}{12\pi}\Big(\frac{1}{(\lambda_k+2\mu_k)^{3/2}} + \frac{2}{(\mu_k+\alpha_k)^{3/2}}\Big)\delta_{pq}, \qquad (12)$$
$$\lim_{x\to y}\big[\overset{k}{\widetilde{\Gamma}}{}^{(4)}_{pq}(x-y,-\varkappa^2) - \overset{k}{\widetilde{\Gamma}}{}^{(4)}_{pq}(x-y,-\varkappa_0^2)\big] =$$
$$= \frac{(\varkappa_0 - \varkappa)I_k^{3/2}}{12\pi}\Big(\frac{1}{(\varepsilon_k+2\nu_k)^{3/2}} + \frac{2}{(\nu_k+\beta_k)^{3/2}}\Big)\delta_{pq}, \quad p,q=\overline{1,3}. \qquad (13)$$

5. Our further investigation will deal with the second boundary-contact problem, when stresses are prescribed on the boundary. Investigation of other problems is similar.

Green's tensor of the second basic boundary-contact problem for the operator $\overset{k}{\widetilde{M}}(\partial x) - E\varkappa_0^2$ (E is the unit matrix of dimension 6×6) will be defined as a 6×6-matrix $G(x,y,-\varkappa_0^2)=\overset{k}{G}(x,y,-\varkappa_0^2)$, $x\in D_k$, $y\in D$ ($D=\bigcup_{k=0}^{m_0}$) satisfying

1) $\forall x\in D_k$, $\forall y\in D$, $x\neq y$:

$$\overset{k}{\widetilde{M}}(\partial x)\overset{k}{G}(x,y,-\varkappa_0^2)-\varkappa_0^2\overset{k}{G}(x,y,-\varkappa_0^2)=0,\quad k=\overline{0,m_0};$$

2) $\forall z\in S_k$, $\forall y\in D$: $\quad(\overset{\circ}{G}{}^+(z,y,-\varkappa_0^2)=\overset{k}{G}{}^-(z,y,-\varkappa_0^2)$,

$$\big(\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{G}(z,y,-\varkappa_0^2)\big)^+=\big(\overset{k}{\widetilde{T}}(\partial z,n(z))\overset{k}{G}(z,y,-\varkappa_0^2)\big)^-,\quad k=\overline{1,m_0};$$

3) $\forall z\in S_k$, $\forall y\in D$: $\big(\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{G}(z,y,-\varkappa_0^2)\big)^+=0$, $k=0,m_0+1,\dots,m$;

4) $\forall x\in D_k$, $\forall y\in D$:

$$\overset{k}{G}(x,y,-\varkappa_0^2)=\overset{k}{\widetilde{\Gamma}}(x-y,-\varkappa_0^2)-\overset{k}{g}(x,y,-\varkappa_0^2),\quad k=\overline{0,m_0};$$

here $\overset{k}{g}(x,y,-\varkappa_0^2)$ is the regular in D_k solution of the following problem:

1) $\forall x\in D_k$, $\forall y\in D$: $\overset{k}{\widetilde{M}}\overset{k}{g}(x,y,-\varkappa_0^2)-\varkappa_0^2\overset{k}{g}(x,y,-\varkappa_0^2)=0$, $k=\overline{0,m_0}$;

2) $\forall z\in S_k$, $\forall y\in D$: $\quad\overset{\circ}{g}{}^+(z,y,-\varkappa_0^2)-\overset{k}{g}{}^-(z,y,-\varkappa_0^2)=$

$$=\overset{\circ}{\widetilde{\Gamma}}(z-y,-\varkappa_0^2)-\overset{k}{\widetilde{\Gamma}}(z-y,-\varkappa_0^2);$$

$$\big(\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{g}(z,y,-\varkappa_0^2)\big)^+-\big(\overset{k}{\widetilde{T}}(\partial z,n(z))(\overset{k}{g}(z,y,-\varkappa_0^2)\big)^-=$$

$$=\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{\widetilde{\Gamma}}(z,y,-\varkappa_0^2)-\overset{k}{\widetilde{T}}(\partial z,n(z))\overset{k}{\widetilde{\Gamma}}(z,y,-\varkappa_0^2),k=\overline{1,m_0};$$

3) $\forall z\in S_k,\forall y\in D$: $\big(\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{g}(z,y,-\varkappa_0^2)\big)^+=$

$$=\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{\widetilde{\Gamma}}(z-y,-\varkappa_0^2),\quad k=0,m_0+1,\dots,m,$$

where $\widetilde{T}=\widetilde{r}^{-1}T\widetilde{r}^{-1}$.

The solvability of this problem is proved in [1], and thus we have the existence of $G(x,y,-\varkappa_0^2)$. As is known [1], $G(x,y,-\varkappa_0^2)$ possesses the symmetry property of the kind

$$G(x,y,-\varkappa_0^2)=G^T(y,x,-\varkappa_0^2),\tag{14}$$

where the symbol $''T''$ denotes the transposition of a matrix. Moreover, the following estimates are valid [2]:

$$\left.\begin{aligned}\forall(x,y)\in D_k\times D_k:\quad & G_{pq}(x,y,-\varkappa_0^2)=O(|x-y|^{-1}),\\ & \frac{\partial}{\partial x_j}G_{pq}(x,y,-\varkappa_0^2)=O(|x-y|^{-2}),\\ & p,q=\overline{1,6};\quad j=\overline{1,3};\quad k=\overline{0,m_0}.\end{aligned}\right\}\tag{15}$$

6. Let $u(x)=\overset{k}{u}(x)$ and $v(x)=\overset{k}{v}(x)$, $x\in D_k$, be arbitrary six-component (regular) vectors of the class $C^1(\overline{D}_k)\cap C^2(D_k)$, $k=\overline{0,m_0}$. Then the following Green's formulas are valid [1]:

$$\sum_{k=0}^{m_0}\int_{D_k}\Big[\overset{k}{v}\overset{k}{\widetilde{M}}\overset{k}{u}+\overset{k}{\widetilde{E}}(\overset{k}{v},\overset{k}{u})\Big]dx=\int_{S_0}\overset{\circ}{v}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^+ds+$$
$$+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{v}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^+ds+\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{\circ}{v}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^+-\overset{k}{v}{}^-(\overset{k}{\widetilde{T}}\overset{k}{u})^-\Big]ds; \quad (16)$$

$$\sum_{k=0}^{m_0}\int_{D_k}\Big(\overset{k}{v}\overset{k}{\widetilde{M}}\overset{k}{u}-\overset{k}{u}\overset{k}{\widetilde{M}}\overset{k}{v}\Big)dx=\int_{S_0}\Big[\overset{\circ}{v}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^+-\overset{\circ}{u}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{v})^+\Big]ds+$$
$$+\sum_{k=m_0+1}^{m}\int_{S_k}\Big[\overset{\circ}{v}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^+-\overset{\circ}{u}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{v})^+\Big]ds+$$
$$+\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{\circ}{v}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^+-\overset{k}{v}{}^-(\overset{k}{\widetilde{T}}\overset{k}{u})^-\Big]ds-\sum_{k=1}^{m_0}\int_{S_k}\Big[\overset{\circ}{u}{}^+(\overset{\circ}{\widetilde{T}}\overset{\circ}{v})^+-\overset{k}{u}{}^-(\overset{k}{\widetilde{T}}\overset{k}{v})^-\Big]ds, \quad (17)$$

where $\overset{k}{\widetilde{E}}(\overset{k}{v},\overset{k}{u})=\overset{k}{E}(\widetilde{r}^{-1}\overset{k}{v},\,\widetilde{r}^{-1}\overset{k}{u})$, and

$$\overset{k}{E}(\overset{k}{v},\overset{k}{u})=\frac{3\lambda+2\mu}{3}\sum_{i,j=1}^{3}\frac{\partial\overset{k}{v}_j}{\partial x_j}\frac{\partial\overset{k}{u}_i}{\partial x_i}+\frac{\mu}{2}\sum_{i,j=1}^{3}\Big[\frac{\partial\overset{k}{v}_i}{\partial x_j}+\frac{\partial\overset{k}{v}_j}{\partial x_i}-$$
$$-\frac{2}{3}\delta_{ij}\sum_{p=1}^{3}\frac{\partial\overset{k}{v}_p}{\partial x_p}\Big]\Big[\frac{\partial\overset{k}{u}_i}{\partial x_j}+\frac{\partial\overset{k}{u}_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\sum_{p=1}^{3}\frac{\partial\overset{k}{u}_p}{\partial x_p}\Big]+$$
$$+\frac{\alpha}{2}\sum_{i,j=1}^{3}\Big[\frac{\partial\overset{k}{v}_j}{\partial x_i}-\frac{\partial\overset{k}{v}_i}{\partial x_j}+2\sum_{p=1}^{3}\varepsilon_{pji}\overset{k}{v}_{p+3}\Big]\Big[\frac{\partial\overset{k}{u}_j}{\partial x_i}-\frac{\partial\overset{k}{u}_i}{\partial x_j}+2\sum_{p=1}^{3}\varepsilon_{pji}\overset{k}{u}_{p+3}\Big]+$$
$$+\frac{3\varepsilon+2\nu}{3}\sum_{i,j=1}^{3}\frac{\partial\overset{k}{v}_{i+3}}{\partial x_i}\frac{\partial\overset{k}{u}_{j+3}}{\partial x_j}+\frac{\nu}{2}\sum_{i,j=1}^{3}\Big[\frac{\partial\overset{k}{v}_{i+3}}{\partial x_j}+\frac{\partial\overset{k}{v}_{j+3}}{\partial x_i}-$$
$$-\frac{2}{3}\delta_{ij}\sum_{p=1}^{3}\frac{\partial\overset{k}{v}_{p+3}}{\partial x_p}\Big]\Big[\frac{\partial\overset{k}{u}_{i+3}}{\partial x_j}+\frac{\partial\overset{k}{u}_{j+3}}{\partial x_i}-\frac{2}{3}\delta_{ij}\sum_{p=1}^{3}\frac{\partial\overset{k}{u}_{p+3}}{\partial x_p}\Big]+$$
$$+\frac{\beta}{2}\sum_{i,j=1}^{3}\Big[\frac{\partial\overset{k}{v}_{j+3}}{\partial x_i}-\frac{\partial\overset{k}{v}_{i+3}}{\partial x_j}\Big]\Big[\frac{\partial\overset{k}{u}_{j+3}}{\partial x_i}-\frac{\partial\overset{k}{u}_{i+3}}{\partial x_j}\Big]. \quad (18)$$

It follows from (18) that $\overset{k}{E}(\overset{k}{v},\overset{k}{u})=\overset{k}{E}(\overset{k}{u},\overset{k}{v})$ and $\overset{k}{E}(\overset{k}{v},\overset{k}{v})\geq 0$.

For a regular in $D_k(k=\overline{0,m_0})$ vector $u(x)$, the following general integral representation is valid [1]:

$$\forall y \in D_k : u_j(y) = -\sum_{k=0}^{m_0} \int_{D_k} \overset{k}{\tilde{\Gamma}}_j(x-y,-\varkappa^2)\,[\overset{k}{\widetilde{M}}(\partial x)\overset{k}{u}(x)\varkappa^2\overset{k}{u}(x)]\,dx +$$

$$+\int_{S_0} [\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)\,(\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{u}(z))^+ -$$

$$-\overset{\circ}{u}{}^+(z)\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)]\,d_z s+$$

$$+\sum_{k=m_0+1}^{m} \int_{S_k} [\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)(\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{u}(z))^+ -$$

$$-\overset{\circ}{u}{}^+(z)\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)]\,d_z s +$$

$$+\sum_{k=1}^{m_0} \int_{S_k} [\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)(\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{u}(z))^+ -$$

$$-\overset{k}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)(\overset{k}{\tilde{T}}(\partial z,n(z))\overset{k}{u}(z))^-]\,d_z s -$$

$$-\sum_{k=1}^{m_0} \int_{S_k} [\overset{\circ}{u}{}^+(z)\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2) -$$

$$-\overset{k}{u}{}^-(z)\overset{k}{\tilde{T}}(\partial z,n(z))\overset{k}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)]\,d_z s, \quad j=\overline{1,6}. \tag{19}$$

7. To determine the asymptotic behaviour of eigenfunctions and eigenvalues by T. Carleman's method, it is necessary to estimate the regular part of Green's tensor $g(x,y,-\varkappa^2)$ as $\varkappa \to \infty$. To this end, we consider the functional

$$L[u]=\sum_{k=0}^{m_0} \int_{D_k} [\overset{k}{E}(\overset{k}{u},\overset{k}{u})+\varkappa^2\overset{k}{u}{}^2]\,dx-2\int_{S_0} \overset{\circ}{u}{}^+\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s-$$

$$-2\sum_{k=m_0+1}^{m} \int_{S_k} \overset{\circ}{u}{}^+\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s -$$

$$-2\sum_{k=1}^{m_0} \int_{S_k} [\overset{\circ}{u}{}^+\overset{\circ}{\tilde{T}}(\partial z,n(z))\overset{\circ}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s -$$

$$-\overset{k}{u}{}^-\overset{k}{\tilde{T}}(\partial z,n(z))\overset{k}{\tilde{\Gamma}}_j(z-y,-\varkappa^2)]\,d_z s, \tag{20}$$

where $j=\overline{1,6}$ is a fixed number and y is an arbitrary fixed point in D_k, $k=\overline{0,m_0}$. We define the functional L in the class of regular in $D_k(k=\overline{0,m_0})$ vector functions satisfying the following conditions:

1) $\forall z \in S_k$: $\overset{\circ}{u}{}^{+}(z) - \overset{k}{u}{}^{-}(z) = \overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2) - \overset{k}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)$,

$$\big(\overset{\circ}{\widetilde{T}}\overset{\circ}{u}(z)\big)^{+} - \big(\overset{k}{\widetilde{T}}\overset{k}{u}(z)\big)^{-} = \overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2) - \overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2), \quad k=\overline{1,m_0};$$

2) $\forall z \in S_k$: $\big(\overset{\circ}{\widetilde{T}}\overset{\circ}{u}(z)\big)^{+} = \overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)$, $k=0,m_0+1,\dots,m$.

Theorem 1. *The functional L takes its minimal value for $u=g_j(x,y,-\varkappa^2)$.*

Proof. Let u be an arbitrary vector from the domain of definition of the functional L, and let $v=u-g_j$. Then, taking into account (18), we find from (20) that

$$L[u] = L[v+g_j] = \sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v}+\overset{k}{g}_j,\overset{k}{v}+\overset{k}{g}_j) + \varkappa^2(\overset{k}{v}+\overset{k}{g}_j)^2\big]\,dx -$$
$$-2\int_{S_0}\big[(\overset{\circ}{v}{}^{+}(z)+\overset{\circ}{g}{}_j^{+}(z))\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)\big]\,d_z s -$$
$$-2\sum_{k=m_0+1}^{m}\int_{S_k}\big[(\overset{\circ}{v}{}^{+}(z)+\overset{\circ}{g}{}_j^{+}(z))\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)\big]\,d_z s -$$
$$-2\sum_{k=1}^{m_0}\int_{S_k}\big[(\overset{\circ}{v}{}^{+}+\overset{\circ}{g}{}_j^{+})\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j - (\overset{k}{v}{}^{-}+\overset{k}{g}{}_j^{-})\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j\big]\,ds =$$
$$= \sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{v}) + 2\overset{k}{E}(\overset{k}{v},\overset{k}{g}_j) + \overset{k}{E}(\overset{k}{g}_j,\overset{k}{g}_j) + \varkappa^2(\overset{k}{v}{}^2 + 2\overset{k}{v}\overset{k}{g}_j + \overset{k}{g}{}_j^2)\big]\,dx -$$
$$-2\int_{S_0}\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds - 2\int_{S_0}\overset{\circ}{g}{}_j^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds - 2\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds -$$
$$-2\sum_{k=m_0+1}^{m_0}\int_{S_k}\overset{\circ}{g}{}_j^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds - 2\sum_{k=1}^{m_0}\int_{S_k}(\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j - \overset{k}{v}{}^{-}\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j\,ds -$$
$$-2\sum_{k=1}^{m_0}\int_{S_k}(\overset{\circ}{g}{}_j^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j - \overset{k}{g}{}_j^{-}\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j\,ds = L[g_j] + \sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{v}) + \varkappa^2\overset{k}{v}{}^2\big]\,dx +$$
$$+2\sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{g}_j) + \varkappa^2\overset{k}{v}\overset{k}{g}_j\big]\,dx - 2\int_{S_0}\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds -$$
$$-2\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds - 2\sum_{k=1}^{m_0}\int_{S_k}\big(\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j - \overset{k}{v}{}^{-}\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j\big)\,ds. \tag{21}$$

Using Green's formula (16) for $v=u-g_j$ and $u=g_j$, also taking into consideration that $\overset{k}{\widetilde{M}}\overset{k}{g}_j = \varkappa^2\overset{k}{g}_j$, $\overset{\circ}{\widetilde{T}}\overset{\circ}{g}_j = \overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j$ for $z\in S_k$ $(k=0,m_0+$

$\overline{1,\ldots,m}$) and $\overset{\circ}{v}{}^{+}(z)=\overset{k}{v}{}^{-}(z)$, $(\overset{\circ}{\widetilde{T}}\overset{\circ}{g}_j)^{+}-(\overset{k}{\widetilde{T}}\overset{k}{g}_j)^{-}=\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j-\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j$ for $z\in S_k$ ($k=\overline{1,m_0}$), we obtain

$$\sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{g}_j)+\varkappa^2\overset{k}{v}\overset{k}{g}_j\big]\,dx=\int_{S_0}\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds+$$
$$+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds+\sum_{k=1}^{m_0}\int_{S_k}\big(\overset{\circ}{v}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j-\overset{k}{v}{}^{-}\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j\big)\,ds. \tag{22}$$

Owing to (22), we find from (21) that

$$L[u]=L[g_j]+\sum_{k=0}^{m_0}\int_{D_k}\big[\overset{k}{E}(\overset{k}{v},\overset{k}{v})+\varkappa^2\overset{k}{v}{}^2\big]\,dx\geq L[g_j].\quad\blacksquare$$

Theorem 2. *For the function $g_{jj}(y,y-\varkappa^2)$, the estimate*

$$\big|g_{jj}(y,y-\varkappa^2)-g_{jj}(y,y-\varkappa_0^2)\big|\leq \mathrm{const}\,/l_y^{1+\delta},\quad y\in D,\quad \delta>0, \tag{23}$$

is valid.

Proof. Writing the formula (19) for $u_j(x)=g_{jj}(x,y,-\varkappa^2)$ and $\widetilde{\Gamma}_j(x-y,-\varkappa^2)=G_j(x,y,-\varkappa^2)$ and taking into account the boundary and contact conditions for g and G, we obtain

$$\forall y\in D_k:\ g_{jj}(y,y,-\varkappa^2=\int_{S_0}\overset{\circ}{G}{}_j^{+}(z,y,-\varkappa^2)\overset{\circ}{\widetilde{T}}(\partial z,n(z))\overset{\circ}{\widetilde{\Gamma}}_j(z,y,-\varkappa^2)d_zs+$$
$$+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{G}{}_j^{+}(z,y,-\varkappa^2)\overset{\circ}{\widetilde{T}}(\partial_z,n(z))\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_zs+$$
$$+\sum_{k=1}^{m_0}\int_{S_k}\big[(\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\Gamma}_j-\overset{k}{\widetilde{G}}{}_j^{-}\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j)-(\overset{\circ}{\widetilde{\Gamma}}_j(\overset{\circ}{\widetilde{T}}\overset{\circ}{G}_j)^{+}-\overset{k}{\widetilde{\Gamma}}_j(\overset{k}{\widetilde{T}}\overset{k}{G}_j)^{-})\big]\,ds. \tag{24}$$

Using the formula (16) for $u=v$, we can write (20) as follows:

$$L[u]=-\sum_{k=0}^{m_0}\int_{D_k}\overset{k}{u}(x)\big[\overset{k}{\widetilde{M}}(\partial x)\overset{k}{u}(x)-\varkappa^2\overset{k}{u}(x)\big]\,dx+\int_{S_0}\overset{\circ}{u}{}^{+}(z)(\overset{\circ}{\widetilde{T}}\overset{\circ}{u}(z))^{+}\,ds+$$
$$+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{u}{}^{+}(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^{+}\,ds-2\int_{S_0}\overset{\circ}{u}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds-2\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{u}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds+$$
$$+\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{u}{}^{+}(\overset{\circ}{\widetilde{T}}\overset{\circ}{u})^{+}-\overset{k}{u}{}^{-}(\overset{k}{\widetilde{T}}\overset{k}{u})^{-}\big]\,ds-2\sum_{k=1}^{m_0}\int_{S_k}\big[\overset{\circ}{u}{}^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j-\overset{k}{u}{}^{-}\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j\big]\,ds. \tag{25}$$

From (25), for $u(x)=g_j(x,y-\varkappa^2)=\Gamma_j(x-y,-\varkappa^2)-G(x,y,-\varkappa^2)$ we obtain

$$L[g_j]=\int_{S_0}\overset{\circ}{\widetilde{\Gamma}}_j\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds-\int_{S_0}\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{\widetilde{\Gamma}}_j\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j\,ds-$$

$$-\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds-2\int_{S_0}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+2\int_{S_0}\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds-$$
$$-2\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+2\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+$$
$$+\sum_{k=1}^{m_0}\int_{S_k}[(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)-(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{G}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{G}_j)-(\overset{\circ}{G}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{G}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)+$$
$$+(\overset{\circ}{G}_j\overset{\circ}{\widetilde T}\overset{\circ}{G}_j-\overset{k}{G}_j\overset{k}{\widetilde T}\overset{k}{G}_j)]ds-2\sum_{k=1}^{m_0}\int_{S_k}[(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)-(\overset{\circ}{G}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{G}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)]ds=$$
$$=-\int_{S_0}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+\int_{S_0}\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds-\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+$$
$$+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+\sum_{k=1}^{m_0}\int_{S_k}[(\overset{\circ}{G}{}_j^{+}\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{G}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)-$$
$$-(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)-(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{G}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{G}_j)]\,ds. \qquad (26)$$

By virtue of (26), we find from (24) that

$$g_{jj}(y,y-\varkappa^2)=L[g_j]+\int_{S_0}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+$$
$$+\sum_{k=1}^{m_0}\int_{S_k}(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j\,ds. \qquad (27)$$

The vector $\overset{\widehat{k}}{\Gamma}(x-y,-\varkappa^2)$ defined by (8) belongs to the domain of definition of the functional L, and since $g_j(x,y,-\varkappa^2)$ gives a minimum to the functional L, it becomes evident that $L[g_j]\le L[\widehat\Gamma_j]$. Then from (27) we get

$$g_{jj}(y,y-\varkappa^2)\le L[\widehat\Gamma_j]+\int_{S_0}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds+$$
$$+\sum_{k=1}^{m_0}\int_{S_k}(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)ds,\quad y\in D_k. \qquad (28)$$

By virtue of properties of $\widehat\Gamma$, (25) yields

$$L[\widehat\Gamma_j]=-\int_{\mathbb{B}(y,l_y)}\widehat\Gamma_j(\widetilde M\widehat\Gamma_j-\varkappa^2\widehat\Gamma_j)ds-\int_{S_0}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds-$$
$$-\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j\,ds-\sum_{k=1}^{m_0}\int_{S_k}(\overset{\circ}{\widetilde\Gamma}_j\overset{\circ}{\widetilde T}\overset{\circ}{\widetilde\Gamma}_j-\overset{k}{\widetilde\Gamma}_j\overset{k}{\widetilde T}\overset{k}{\widetilde\Gamma}_j)ds. \qquad (29)$$

Bearing (29) in mind, from (28) we obtain

$$g_{jj}(y,y-\varkappa^2)\le-\int_{\mathbb{B}(y,l_y)}\widehat{\Gamma}_j(\widetilde{M}\widehat{\Gamma}_j-\varkappa^2\widehat{\Gamma}_j)ds,\ \ y\in D_k,\ \ k=\overline{0,m_0}. \quad (30)$$

Taking into account the estimates (11), we have

$$\left|\widehat{\Gamma}_{ij}(x-y,-\varkappa^2)\right|\le \text{const}\cdot l_y^{-1};$$

$$\left|\varkappa^2\widehat{\Gamma}_{ij}(x-y,-\varkappa^2)\right|\le \varkappa^2\frac{\text{const}\cdot e^{-a\varkappa r}}{l_y^5}r^4=\frac{\text{const}}{l_y^5}r^2(\varkappa r)^2e^{-a\varkappa r}\le\frac{\text{const}}{l_y^3};$$

$$\left|\widetilde{M}\widetilde{\Gamma}_j(x-y,-\varkappa^2)\right|\le\frac{\text{const}}{l_y^3},\quad i,j=\overline{1,6};\quad y\in D_k,\quad k=\overline{0,m_0}.$$

From this and (30) it follows that

$$g_{jj}(y,y-\varkappa^2)\le\frac{\text{const}}{l_y^4}\cdot\frac{4}{3}\pi l_y^3\le\frac{\text{const}}{l_y}\le\frac{\text{const}}{l_y^{1+\delta}}, \quad (31)$$

where $\delta>0$ is an arbitrary number.

Estimate $g_{jj}(y,y-\varkappa^2)$ from below. To this end, we introduce the following notation:

$$Q[u]=\sum_{k=0}^{m_0}\int_{D_k}(\overset{k}{E}(\overset{k}{u},\overset{k}{u})+\varkappa^2\overset{k}{u}{}^2)dx,\quad Q_0[u]=\sum_{k=0}^{m_0}\int_{D_k}(\overset{k}{E}(\overset{k}{u},\overset{k}{u})+\varkappa_0^2\overset{k}{u}{}^2)dx,$$

$$R[u]=\int_{S_0}\overset{\circ}{u}{}^+\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_zs+\sum_{k=m_0+1}^{m}\int_{S_k}\overset{\circ}{u}{}^+\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_zs+$$

$$+\sum_{k=1}^{m_0}\int_{S_k}\left[\overset{\circ}{u}{}^+\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)-\overset{k}{u}{}^-\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)\right]d_zs.$$

Since $\varkappa_0^2\le\varkappa^2$, we have $Q_0[u]\le Q[u]$. Then

$$L[g_j]=\min L[u]=\min(Q[u]-2R[u])\ge\min(Q_0[u]-2R[u]).$$

Let the vector function $\varphi(x,y)$ give a minimum to the functional $N[u]=Q_0[u]-2R[u]$. Then $\varphi(x,y)$ will be a regular in D_k solution of the following problem:

$$\forall x\in D_k,\ \ \forall y\in D:\overset{k}{\widetilde{M}}(\partial x)\overset{k}{\varphi}(x,y)-\varkappa_0^2\overset{k}{\varphi}(x,y)=0,\quad k=\overline{0,m_0}$$

$$\forall z\in S_k,\ \ \forall y\in D:\overset{\circ}{\varphi}{}^+(z,y)-\overset{k}{\varphi}{}^-(z,y)=\overset{\circ}{\widetilde{\Gamma}}(z-y,-\varkappa^2)-\overset{k}{\widetilde{\Gamma}}(z-y,-\varkappa^2);$$

$$\big(\overset{\circ}{\widetilde{T}}\overset{\circ}{\varphi}(z,y)\big)^+-\big(\overset{k}{T}\overset{k}{\varphi}(z,y)\big)^-=\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}(z-y,-\varkappa^2)-\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}(z-y,-\varkappa^2),\ \ k=\overline{1,m_0};$$

$$\forall z\in S_k,\ \forall y\in D:\big(\overset{\circ}{\widetilde{T}}\overset{\circ}{\varphi}(z,y)\big)^+=\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}(z-y,-\varkappa^2),\ k=0,m_0+1,\dots,m;$$

Writing the formula (19) for φ, where $\widetilde{\Gamma} = G$, we obtain

$$\begin{aligned}\varphi(x,y) = \int_{S_0} \overset{\circ}{G}{}_j^{+}(z,x,-\varkappa_0^2)\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s + \\ + \sum_{k=m_0+1}^{m} \int_{S_k} \overset{\circ}{G}{}_j^{+}(z,x-\varkappa_0^2)\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s + \\ + \sum_{k=1}^{m_0} \int_{S_k} \big[\overset{\circ}{G}{}_j^{+}(z,x-\varkappa_0^2)\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2) - \\ - \overset{k}{G}{}_j^{-}(z,x-\varkappa_0^2)\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)\big]\, d_z s - \\ - \sum_{k=1}^{m_0} \int_{S_k} \big[\overset{\circ}{\widetilde{\Gamma}}_j(z-x,-\varkappa^2)(\overset{\circ}{\widetilde{T}}\overset{\circ}{G}_j(z,y-\varkappa_0^2)^{+} - \\ - \overset{k}{\widetilde{\Gamma}}_j(z-x,-\varkappa^2)(\overset{k}{\widetilde{T}}\overset{k}{G}_j(z,y,-\varkappa_0^2))\big]\, d_z s. \qquad (32)\end{aligned}$$

Using (9), (15) and the theorem on composition of kernels [3], we obtain from (32) that

$$\forall (x,y) \in \overline{D}_k \times D_k : \ |\varphi(x,y)| \le \frac{\text{const}}{r_{xy}}, \quad r_{xy} = |x-y|, \quad k = \overline{0, m_0}. \qquad (33)$$

On the other hand,

$$\begin{aligned}L[g_j] \ge Q_0[\varphi] - 2R[\varphi] \ge -2R[\varphi] = -2\int_{S_0} \overset{\circ}{\varphi}{}^{+}(z,y)\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s - \\ -2\sum_{k=m_0+1}^{m} \int_{S_k} \overset{\circ}{\varphi}{}^{+}(z,y)\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)d_z s - \\ -2\sum_{k=1}^{m_0} \int_{S_k} \big[\overset{\circ}{\varphi}{}^{+}(z,y)\overset{\circ}{\widetilde{T}}\overset{\circ}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2) - \overset{k}{\varphi}{}^{-}(z,y)\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)\big]\, d_z s. \qquad (34)\end{aligned}$$

Taking now into account that

$$\forall (x,y) \in S_k \times D_k : \ |\varphi(x,y)| \le \frac{\text{const}}{r_{xy}} \le \frac{\text{const}}{l_y};$$

$$|\overset{k}{\widetilde{T}}\overset{k}{\widetilde{\Gamma}}_j(z-y,-\varkappa^2)| \le \frac{\text{const}}{r_{zy}^2} = \frac{\text{const}}{r_{zy}^{\delta} r_{zy}^{2-\delta}} \le \frac{\text{const}}{l_y^{\delta} r_{zy}^{2-\delta}}, \quad \delta > 0,$$

from (34) we get

$$L[g_j] \ge \frac{\text{const}}{l_y^{1+\delta}}, \quad \delta > 0.$$

Owing to the representation (27), we readily obtain the estimate

$$\forall y \in D: \ g_{jj}(y, y-\varkappa^2) \geq -\frac{\text{const}}{l_y^{1+\delta}}. \tag{35}$$

From (31) and (35) it follows (23). ■

8. Consider the second boundary-contact eigenvalue problem which is formulated as follows: find in $D_k(k=0,\overline{m_0})$ a regular six-component vector $w(x) = \overset{k}{w}(x)$, $x \in D_k$, $k = \overline{0, m_0}$, which is a non-trivial solution of the equation

$$\forall x \in D_k: \ \overset{k}{\widetilde{M}}(\partial x)\overset{k}{w} + \gamma \overset{k}{w}(x) = 0, \quad k = \overline{0, m_0},$$

and satisfies the contact and boundary conditions

$$\forall z \in S_k: \ \overset{\circ}{w}^+(z) = \overset{k}{w}^-(z), \ (\overset{\circ}{\widetilde{T}}\overset{\circ}{w}(z))^+ = (\overset{k}{\widetilde{T}}\overset{k}{w}(z))^-, \quad k = \overline{1, m_0},$$

and

$$\forall z \in S_k: \ (\overset{\circ}{\widetilde{T}}\overset{\circ}{w}(z))^+ = 0, \quad k = 0, m_0+1, \dots, m,$$

respectively.

Denote this problem by II^c_γ. In the way described in [1], we can show that the problem II^c_γ is equivalent to a system of integral equations, namely

$$w(x) = (\gamma + \varkappa_0^2)\int_D G(x, y-\varkappa^2)w(y)dy. \tag{36}$$

By virtue of (14) and (15), the equation (36) is an integral one with a symmetric kernel of the class $L_2(D)$. This implies that there exists a countable system of eigenvalues $(\gamma_n + \varkappa_0^2)_{n=1}^\infty$ and the corresponding orthonormalized in D system of eigenvectors $[w^{(n)}(x)]_{n=1}^\infty = [\overset{k}{w}{}^{(n)}(x)]_{n=1}^\infty$, $x \in D_k$, $k = \overline{0, m_0}$ of the equation (36). This in its turn means that $(\gamma_n)_{n=1}^\infty$ and $[w^{(n)}(x)]_{n=1}^\infty$ are respectively eigenvalues and eigenvectors of the problem II^c_γ. It is known [1] that all $\gamma_n \geq 0$. Moreover, the system $[w^{(n)}(x)]_{n=1}^\infty$ is complete in $L_2(D)$ [4]. The regularity of the eigenvectors follows from the properties of the solid potential [1].

9. In deriving asymptotic formulas, a Tauber-type theorem due to Hardy and Littlewood [5] is of importance.

Theorem 3. *If a non-decreasing function $\Phi(t)$ is Stieltjes summable, and the asymptotic representation*

$$\int_0^{+\infty} \frac{d\Phi(t)}{(x+t)^l} \sim \frac{P}{x^m}$$

holds as $x \to \infty$, where the constants l, m and p satisfy $0 < m < l$, $p \neq 0$, then

$$\Phi(t) \sim \frac{P \cdot \Gamma(l)}{\Gamma(m)\Gamma(l-m+1)} t^{l-m}.$$

Here Γ *is the Euler function.*

From the expansion of the kernel in terms of the eigenvectors, we find that

$$G(x,y,-\varkappa^2)-G(x,y,-\varkappa_0^2)=(\varkappa_0^2-\varkappa^2)\sum_{n=1}^{\infty}\frac{w^{(n)}(x)\times w^{(n)}(y)}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}, \qquad (37)$$

where $x,y\in D_k$ $(k=\overline{(0,m_0)})$, and the symbol $''\times''$ in (37) denotes the matrix product of a column vector by a row vector (dyadic product):

$$w^{(n)}(x)\times w^{(n)}(y)=\|w_i^{(n)}(x)w_k^{(n)}(y)\|_{6\times 6},\quad i,k=\overline{1,6}.$$

Passing in (37) to limit as $x\to y$, we obtain

$$(\varkappa_0^2-\varkappa^2)\sum_{n=1}^{\infty}\frac{[w_j(y)]^2}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}=\lim_{x\to y}\,[\overset{k}{\widetilde{\Gamma}}_{jj}(x-y,-\varkappa^2)-$$

$$-\overset{k}{\widetilde{\Gamma}}_{jj}(x-y,-\varkappa_0^2)]-[g_{jj}(y,y-\varkappa^2)-g_{jj}(y,y-\varkappa_0^2)], \qquad (38)$$

$$x,y\in D_k,\quad k=\overline{0,m_0};\quad j=\overline{1,6}.$$

From (12) and (13) we have

$$\lim_{x\to y}[\overset{k}{\widetilde{\Gamma}}_{jj}(x-y,-\varkappa^2)-\overset{k}{\widetilde{\Gamma}}_{jj}(x-y,-\varkappa_0^2)]=(\varkappa_0-\varkappa)B_k(j),\ k=\overline{1,6},\ j=\overline{1,6}, \qquad (39)$$

where

$$B_k^{(j)}=\begin{cases}\overset{1}{B}_k &= \frac{\rho_k^{3/2}}{12\pi}\left[\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{(\mu_k+\alpha_k)^{3/2}}\right],\quad j=1,2,3\\ \overset{2}{B}_k &= \frac{I_k}{12\pi}\left[\frac{1}{(\varepsilon_k+2\nu_k)^{3/2}}+\frac{2}{(\nu_k+\beta_k)^{3/2}}\right],\quad j=4,5,6\end{cases}$$

Taking into consideration (23) and (39), we obtain from (38) that

$$\sum_{n=1}^{\infty}\frac{[w_j^{(n)}(y)]^2}{(\gamma_n+\varkappa^2)(\gamma_n+\varkappa_0^2)}\sim\frac{B_k(j)}{\varkappa+\varkappa_0}\,, \qquad (40)$$

$$y\in D_k,\quad k=\overline{0,m_0},\quad j=\overline{1,6}.$$

Consider the function

$$\Phi_j(t)=\sum_{\gamma_n\le t}\frac{[w_j^{(n)}(y)]^2}{\gamma_n+\varkappa_0^2},\quad y\in D_k,\quad k=\overline{0,m_0}\quad j=\overline{1,6}.$$

It can be easily seen that

$$\int_0^{\infty}\frac{d\Phi_j(t)}{t+\varkappa^2}=\sum_{n=1}^{\infty}\frac{[w_j^{(n)}(y)]^2}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}.$$

By (40) we have

$$\int_0^\infty \frac{d\Phi_j(t)}{t+\varkappa^2} \sim \frac{B_k(j)}{\varkappa}\,, \tag{41}$$

whence, according to Theorem 3, we get

$$\Phi_j(t) = \sum_{\gamma_n \le t} \frac{[w_j^{(n)}(y)]^2}{\gamma_n + \varkappa_0^2} \sim \frac{B_k^{(j)}\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} t^{3/2} = \frac{2}{\pi} B_k^{(j)} t^{3/2}. \tag{42}$$

Obviously,

$$\sum_{\gamma_n \le t} [w_j^{(n)}(y)]^2 = \int_0^t (\xi + \varkappa_0^2)\, d\Phi_j(\xi) = (\xi + \varkappa_0^2)\Phi_j(\xi)\Big|_0^t - \int_0^t \Phi_j(\xi)\, d\xi,$$

Taking into account (42), we get

$$\sum_{\gamma_n \le t} [w_j^{(n)}(y)]^2 \sim \frac{2}{3\pi} B_k^{(j)} t^{3/2}, \quad y \in D_k, \quad k = \overline{0, m_0}, \quad j = \overline{1,6}.$$

Summing these relations with respect to j, we conclude that

$$\sum_{\gamma_n \le t} \sum_{j=1}^{3} [\overset{k}{w}{}_j^{(n)}(y)]^2 \sim \frac{\rho_k^{3/2}}{6\pi^2} \Big[\frac{1}{(\lambda_k + 2\mu_k)^{3/2}} + \frac{2}{(\mu_k + \alpha_k)^{3/2}} \Big] \cdot t^{3/2}, \tag{43}$$

$$\sum_{\gamma_n \le t} \sum_{j=4}^{6} [\overset{k}{w}{}_j^{(n)}(y)]^2 \sim \frac{I_k^{3/2}}{6\pi^2} \Big[\frac{1}{(\varepsilon_k + 2\nu_k)^{3/2}} + \frac{2}{(\nu_k + \beta_k)^{3/2}} \Big] \cdot t^{3/2}, \tag{44}$$

$$y \in D_k, \quad k = \overline{0, m_0}.$$

Thus the relations (43) and (44) provide us with the asymptotic distribution of the eigenvector functions.

10. Taking into consideratrion (39), it follows from (38) that

$$(\varkappa^2 - \varkappa_0^2) \sum_{n=1}^{\infty} \frac{[w^{(n)}(y)]^2}{(\gamma_n + \varkappa_0^2)(\gamma_n + \varkappa^2)} = 3(x - \varkappa_0)(\overset{1}{B}_k + \overset{2}{B}_k) +$$

$$+ \sum_{j=1}^{6} \big[\overset{k}{g}_{jj}(y, y, -\varkappa_0^2) - \overset{k}{g}_{jj}(y, y, -\varkappa^2)\big]. \tag{45}$$

Denote

$$\psi(y, \varkappa) = 3(\varkappa - \varkappa_0)(\overset{1}{B}_k + \overset{2}{B}_k) + \sum_{j=1}^{6} \big[g_{jj}(y, y, -\varkappa_0^2) - g_{jj}(y, y, -\varkappa^2)\big].$$

Then (45) implies

$$\frac{\psi(y, \varkappa)}{\varkappa^2 - \varkappa_0^2} = \sum_{n=1}^{\infty} \frac{[w^{(n)}(y)]^2}{(\gamma_n + \varkappa_0^2)(\gamma_n + \varkappa^2)} \le \sum_{n=1}^{\infty} \frac{[w^{(n)}(y)]^2}{(\gamma_n + \varkappa_0^2)^2}. \tag{46}$$

According to Bessel's inequality, we have

$$\sum_{n=1}^{\infty}\frac{[w^{(n)}(x)]^2}{(\gamma_n+\varkappa_0^2)^2}\le\int_D\left|G(x,y-\varkappa_0^2)\right|^2dy,\quad x\in D_k. \tag{47}$$

Taking into account estimates (15), from (47) we deduce the existence and the uniform boundedness in $\overline{D}_k$ ($k=\overline{0,m_0}$) of the sum of the series

$$\sum_{n=1}^{\infty}\frac{[w^{(n)}(x)]^2}{(\gamma_n+\varkappa_0^2)^2}$$

From this and the inequality (46) we find that

$$\forall y\in\overline{D}_k(k=\overline{0,m_0}):|\psi(y,\varkappa)|\le\mathrm{const}\cdot(\varkappa^2-\varkappa_0^2). \tag{48}$$

Integrating (46) over the domain D, we find in view of the orthonormality of the vectors $[w^{(n)}(x)]_{n=1}^{\infty}$ in D that

$$\int_D\psi(y,\varkappa)dy=(\varkappa^2-\varkappa_0^2)\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}. \tag{49}$$

$$\int_D\psi(y,\varkappa)dy=\int_D3(\varkappa-\varkappa_0)(\overset{1}{B}_k+\overset{2}{B}_k)dy+\int_D\Big[\sum_{j=1}^{6}\big(g_{jj}(y,y,-\varkappa_0^2)-$$
$$-g_{jj}(y,y,-\varkappa^2)\big)\Big]dy=3(\varkappa-\varkappa_0)\sum_{k=0}^{m_0}(\overset{1}{B}_k+\overset{2}{B}_k)\,\mathrm{mes}\,D_k+$$
$$+\sum_{j=1}^{6}\int_D\left[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa^2)\right]dy. \tag{50}$$

By virtue of (50), from (49) we get

$$\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}-\frac{3}{\varkappa+\varkappa_0}\sum_{k=0}^{m_0}(\overset{1}{B}_k+\overset{2}{B}_k)\,\mathrm{mes}\,D_k=$$
$$=\frac{1}{\varkappa^2-\varkappa_0^2}\sum_{j=1}^{6}\int_D\left[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa^2)\right]dy. \tag{51}$$

Denote by $(D_k)_\eta$ the part of the domain D_k ($k=0,m_0$) the distance from the points of which to the boundary D is less than η, and set $D_\eta=\bigcup_{k=0}^{m_0}(D_k)_\eta$. Then

$$\int_D\left[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa^2)\right]dy=\int_{D\setminus D_\eta}\left[g_{jj}(y,y,-\varkappa_0^2)-\right.$$
$$\left.-g_{jj}(y,y,-\varkappa^2)\right]dy+\int_{D_\eta}\psi(y,\varkappa)dy-\int_{D_\eta}3(\varkappa-\varkappa_0)(\overset{1}{B}_k+\overset{2}{B}_k)dy.$$

This and (51) imply that

$$\Big|\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}-\frac{3}{\varkappa+\varkappa_0}\sum_{k=0}^{m_0}(\overset{1}{B}_k+\overset{2}{B}_k)\,\mathrm{mes}\,D_k\Big|\le$$
$$\le\frac{1}{\varkappa^2-\varkappa_0^2}\Big|\int_{D_\eta}\psi(y,\varkappa)dy\Big|+\frac{1}{\varkappa^2-\varkappa_0^2}\sum_{j=1}^{6}\Big|\int_{D\setminus D_\eta}\big[g_{jj}(y,y,-\varkappa_0^2)-$$
$$-g_{jj}(y,y,-\varkappa^2)\big]\,dy\Big|+\frac{3}{\varkappa+\varkappa_0}\Big|\int_{D_\eta}(\overset{1}{B}_k+\overset{2}{B}_k)dy\Big|. \tag{52}$$

The following estimates are valid:

$$\frac{3}{\varkappa+\varkappa_0}\Big|\int_{D_\eta}(\overset{1}{B}_k+\overset{2}{B}_k)dy\Big|=\frac{3}{\varkappa+\varkappa_0}\Big|\sum_{k=0}^{m_0}\int_{(D_k)_\eta}(\overset{1}{B}_k+\overset{2}{B}_k)dy\Big|\le$$
$$\le\frac{3(\overset{1}{B}_k+\overset{2}{B}_k)}{\varkappa+\varkappa_0}\,\mathrm{const}\cdot\eta \tag{53}$$

$$\frac{1}{\varkappa^2-\varkappa_0^2}\Big|\int_{D_\eta}\psi(y,\varkappa)dy\Big|\le\frac{\mathrm{const}}{\varkappa^2-\varkappa_0^2}(\varkappa^2-\varkappa_0^2)\eta=\mathrm{const}\cdot\eta, \tag{54}$$

$$\frac{1}{\varkappa^2-\varkappa_0^2}\sum_{j=1}^{6}\Big|\int_{D\setminus D_\eta}\big[g_{jj}(y,y,-\varkappa_0^2)-g_{jj}(y,y,-\varkappa^2)\big]\,dy\le\frac{\mathrm{const}}{\varkappa^2-\varkappa_0^2}\cdot\frac{1}{\eta^\delta}. \tag{55}$$

The validity of (53) is obvious. The inequalities (54) and (55) hold by virtue of (48) and (23), respectively. It should be noted that the constants in (53), (54) and (55) do not depend on $\varkappa$ and y.

Consider the function

$$\Phi(t)=\sum_{\gamma_n\le t}\frac{1}{\gamma_n+\varkappa_0^2}.$$

We can easily see that

$$\int_0^\infty\frac{d\Phi(t)}{t+\varkappa^2}=\sum_{n=1}^{\infty}\frac{1}{(\gamma_n+\varkappa_0^2)(\gamma_n+\varkappa^2)}.$$

Owing to (53), (54) and (55), from (52) we obtain for $\eta=\frac{1}{\varkappa^2-\varkappa_0^2}$ that

$$\int_0^\infty\frac{d\Phi(t)}{t+\varkappa^2}\sim\frac{3\sum\limits_{k=0}^{m_0}(\overset{1}{B}_k+\overset{2}{B}_k)\,\mathrm{mes}\,D_k}{\varkappa}. \tag{56}$$

By Theorem 3, we have

$$\Phi(t)\sim\frac{6\sum\limits_{k=0}^{m_0}(\overset{1}{B}_k+\overset{2}{B}_k)\,\mathrm{mes}\,D_k}{\pi}\cdot t^{1/2}. \tag{57}$$

Denote $N(t)=\sum\limits_{\gamma_n\leq t}1$. Then for the number of the eigenvalues not greater than t, we obtain

$$N(t)=\int_0^t(\xi+\varkappa_0^2)d\Phi(\xi)=(\xi+\varkappa_0^2)\Phi(\xi)\big|_0^t-\int_0^t\Phi(\xi)d\xi.$$

which with regard for (56) yields $N(t)\sim\frac{2}{\pi}\sum\limits_{k=0}^{m_0}(\overset{1}{B}_k+\overset{2}{B}_k)\,\mathrm{mes}\,D_k t^{3/2}$, or finally

$$N(t)\sim\frac{1}{6\pi^2}\sum_{k=0}^{m_0}\mathrm{mes}\,D_k\bigg[\rho_k^{3/2}\Big(\frac{1}{(\lambda_k+2\mu_k)^{3/2}}+\frac{2}{(\mu_k+\alpha_k)^{3/2}}\Big)+$$
$$+I_k^{3/2}\Big(\frac{1}{(\varepsilon_k+2\nu_k)^{3/2}}+\frac{2}{(\nu_k+\beta_k)^{3/2}}\Big)\bigg]t^{3/2}. \qquad (58)$$

Thus the results of the present paper can be formulated as the following

Theorem 4. *The asymptotic distribution of the eigenvector functions and the eigenvalues of the basic boundary-contact oscillatory problems of the couple-stress elasticity is given by the formulas* (43), (44) *and* (58).

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