Memoirs on Differential Equations and Mathematical Physics
Volume 20, 2000, 113–126

T. Chantladze, N. Kandelaki, A. Lomtatidze, and D. Ugulava

THE P-LAPLACIAN AND CONNECTED PAIRS OF FUNCTIONS

Abstract. The present investigation is stimulated by the works [1], [2] and [3] in which the authors study oscillatory properties of half-linear ordinary differential equations, of the so-called P-Laplacian. Here we consider its two-dimensional version. Moreover, it turns out that the fundamentals of the P-Laplacian theory can be successfully involved in a more general scheme which we call the theory of connected pairs. We will see that it contains, on the one hand, the notion of analyticity and, on the other hand, it allows one to apply a very important transformation of the analysis, the Legendre transformation to the investigation of the P-Laplacian.

1991 Mathematics Subject Classification. 34K15, 34K40.

Key words and phrases: Half-linear ordinary differential equations, the P-Laplacian, connected pairs.

რეზიუმე. წინამდებარე გამოკვლევას სტიმული მისცეს [1], [2], [3] და ნაშრომებმა, რომლებშიც შესწავლილია ნახევრადწრფივი ჩვეულებრივი დიფერანციალური განტოლებების, ე.წ. P–ლაპლასიანის რხევადი თვისებები. აქ განხილულია მისი ორგანზომილებიანი ვარიანტი. უფრო მეტიც, აღმოჩნდა, რომ P–ლაპლასიანის თეორიის საფუძვლები წარმატებით შეიძლება იქნას ჩართული უფრო ზოგად სქემაში, რომელსაც ჩვენ ვუწოდებთ ბმულ წყვილთა თეორიას. ნაშრომში ნაჩვენებია, რომ ეს თეორია ერთის მხრივ მოიცავს ანალიზურობის ცნებას, ხოლო მეორეს მხრივ საშუალებას იძლევა ანალიზის ერთერთი უმნიშვნელოვანესი გარდაქმნა – ლეჟანდრის გარდაქმნა გამოყენებული იქნას P–ლაპლასიანის შესასწავლად.

1. General Statements

Let G be a connected domain in R^n. In what follows, we will consider a pair (u, v) of twice differentiable real functions defined on G and also a continuous and almost everywhere differentiable function T which maps R^n into R^n.

Definition 1.1. A pair of functions (u, v) is said to be T-connected or, simply, connected if in the domain G the condition

$$u' = T(v') \tag{1.1}$$

is fulfilled. The latter means that the diagram

$$\begin{tikzcd} R^n \arrow[rr, "T"] & & R^n \\ & G \arrow[ul, "v'"] \arrow[ur, "u'"'] & \end{tikzcd}$$

is commutative.

Sometimes we will say that the pair (u, v) is T-subordinate. For such a pair the following relations are valid:

$$\begin{aligned} u'' &= T'(v'), \\ D_u &= J_T(v')D_v, \end{aligned} \tag{1.2}$$

where J_T is the Jacobian of the mapping T, and D_u and D_v are the determinants of the matrices u'' and v'', respectively.

We will consider the T-connected pairs such that on the whole domain G

$$D_u \neq 0 \tag{1.3}$$

and J_T is defined everywhere. Then in the entire domain G we will have

$$J_T \neq 0, \qquad D_v \neq 0. \tag{1.4}$$

Let now $\xi = u'(x)$. Since $D_u \neq 0$ on G, by virtue of the well-known theorem on the local isomorphism, on some neighborhood Θ of the point x there exists an inverse to $\xi = u'(x)$ mapping $x = F(\xi)$. Define on the set $u'(\Theta)$ the function

$$\omega(\xi) = (x|\xi) - u(x), \tag{\#}$$

where $(\,|\,)$ is the inner product in R^n. Since $x = F(\xi)$, the function $\omega(\xi)$ is, in fact, is a function of ξ only. Further we have

$$\frac{\partial\omega(\xi)}{\partial\xi_i} = \sum_{k=1}^{n} \frac{\partial x_k}{\partial\xi_i}\xi_k + x_i - \sum_{k=1}^{n} \frac{\partial u(x)}{\partial x_k}\frac{\partial x_k}{\partial\xi_i}.$$

Since $\frac{\partial \omega(\xi)}{\partial \xi_i} = \xi_k$, we obtain $\frac{\partial \omega(\xi)}{\partial \xi_i} = x_i$, i.e., $x = \omega'(\xi)$, and it follows from (#) that $u(x) + \omega(\xi) = (x|\xi)$.

Consequently, every point of the domain G has a neighborhood Θ such that one can define on $u'(\Theta)$ a real function ω such that

$$\begin{aligned} \xi &= u'(x) \\ x &= \omega'(\xi), \\ u(x) + \omega(\xi) &= (x|\xi) \end{aligned} \tag{1.5}$$

for every $x \in \Theta$ and $\xi \in u'(\Theta)$. The triplet of relations (1.5) is called the Legendre transformation.

According to (1.4), since $D_v \neq 0$ on G, just the same can be said about the function v and its Legendre transformation

$$\begin{aligned} \lambda &= v'(x), \\ x &= \Psi'(\lambda), \\ v(x) + \Psi(\lambda) &= (x|\lambda) \end{aligned} \tag{1.6}$$

for every $x \in \Theta$ and $\lambda \in v'(\Theta)$. Determine now a new function w by

$$w = \omega \circ T. \tag{1.7}$$

Hence for every point $\lambda \in v'(\Theta)$ we have

$$w(\lambda) = \omega(T(\lambda)).$$

In this connection it is useful to comprehend the commutative diagrams

$$\begin{tikzcd} v'(\Theta) \arrow[rr, "T"] & & u'(\Theta) \\ & \Theta \arrow[ul, "v'"] \arrow[ur, "u'"'] & \end{tikzcd} \qquad \begin{tikzcd} v'(\Theta) & & u'(\Theta) \arrow[ll, "T^{-1}"'] \\ & \Theta \arrow[ul, "\Psi'"] \arrow[ur, "\omega'"'] & \end{tikzcd} \tag{1.8}$$

According to (1.7), for the derivative of the function w we have $w' = \omega'(T)T'$, on the one hand, and according to (1.1), (1.2) and (1.6) we get on the other hand

$$\xi = T(\lambda) \tag{1.9}$$

for every

$$x = \omega'(\xi) = \Psi'(\lambda) \tag{1.10}$$

which finally gives for the derivative of w a very significant linear system: for every point $\lambda \in v'(\Theta)$,

$$w'(\lambda) = \Psi'(\lambda)T'(\lambda). \tag{1.11}$$

Let us throw the light on the meaning of the relation (1.11). On the left there is a linear functional defined by the vector $w'(\lambda)$ and on the right the composition of the operator $T'(\lambda)$ and the vector $\Psi'(\lambda)$. Write this equality at an arbitrary point x as follows:

$$\big(w'(\lambda)|x\big) = \big(\Psi'(\lambda)|T'(\lambda)x\big) = \big(T^{'*}(\Lambda)\Psi'(\Lambda)|x\big),$$

where $T^{'*}$ is the operator conjugate to T'. Since the above equality is valid at every point x, we obtain the linear equation

$$w'(\lambda) = T^{'*}(\lambda)\Psi'(\lambda).$$

This is the main idea of the local linearization of an equation.

All the above-said we formulate in the following

Proposition 1.1. *Let for a T-connected pair (u, v) the condition (1.3) be fulfilled on G. Then each point of G has a neighborhood Θ such that the conditions (1.4), (1.5), (1.6), (1.9), (1.10) and (1.11) are fulfilled.*

Now let us make sure that Proposition 1.1 is inversible. Namely, the proposition below is valid.

Proposition 1.2. *In the domain Ω of the space R^n let the pair of functions (w, Ψ) satisfy the condition (1.11) and let there be given a mapping T of the space R^n into R^n such that in the entire domain Ω we have $J_T \neq 0$ and $D_\Psi \neq 0$. Then every point of the domain Ω possesses a neighborhood Λ in which the diagrams*

$$\begin{tikzcd} \Lambda \arrow[dr, "\Psi'"'] & & T(\Lambda) \arrow[ll, "T^{-1}"'] \arrow[dl, "\omega'"] \\ & \Psi''(\lambda) & \end{tikzcd} \qquad \begin{tikzcd} \Lambda \arrow[rr, "T"] & & T(\Lambda) \\ & \Psi'(\Lambda) \arrow[ul, "v'"] \arrow[ur, "u'"'] & \end{tikzcd} \tag{1.12}$$

are commutative.

The function ω appearing in one of the diagrams is defined on $T(\Lambda)$ by the formula

$$\omega(\xi) = w(T^{-1}(\xi)) \tag{1.13}$$

and the functions u and v being the Legendre transformation of the functions ω and Ψ, respectively, compose a T-connected pair in the domain $\Psi'(\Lambda)$.

Proof. From (1.13) we have that $w(\lambda) = \omega(\xi) = \omega(T(\lambda))$ for $\lambda = T^{-1}(\xi)$, $\xi = T(\lambda)$, where respectively $\lambda \in \Lambda$, $\xi \in T(\Lambda)$. Differentiating the function w, we obtain $w'(\lambda) = \omega'(T(\lambda))\ T'(\lambda)$. Now from (1.11) it follows that $(\Psi'(\lambda) - \omega'(T(\lambda)))\ T'(\lambda) = 0$. By the condition of the theorem, since $\det J_T \neq 0$ on G, for every $\lambda \in \Lambda$ we have $\Psi'(\lambda) = \omega'(\xi)$. This means that for the pair (u, v), which is the Legendre transformation of the pair (ω, Ψ), the relation $u' = T(v')$ is valid, i.e., the pair (u, v) is T-connected. $\square$

Remark 1.1. Thus, if the T-connectedness of the pair (u, v) is defined by the nonlinear relation $u' = T(v')$, then the pair of its Legendre transformations $(\omega; \Psi)$, or more exactly (w, Ψ), is linearly connected and $w'(\lambda) = \Psi'(\lambda)T'(\lambda)$. This property of T-connected pairs in the case where $D_u \neq 0$, is the most remarkable and basic property.

Remark 1.2. If in the first proposition the mapping u' is a homeomorphism, then the functions ω and Ψ will be defined on the entire $u'(\Theta)$ and $v'(\Theta)$, respectively. The same can be said about the second proposition: if Ψ and T are homeomorphisms, then the Legendre transformations will be defined on the entire domain.

In the sequel, we will be engaged with the special type of connectedness, namely

$$T = OF, \tag{1.14}$$

where F is a defined on R^n twice continuously differentiable function with values from R, and O is an orthogonal operator in R^n.

2. Connected Pairs in R^2

Consider the case $n = 2$, $T = OF'$, where

$$O = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1}$$

Consequently,

$$T = \begin{pmatrix} F_\mu \\ -F_\lambda \end{pmatrix}. \tag{2.2}$$

In what follows, F_λ and F_μ will denote partial derivatives in the first and second arguments, respectively. For the sake of convenience, these functions will be often denoted by corresponding subscripts. To denote vectors from R^2, the use will be made of the conventional notation (x, y), (ξ, η), λ, μ and so on.

Let (u, v) be a T-connected pair defined in the domain G. Taking into account the above-said and the relations (2.1) and (2.2), the equation (1.1) takes the form

$$\begin{aligned} u_x &= F_\mu(v_x, v_y), \\ u_y &= -F_\lambda(v_x, v_y). \end{aligned} \tag{2.3}$$

For the function v defined in the domain G the equivalent condition must be the one which would satisfy the equation

$$\frac{\partial}{\partial x} F_\lambda(v_x, v_y) + \frac{\partial}{\partial y} F_\mu(v_x, v_y) = 0 \tag{2.4}$$

and be represented in the form of the curvilinear integral

$$u(x,y)=\int^{(x,y)} F_\mu(v_x,v_y)dx - F_\lambda(v_x,v_y)dy + C. \tag{2.5}$$

Let a defined in the domain G twice continuously differentiable function v be continuous on the closure of the domain $\overline{G}$, on its boundary Γ ($\overline{G} = GU\Gamma$) coincide with the preassigned continuous function f and minimize the functional

$$I(v)=\int_G F(v_x,v_y)dxdy. \tag{2.6}$$

Then it satisfies the equation (2.4) and hence together with the function u defined by the formula (2.5) forms the T-connected pair (u,v). The equation (2.4) is the well-known Euler equation for the functional (2.6).

The equation (2.4) can be expressed in the form

$$F_{\lambda\lambda}(v_x,v_y)v_{xx} + 2F_{\lambda\mu}(v_x,v_y)v_{xy} + F_{\mu\mu}(v_x,v_y)v_{yy} = 0, \tag{2.7}$$

or more precisely

$$TrF''(v')v'' = 0. \tag{2.8}$$

It represents the so-called quasi-linear partial differential equation.

Now, just as in Section 1, the function u is required to satisfy the condition (1.3) on the entire plane domain G. Then, as it has been shown, every point from G has a neighborhood Θ such that the conditions (1.5) and (1.6) are fulfilled. It is advantageous here to write them in conventional notation, i.e., to express them in the form

$$\begin{aligned} \xi &= u_x(x,y), & \lambda &= v_x(x,y),\\ \eta &= u_y(x,y), & \mu &= v_y(x,y),\\ x &= \omega_\xi(\xi,\eta), & x &= \Psi_\lambda(\lambda,\mu),\\ y &= \omega_\eta(\xi,\eta), & y &= \Psi_\mu(\lambda,\mu),\\ u(x,y)+\omega(\xi,\eta) &= x\xi + y\eta, & v(x,y)+\Psi(\lambda,\mu) &= x\lambda + y\mu \end{aligned} \tag{2.9}$$

for every point $(x,y)\in\Theta$, $(\xi,\eta)\in u'(\theta)$ and $(\lambda,\mu)\in v'(\Theta)$. According to (1.7), the function w must be of the form

$$w(\lambda,\mu) = \omega\big(F_\mu(\lambda,\mu) - F_\lambda(\lambda,\mu)\big). \tag{2.10}$$

Clearly, the diagrams (1.8) will have the same form and therefore we do not draw them here. As for (1.9), in the just accepted notation we will have

$$\begin{aligned} \xi &= F_\mu(\lambda,\mu),\\ \mu &= -F_\lambda(\lambda,\mu) \end{aligned} \tag{2.11}$$

for every point $(\lambda, \mu) \in v'(\Theta)$. Similarly, for (1.10) we get

$$x = \omega_\xi(\xi, \eta) = \Psi_\lambda(\lambda, \mu),$$
$$y = \omega_\eta(\xi, \eta) = \Psi_\mu(\lambda, \mu). \tag{2.12}$$

Finally, the basic equation (1.11) will take the form

$$w_\lambda(\lambda, \mu) = F_{\lambda\mu}(\lambda, \mu)\Psi_\lambda(\lambda, \mu) - F_{\lambda\lambda}(\lambda, \mu)\Psi_\mu(\lambda, \mu),$$
$$w_\mu(\lambda, \mu) = F_{\mu\mu}(\lambda, \mu)\Psi_\lambda(\lambda, \mu) - F_{\lambda\mu}(\lambda, \mu)\Psi_\mu(\lambda, \mu). \tag{2.13}$$

Suppose first that the function F is thrice differentiable. Then from (2.13) and due to the fact that $w_{\mu\lambda} = w_{\lambda\mu}$, we obtain that Ψ satisfies the linear second order differential equation:

$$F_{\mu\mu}(\lambda, \mu)\Psi_{\lambda\lambda}(\lambda, \mu) - 2F_{\lambda\mu}(\lambda, \mu)\Psi_{\lambda\mu}(\lambda, \mu) + F_{\lambda\lambda}(\lambda, \mu)\Psi_{\mu\mu}(\lambda, \mu) = 0. \tag{2.14}$$

The fact that the same equation can be obtained directly from (2.7) by applying the Legendre transformation is worth mentioning. Indeed, according to (2.9), $\Psi'(v') = id$ which implies

$$\Psi''(\lambda, \mu)v''(x, y) = I,$$

or

$$v''(x, y) = \left[\Psi''(\lambda, \mu)\right]^{-1}.$$

Thus we obtain

$$v_{xx} = [D_\Psi]^{-1}\Psi_{\mu\mu}, \quad v_{yy} = [D_\Psi]^{-1}\Psi_{\lambda\lambda}, \quad v_{xy} = -[D_\Psi]^{-1}\Psi_{\lambda\mu}. \tag{2.15}$$

If we take into consideration that by our assumption $D_\Psi \neq 0$, then applying to (2.7) the Legendre transformation, we obtain exactly (2.14) which will be valid on the entire $v'(\Theta)$.

Before elucidating the meaning of linear equations (2.13) we briefly recall the following: let in some domain $\Omega \subset R^2$ a positive definite matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{2.16}$$

$$D = ac - b^2 > 0 \tag{2.17}$$

be given. The elements of the matrix are continuously differentiable in the domain Ω, and $D > 0$ in the same domain. The differential operator

$$a(\lambda, \mu)\frac{\partial^2}{\partial\lambda^2} + 2b(\lambda, \mu)\frac{\partial^2}{\partial\lambda\partial\mu} + c(\lambda, \mu)\frac{\partial^2}{\partial\mu^2} \tag{2.18}$$

defined in Ω is naturally connected with the quadratic form of the matrix A

$$aX^2 + 2bXY + cY^2. \tag{2.19}$$

The problem is to find a (w, Ψ) transformation of the variable and to reduce the operator (2.19) to the normal type

$$\widetilde{A} = \begin{pmatrix} a & o \\ o & a \end{pmatrix}.$$

The fact that for (2.19) we have $D > 0$ means its ellipticity. Moreover, it is well-known that the problem of finding an unknown pair of functions (w, Ψ) is reduced to the investigation of the so-called system of Beltrami equations in the domain Ω:

$$\begin{aligned} w_\lambda &= D^{-1}(-b\Psi_\lambda - c\Psi_\mu), \\ w_\mu &= D^{-1}(a\Psi_\lambda + b\Psi_\mu), \end{aligned} \tag{2.20}$$

which in its turn is equivalent to the study of the second order equation with respect to Ψ in the domain Ω:

$$\frac{\partial}{\partial\lambda}\Big[D^{-1}(a\Psi_\lambda + b\Psi_\mu)\Big] + \frac{\partial}{\partial\lambda}\Big[D^{-1}(b\Psi_\lambda + c\Psi_\mu)\Big]. \tag{2.21}$$

Take now as A the matrix of the type

$$A = \frac{1}{\sqrt{D_F}} \begin{pmatrix} F_{\mu\mu} & -F_{\lambda\mu} \\ -F_{\lambda\mu} & F_{\lambda\lambda} \end{pmatrix}. \tag{2.22}$$

Obviously, for every point $(\lambda, \mu) \in \Omega$

$$D_F = F_{\mu\mu}(\lambda,\mu)F_{\lambda\lambda}(\lambda,\mu) - F^2_{\lambda\mu}(\lambda,\mu) > 0 \tag{2.23}$$

and the corresponding Beltrami's system has the form (2.13), while the equivalent second order equation takes the form (2.14).

Consequently, with the twice continuously differentiable function F from R^2 in R are associated: on the one hand, the functional $I(v)$ from (2.6) defined in the domain G, and on the other hand, the matrix A from (2.22) defined in the domain Ω and satisfying (2.23). The investigation of the former is reduced directly to that of the T-connected pair (u, v) with $T = OF'$, while to study the latter we use the pair of functions (ω, Ψ) which is obtained by the Legendre transformation of the pair (u, v). If in the first case we deal with a second order elliptic quasi-linear equation (2.7), in the second case we are concerned with a second order elliptic quasi-linear partial differential equation (equation (2.14)). Thus the investigation of the functional $I(v)$ is reduced to that of the matrix A, and vice versa. The basic problem in both cases consists in finding a function v (respectively Ψ) defined in G (respectively in Ω) such that v' and Ψ' realize homeomorphic embeddings of the domains G and Ω, respectively.

3. The P-Laplacian

Again, we will deal with the T-connected pair (u, v) with $T = OF'$ defined in the two-dimensional domain G. Consider the function of the type

$$F(\lambda, \mu) = (p+1)^{-1}(\lambda^2 + \mu^2)^{(p+1)/2}, \tag{3.1}$$

$(\lambda, \mu) \in R^2$, and the parameter $p > 0$. We have

$$\begin{aligned} F_\lambda(\lambda, \mu) &= (\lambda^2 + \mu^2)^{(p-1)/2}\lambda, \\ F_\mu(\lambda, \mu) &= (\lambda^2 + \mu^2)^{(p-1)/2}\mu, \\ F_{\lambda\mu}(\lambda, \mu) &= (p-1)(\lambda^2 + \mu^2)^{(p-3)/2}\lambda\mu, \\ F_{\lambda\lambda}(\lambda, \mu) &= (\lambda^2 + \mu^2)^{(p-3)/2}(p\lambda^2 + \mu^2), \\ F_{\mu\mu}(\lambda, \mu) &= (\lambda^2 + \mu^2)^{(p-3)/2}(\lambda^2 + p\mu^2), \\ D_F &= p(\lambda^2 + \mu^2)^{(p-1)}. \end{aligned} \tag{3.2}$$

The matrix (2.2) considered in the previous section has the form

$$A_p = \left[p^{1/2}(\lambda^2 + \mu^2)\right]^{-1} \begin{pmatrix} \lambda^2 + p\mu^2 & (1-p)\lambda\mu \\ (1-p)\lambda\mu & \mu^2 + p\lambda^2 \end{pmatrix}. \tag{3.3}$$

Since $p > 0$, F is continuous everywhere in R^2, and the mapping $F' = (F_\lambda, F_\mu)$ from $R\backslash\{0\}$ into R^2 for $p > 0$ can be continuously extended to the entire domain, and F'' (except at zero) is infinitely differentiable. D_F is just the same. Moreover, according to (3.2), since $D_F > 0$, the considered in Section 2 equations in $R^2\backslash\{0\}$ belong to the elliptic type. Introduce the notation $\vec{x} = (x, y)$, $\vec{\xi} = (\xi, \eta)$, $\vec{\lambda} = (\lambda, \mu)$, $(x^2 + y^2)^{0,5} = |\vec{x}|$ and $F' = \Psi_p$. Thus for every $\vec{\lambda} \in R^2$ we have

$$F(\vec{\lambda}) = (p+1)^{-1}|\vec{\lambda}|^{(p+1)}, \tag{3.4}$$

$$F'(\vec{\lambda}) = \Psi_p(\vec{\lambda}) = |\vec{\lambda}|^{(p-1)}\vec{\lambda}. \tag{3.5}$$

Proposition 3.1. *For the mapping $\Psi_p : R^2 \to R^2$ the following is valid: for every $p > 0$, $q > 0$,*

$$\Psi_p \Psi_q = \Psi_{pq}, \tag{3.6}$$

the mapping Ψ_p is a homeomorphism and its inverse is given by the formula

$$\Psi_p^{-1}(\vec{\lambda}) = \Psi_{1/p}(\vec{\lambda}) = \Phi'(\lambda), \tag{3.7}$$

where

$$\Phi(\vec{\lambda}) = \frac{p}{p+1}|\vec{\lambda}|^{\frac{p+1}{p}}. \tag{3.8}$$

The mapping Ψ_p is permutable with any orthogonal operator O,

$$O\Psi_p = \Psi_p O. \tag{3.9}$$

Proof. We have

$$\begin{aligned}\Psi_p\Psi_q(\vec{\lambda})&=\Psi_p(\Psi_q(\vec{\lambda}))=|\Psi_q(\vec{\lambda})|^{(p-1)}\Psi_q(\vec{\lambda})=||\vec{\lambda}||^{(q-1)}|\vec{\lambda}|^{(q-1)}|\vec{\lambda}|^{(p-1)}\vec{\lambda}=\\&=|\vec{\lambda}|^{(q-1)(p-1)}|\vec{\lambda}|^{(p-1)}|\vec{\lambda}|^{(q-1)}\vec{\lambda}=|\vec{\lambda}|^{(pq-1)}\vec{\lambda}=\Psi_{pq}(\vec{\lambda}).\end{aligned}$$

It is clear that $\Psi_i = id$. Therefore $\Psi_p^{-1} = \Psi_{1/p}$. Since $\Psi_{1/p}(\vec{\lambda}) = |\vec{\lambda}|^{(1-p)/p}\vec{\lambda}$ and $p > 0$, the mapping $\Psi_p^{-1} = \Psi_{1/p}$, just as Ψ_p, can be continuously extended to the entire R^2, and hence Ψ_p is the homeomorphism, and its inverse is $\Psi_{1/p}$.

Consider now the function $\Phi(\vec{\lambda}) = p(p+1)^{-1}|\vec{\lambda}|^{(p+1)/p}$. We can easily see that $\Phi'(\vec{\lambda}) = \Psi_{1/p}(\vec{\lambda})$. Finally we prove that Ψ_p is permutable with any orthogonal operator. Indeed,

$$\begin{aligned}\Psi_p O(\vec{\lambda}) &= \Psi_p(O\vec{\lambda}) = |O\vec{\lambda}|^{(p-1)}O\vec{\lambda} = |\vec{\lambda}|^{(p-1)}O\vec{\lambda} =\\&= O|\vec{\lambda}|^{(p-1)}\vec{\lambda} = O\Psi_p(\vec{\lambda}). \quad \square\end{aligned}$$

Consequently we have that if the functions F and Φ are defined as

$$F(\vec{\lambda}) = (p+1)^{-1}|\vec{\lambda}|^{(p+1)} \text{ and } \Phi(\vec{\lambda}) = p(p+1)^{-1}|\vec{\lambda}|^{(p+1)/p}, \tag{3.10}$$

then the mappings

$$F'(\vec{\lambda})=\Psi_p(\vec{\lambda})=|\vec{\lambda}|^{(p-1)}\vec{\lambda} \text{ and } \Phi'(\vec{\lambda})=\Psi_{1/p}(\vec{\lambda})=|\vec{\lambda}|^{(p-1)/p}\vec{\lambda} \tag{3.11}$$

are inverse to each other homeomorphisms. It should be noted that $p' = p+1$ and $q' = (p+1)/p$ are the conjugate parameters

$$1/p' + 1/q' = 1. \tag{3.12}$$

Owing to certain reasons, for the pair (u, v) instead of speaking about their $T = OF'$ connectedness, connectedness or subordination, we will speak about their P-connectedness, connectedness and subordination. Hence, in accordance with Sections 1 and 2, the pair of functions (u, v) is called P-connected if for every point $(x, y) \in G$ there take place the equalities

$$\begin{aligned}u_x &= (v_x^2 + v_y^2)^{(p-1)/2}v_y,\\ u_y &= -(v_x^2 + v_y^2)^{(p-1)/2}v_x,\end{aligned} \tag{3.13}$$

the functional (2.6) has the form

$$I(v) = (p+1)^{-1}\int\limits_G (v_x^2 + v_y^2)^{(p+1)/2}dxdy \tag{3.14}$$

and the corresponding second order quasi-linear equation (2.7) is written as

$$\begin{gathered}(v_x^2+v_y^2)^{(p-3)/2}\big[(v_x^2+pv_x^2)v_{xx}+\\ +2(p-1)v_xv_yv_{xy}+(v_x^2+pv_y^2)v_{yy}\big]=0.\end{gathered} \tag{3.15}$$

Let us now consider such P-connected pairs (u,v) for which in the entire domain G we have $D_u\neq 0$. For such a pair (u,v), in the corresponding domain Θ under the corresponding Legendre transformations of (ω,Ψ) and hence of (w,Ψ) the basic equations (2.13) and (2.14) will take the form

$$\begin{aligned}w_\lambda(\lambda,\mu)&=(\lambda^2+\mu^2)^{(p-3)/2}\big[(p-1)\lambda\mu\Psi_\lambda(\lambda,\mu)-(\mu^2+p\lambda^2)\Psi_\mu(\lambda,\mu)\big],\\ w_\mu(\lambda,\mu)&=(\lambda^2+\mu^2)^{(p-3)/2}\big[(\lambda^2+p\mu^2)\Psi_\lambda(\lambda,\mu)-(p-1)\lambda\mu\Psi_\mu(\lambda,\mu)\big]\end{aligned} \tag{3.16}$$

and

$$\begin{gathered}(\lambda^2+\mu^2)^{(p-3)/2}\big[(\lambda^2+p\mu^2)\Psi_{\lambda\lambda}-\\ -2(p-1)\lambda\mu\Psi_{\lambda\mu}+(\mu^2+p\lambda^2)\Psi_{\mu\mu}\big]=0,\end{gathered} \tag{3.17}$$

and the corresponding matrix A, as is noted above, is of the form (3.3).

It is notable that I-connectedness of the pair (u,v) means that

$$\begin{aligned}u_x&=v_y,\\ u_y&=-v_y,\end{aligned}$$

i.e., the pair (u,v) in the domain G is analytic. In other words, I-connectedness is equivalent to the Cauchy-Riemann conditions, and the functional $I(v)$ for $P=1$ coincides with the well-known Dirichlet integral

$$I(v)=1/2\int_G(v_x^2+v_y^2)dxdy,$$

while the above-given quasi-linear partial differential equation of the second order is none other than the Laplace equation

$$\Delta v=v_{xx}+v_{yy}=0.$$

For the case $p=1$, the same can be said about the formulas (3.16) and (3.17); the former coincides again with the Cauchy-Riemann conditions and the latter with the Laplace equation. As for the matrix A_p for $p=1$, it is easy to see that it is unique.

Get again back to the quasi-linear differential equation written in the form (2.4):

$$\frac{\partial}{\partial x}\Big[\big(v_x^2+v_y^2\big)^{(p-1)/2}v_x\Big]+\frac{\partial}{\partial y}\Big[\big(v_x^2+v_y^2\big)^{(p-1)/2}v_y\Big]=0 \tag{3.18}$$

or, what comes to the same thing

$$Tr\big(|v'|^{(p-1)}v'\big)'=0 \tag{3.19}$$

and moreover,

$$\Delta_p v = \operatorname{div}\left(|\operatorname{grad} v|^{(p-1)}\operatorname{grad} v\right) = 0. \tag{3.20}$$

This equation is known as the P-Laplace equation and the corresponding operator Δ_p is called the P-Laplacian.

Proposition 3.2. *Let (u, v) be a P-connected pair in the domain G. Then the pair (u, v) is $1/p$-connected. Hence along with $\Delta_p v = 0$ we have also $\Delta_{1/p} u = 0$.*

Proof. Since the pair (u, v) is P-subordinated, we have $u' = O\Psi_p(v')$. By Proposition 3.1, $\Psi_{1/p}(u') = \Psi_{1/p}(O\Psi_p(v')) = O\Psi_{1/p}(\Psi_p(v')) = O\Psi_1(v') = Ov'$, i.e., $v' = O^{-1}\Psi_{1/p}(u')$. This yields

$$\begin{aligned} v_x &= -(u_x^2 + u_y^2)^{(1-p)/p} u_y, \\ v_y &= -(u_x^2 + u_y^2)^{(1-p)/p} u_x. \end{aligned} \tag{3.21}$$

This in its turn means that the pair $(-v, u)$ is already $1/p$-subordinated, the corresponding functional and the differential equation will have the form

$$I(u) = p(p+1)^{-1}\int_G (u_x^2 + u_y^2)^{(p+1)/2} dx dy, \tag{3.22}$$

$$\begin{gathered} (u_x^2 + u_y^2)^{(p-1)/2}\big[(u_y^2 + 1/pu_x^2)u_{xx} + \\ +2(1-p)p^{-1}u_x u_y u_{xy} + (u_x^2 + 1/pu_y^2)\big] = 0 \end{gathered} \tag{3.23}$$

and this in fact means that $\Delta_{1/p} u = 0$.

It is not difficult to note that when considering the P-connected pair (u, v) and the $1/p$-connected pair $(-v, u)$, there appear by Proposition 3.2 the following functions:

$$F(\lambda,\mu) = (p+1)^{-1}(\lambda^2+\mu^2)^{(p+1)/2}, \quad \Phi(\lambda,\mu) = p(p+1)^{-1}(\lambda^2+\mu^2)^{(p+1)/2p}$$

with the corresponding functionals $I(u)$ and $I(v)$

$$I(u) = p(p+1)^{-1}\int_G |u'|^{(p+1)/2p} dx dy, \quad I(v) = (p+1)\int_G |v'|^{(p+1)/2p} dx dy,$$

where the parameters $p' = p+1$ and $q' = q+1$ are conjugate: $1/p' + 1/q' = 1$, F and Φ are the Legendre transformations of each other. Indeed, since $F' = \Psi_p$, $\Phi' = \psi_{1/p}$ and $\Psi_p\Psi_{1/p} = id$, it remains only to show that

$$F(\vec{x}) + \Phi(\vec{\lambda}) = (\vec{x}\,|\vec{\lambda}) \tag{3.24}$$

or that

$$(p+1)^{-1}|\vec{x}\,|^{(p+1)} + p(p+1)^{-1}|\vec{\lambda}|^{(p+1)/p} = (\vec{x}\,|\vec{\lambda}).$$

But since $\vec{\lambda} = F'(\vec{x}) = |\vec{x}\,|^{(p-1)}\vec{x}$, this yields

$$(p+1)^{-1}|\vec{x}\,|^{(p+1)} + p(p+1)^{-1}|\vec{x}\,|^{(p-1)(p+1)/p}|\vec{x}\,|^{(p+1)/p} = (\vec{x}\,|\vec{x})|\vec{x}\,|^{(p-1)},$$

i.e., $|\vec{x}|^{(p+1)} = |\vec{x}|^{(p+1)}$.

Turn now to the matrix A_p defined by the formula (3.3). The relation (3.6) points to the fact that for such matrices the multiplicative law

$$A_p A_q = A_{pq} \tag{3.25}$$

should be fulfilled. Indeed,

$$p^{-1/2}(x^2+y^2)\begin{pmatrix} x^2+py^2 & (1-p)xy \\ (1-p)xy & y^2+px^2 \end{pmatrix} q^{-1/2}(x^2+y^2)^{-1} \times$$
$$\times \begin{pmatrix} x^2+qy^2 & (1-q)xy \\ (1-q)xy & y^2+qx^2 \end{pmatrix} = (pq)^{-1/2}(x^2+y^2)^{-1}\begin{pmatrix} x^2+pqy^2 & (1-pq)xy \\ (1-pq)xy & y^2+pqx^2 \end{pmatrix}$$

which proves (3.25). $\square$

Acknowledgement

The paper was supported by Grant No. 1.41 of Georgian Academy of Sciences.

References

1. T. Chantladze, N. Kandelaki, and A. Lomtatidze, On zeros of solutions of a second order singular semilinear equation. *Mem. Differential Equations Math. Phys.* **17**(1999), 127–155.

2. T. Chantladze, N. Kandelaki, and A. Lomtatidze, Oscillation and nonoscillation criteria for a second order equation. *Georgian Math. J.* **6**(1999), No. 5, 401–414.

3. N. Kandelaki, A. Lomtatidze, and D. Ugulava, On oscillation and nonoscillation of a second order half-linear equation. *Georgian Math. J.* **7**(2000), No. 2, 329–346.

(Received 14.12.1999)

Authors' addresses:

T. Chantladze, N. Kandelaki, D. Ugulava
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., Tbilisi 380093
Georgia

A. Lomtatidze
Department of Mathematical Analysis
Faculty of Natural Sciences
Masaryk University
Janačkovo nám. 2a, 662 95 Brno
Czech Republic