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THE DESCRIPTION OF THE ARBITRARY LOWER BOUNDARY DEGREE SET OF THE LINEAR PFAFF SYSTEM SOLUTION

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Consider the linear Pfaff system

$$\partial x/\partial t_i = A_i(t)x, \quad x \in R^n, \quad t = (t_1, t_2) \in R^2_{>1}, \quad i = 1, 2, \tag{1}$$

with continuously differentiable matrix functions $A_1(t)$ and $A_2(t)$ bounded in $R^2_{>1}$ and satisfying the complete integrability condition [1, pp. 16–26] $\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t)$, $t \in R^2_{>1}$.

Let $p = p[x]$ be a lower characteristic vector [2] of a nontrivial solution $x : R^2_{>1} \to R^n \setminus \{0\}$ of the system (1), and let P_x be the lower characteristic set [2] of $x(t)$. The notion of the lower characteristic degree $d = d_x(p) \in R^2$ of $x(t)$ associated with the lower characteristic vector $p \in P_x$ was defined in [3] by the conditions

$$\underline{ln}_x(p, d) \equiv \varliminf_{t\to\infty} \frac{\ln\|x(t)\| - (p, t) - (d, \ln t)}{\|\ln t\|} = 0, \ \ln t \equiv (\ln t_1, \ln t_2) \in R^2_+, \tag{2_1}$$

$$\underline{ln}_x(p, d + \varepsilon e_i) < 0, \quad e_i = (2 - i, i - 1) \in R^2, \quad \forall \varepsilon > 0, \quad i = 1, 2. \tag{2_2}$$

The union $\underline{D}_x(p) \equiv \cup d_x(p)$ of all lower characteristic degrees $d_x(p)$ is referred to as the lower degree set of the solution $x(t)$ corresponding to the lower characteristic vector p. The lower degree set $\underline{D}_x(p)$ is referred to as an interior lower degree set if the point $p \in P_x$ is an interior point of the lower characteristic set P_x and as a left (respectively, right) boundary lower degree set if p is a "left" (respectively, "right") boundary point of the lower characteristic set.

An arbitrary nonempty interior lower degree set of a nontrivial solution $x(t)$ of the system (1) is completely described in [3]. It is a line of the form $d_1 + d_2 = c_x(p)$ on the plane R^2.

Let the lower characteristic set P_x consist of more than one point, and let p' be its left boundary point. Necessary properties of the boundary lower degree sets are obtained in [4]. The nonempty left boundary lower degree set of the solution $x(t)$ is known [4] to be a closed concave monotone decreasing right and lower unbounded curve on the two-dimensional plane with negative slope ≥ -1 of an arbitrary tangent.

We intend to establish the sufficiency of these properties for the complete description of the left boundary lower degree set $\underline{D}_x(p')$. Moreover, for any curve D on the two-dimensional plane with the above-mentioned properties we construct the linear Pfaff equation

$$\partial x/\partial t_1 = a(t)x, \quad \partial x/\partial t_2 = b(t)x, \quad x \in R, \quad t \in R^2_{>1}, \tag{1_1}$$

with continuously differentiable bounded coefficients $a(t)$ and $b(t)$ satisfying the complete integrability condition $\partial a(t)/\partial t_2 = \partial b(t)/\partial t_1$, $t \in R^2_{>1}$, such that the left boundary lower degree set $\underline{D}_x(p')$ of any nontrivial solution $x(t)$ of this equation coincides with the curve D.

The following statement is valid.

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Theorem. *For any closed concave monotone decreasing right and lower unbounded curve D on the two-dimensional plane with negative slope ≥ -1 of an arbitrary tangent there exists a completely integrable Pfaff equation* (1_1) *with infinitely differentiable bounded coefficients such that the left boundary lower degree set of any nontrivial solution* $x : R^2_{>1} \to R^n \setminus \{0\}$ *of this equation is the curve D.*

Scheme of the proof. We intend to construct the desired Pfaff equation (1_1) by constructing a nontrivial solution of this equation.

Let us first note that the curve D has one of the following three forms: a) unbounded from the left and bounded from above; b) unbounded from the left and from above; c) bounded from the left and from above.

1. Partition of the curve D. Fix a number $\gamma > 0$ and consider the cases a) and b). Let the first partition D_1 of the curve D consist of the points $\Delta(i,1) \in D$, $i = 0,1,2$, of this curve with the first coordinates $\Delta_1(i,1) = (i-1)\gamma$, $i = 0,1,2$, respectively. The second partition $D_2 = \bigcup\limits_{i=0}^{2\times 2^2} \{\Delta(i,2)\} \subset D$ consists of the points $\Delta(i,2) \in D$ with the first components $\Delta_1(i,2) = (i-4)\gamma/2$, $i = 0,1,\ldots,2\cdot 2^2$. Finally, the lth partition $D_l = \bigcup\limits_{i=0}^{l\times 2^l} \{\Delta(i,l)\} \subset D$ consists of the points $\Delta(i,l) \in D$ with the first components $\Delta_1(i,l) = (i2^{1-l} - l)\gamma$, $i = 0,1,\ldots,l\cdot 2^l$. Continuing the process of the partition of the curve D indefinitely we introduce the denumerable set $D_\infty = \bigcup\limits_{l=1}^{+\infty} \bigcup\limits_{i=0}^{l\times 2^l} \{\Delta(i,l)\} \subset D$. By construction, this set is everywhere dense on the curve D.

In the case c) where the curve D is bounded from the left by the finite point $\Delta(0,0) \in D$, the partition D_l of this curve consists of the points $\Delta(i,l) \in D$ with the first components $\Delta_1(i,l) = \Delta_1(0,0) + i\gamma 2^{1-l}$, $i = 0,1,\ldots,l2^l$. Finally, as in the cases a) and b), we obtain the denumerable everywhere dense on the curve D set D_∞: $\overline{D_\infty} = D$.

Let us denote by $D(l)$ a segment of the curve D which lies between the points $\Delta(0,l) \in D_l$ and $\Delta(l2^l,l) \in D_l$ and includes these points.

2. The construction of a solution. We define the desired solution $x(t)$ by the formula $\ln x(t) = \ln\varphi(t) + \ln\psi(t)$, where $\ln\varphi(t) = \ln(e^{-t_1} + e^{-t_2})$. The function $\psi(t)$ is constructed so that the left boundary lower degree set of the solution $x(t)$ coincides with the curve D and the equality $P_x = P_\varphi$ holds.

Fix the ith point $\Delta(i,l) \in D$, $i \in \{0,1,\ldots,l2^l\} \equiv I_l$, of the lth partition, $l \in N$, and construct an arbitrary tangent to the curve D: $d_2 - \Delta_2(i,l) = k(i,l)(d_1 - \Delta_1(i,l))$, $k(i,l) \in [-1,0)$, which lies not lower than this curve. The existence of such a tangent follows from the concavity of the curve D. Moreover, if the point $\Delta \in D$ was already included in the partition, then we construct the same tangent at this point for all the next partitions. This assumption will be needed to guarantee the existence of a sequence realizing the lower limit in the property (2_1) of a lower characteristic degree for the point Δ.

To sew the different infinitely differentiable functions together into a single infinitely differentiable function, we introduce the infinitely differentiable functions:

$$e_{101}(\tau;\alpha_1,\alpha_2,\alpha_3) = e_{01}(\tau;\alpha_2,\alpha_3) + [1 - e_{01}(\tau;\alpha_1,\alpha_2)],$$
$$e_{0110}(\tau;\alpha_1,\alpha_2,\alpha_3,\alpha_4) = e_{01}(\tau;\alpha_1,\alpha_2)\cdot(1 - e_{01}(\tau;\alpha_3,\alpha_4)),$$
$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4, \quad \tau \in R,$$

defined on the basis of the infinitely differentiable function

$$e_{01}(\tau;\tau_1,\tau_2) = \begin{cases} 0, & \tau \in (-\infty,\tau_1], \\ \exp\{-(\tau-\tau_1)^{-2}\exp[-(\tau-\tau_2)^{-2}]\}, & \tau \in (\tau_1,\tau_2), \\ 1, & \tau \in [\tau_2,+\infty), \end{cases}$$

$-\infty < \tau_1 < \tau_2 < +\infty$.

For each $l \in N$ and $i \in I_l$, we define the function

$$\ln \psi_{i,l}(t) \equiv (\Delta(i,l), \ln t) \cdot e_{0110}(\ln t_2/\ln t_1; (\Theta_{i,l} - \tau_{i,l})/2, \Theta_{i,l} - \tau_{i,l}, \Theta_{i,l} + \tau_{i,l}, 2(\Theta_{i,l} + \tau_{i,l})) + ||\ln t||^2 \cdot e_{101}(\ln t_2/\ln t_1; \Theta_{i,l} - \tau_{i,l}, \Theta_{i,l}, \Theta_{i,l} + \tau_{i,l}), \quad t \in R^2_{>1}, \quad \Theta_{i,l} \equiv 1/|k(i,l)|, \quad \tau_{i,l} \equiv \min\{\Theta_{i,l}/2; 2^{-l}||\Delta(l2^l, l) - \Delta(0,l)||^{-1}\}. \tag{3}$$

It is easily seen that for each $i \in I_l$, $l \in N$, there exists a number $T_{i,l} \geq 1$ such that

$$\ln \psi_{i,l}(t) - (d, \ln t) \geq 0, \quad t \in R^2_{\geq 2} \setminus S(i,l), \quad ||t|| \geq T_{i,l}, \quad \forall d \in D(l). \tag{4}$$

We choose an arbitrary number $\eta_1 \geq 2$ and set $\rho_{i,l} = 2(T_{0,l} + \cdots + T_{i,l}) + (||\Delta(0,l)||^2 + \exp(6\tau_{0,l}^{-2})) \cdot \Theta_{0,l}^6(e^{3\Theta_{0,l}} + 1) + \cdots + (||\Delta(i,l)||^2 + \exp(6\tau_{i,l}^{-2})) \cdot \Theta_{i,l}^6(e^{3\Theta_{i,l}} + 1)$, $\alpha_{i,l} = (\eta_l + \rho_{i,l})2^i$, $\beta_{i,l} = \alpha_{i,l} \cdot 2$, $\eta_{l+1} = \beta_{l2^l,l} + 2^{l+1}$, $i \in I_l$, $l \in N$, and define the "basic" strips $\Pi(i,l) = \{t \in R^2_{>1} : \beta_{i,l} \leq t_1 + t_2 \leq \alpha_{i+1,l}\}$, $i = 0, 1, \ldots, l2^l - 1$, $\Pi(l2^l, l) = \{t \in R^2_{>1} : \beta_{l2^l,l} \leq t_1 + t_2 \leq \alpha_{0,l+1}\}$, $l \in N$, the "auxiliary" strips $\tilde{\Pi}(i,l) = \{t \in R^2_{>1} : \alpha_{i,l} < t_1 + t_2 < \beta_{i,l}\}$, $i \in I_l$, $l \in N$, and the closed triangle $T = \{t \in R^2_{>1} : t_1 + t_2 \leq \alpha_{0,1}\}$. Hence, we have divided the quadrant $R^2_{>1}$ (the domain of the solution $x(t)$) into the strips $R^2_{>1} = T \cup (\bigcup_{l \in N} \bigcup_{i \in I_l} (\Pi(i,l) \cup \tilde{\Pi}(i,l)))$.

We introduce the following notation: $\Pi L(i,l) \equiv \tilde{\Pi}(i,l) \cup \Pi(i,l) \cup \tilde{\Pi}(i+1,l)$, $i = 0, 1, \ldots, l2^l - 1$, $l \in N$, $\Pi L(l2^l, l) \equiv \tilde{\Pi}(l2^l, l) \cup \Pi(l2^l, l) \cup \tilde{\Pi}(0, l+1)$, $l \in N$, and $S\Pi(i,l) \equiv S(i,l) \cap \Pi L(i,l)$, $i \in I_l$, $l \in N$. Since $||t|| \geq (t_1 + t_2)/2 \geq T_{i,l}$ in each strip $t \in \Pi L(i,l)$, $i \in I_l$, $l \in N$, from (4) we conclude that

$$\ln \psi_{i,l}(t) - (d, \ln t) \geq 0, \quad \forall t \in \Pi L(i,l) \setminus S\Pi(i,l), \quad \forall d \in D(l). \tag{5}$$

We are now in a position to construct the auxiliary function $\tilde{\psi}(t)$. We set $\ln \tilde{\psi}(t) = 0$, $t \in T$. In each "basic" strip $\Pi(i,l)$, $i \in I_l$, $l \in N$, we define this function by $\ln \tilde{\psi}(t) = \ln \psi_{i,l}(t)$. Thus, in all "basic" strips the function $\ln \tilde{\psi}(t)$ is defined on the basis of the ith point $\Delta(i,l)$ of the lth partition. In each "auxiliary" strip $\tilde{\Pi}(i+1,l)$, $i = 0, 1, \ldots, l2^l - 1$, $l \in N$, we define the function $\ln \tilde{\psi}(t)$ by the formula $\ln \tilde{\psi}(t) = \ln \psi_{i,l}(t) + [\ln \psi_{i+1,l}(t) - \ln \psi_{i,l}(t)] \cdot e_{01}(\ln(t_1 + t_2); \ln \alpha_{i+1,l}, \ln \beta_{i+1,l})$, and in the strip $\tilde{\Pi}(0, l+1)$, $l \in N$, we set $\ln \tilde{\psi}(t) = \ln \psi_{l2^l,l}(t) + [\ln \psi_{0,l+1}(t) - \ln \psi_{l2^l,l}(t)] \cdot e_{01}(\ln(t_1 + t_2); \ln \alpha_{0,l+1}, \ln \beta_{0,l+1})$. Finally, we set $\ln \tilde{\psi}(t) = \ln \psi_{0,1}(t) e_{01}(\ln(t_1 + t_2); \ln \alpha_{0,1}, \ln \beta_{0,1})$, in the strip $\tilde{\Pi}(0,1)$.

In the cases a) and b) where the curve D is unbounded from the left and right, we define the desired function $\psi(t)$ by $\psi(t) = \tilde{\psi}(t)$, $t \in R^2_{>1}$. In the case c), we define this function by the formula $\ln \psi(t) = \ln \tilde{\psi}(t) + [(\Delta(0,0), \ln t) - \ln \tilde{\psi}(t)] \cdot e_{01}(\ln t_2/\ln t_1; 3/|k(0,0)|, 3/|k(0,0)| + 1)$, $t \in R^2_{>1}$. Note that for each point $d \in D$ the function $\ln \psi(t)$ coincides with the scalar product $(d, \ln t)$ in the direction $\ln t_2/\ln t_1 = 1/|k(d)|$.

3. Evaluation of the lower characteristic set Note that the lower characteristic set of the solution $x(t)$ of the equation (1_1) coincides with the lower characteristic set $P_\varphi \equiv \{p \in R^2_- : p_1 + p_2 = -1\}$ of the function $\varphi(t)$. This follows from the existence of the limit $\lim_{t \to \infty} (\ln \psi(t))/||t|| = 0$.

4. Evaluation of the left boundary lower degree set. We take an arbitrary point $\tilde{d} = (\tilde{d}_1, \tilde{d}_2) \in D$ of the curve D. Since the partition is everywhere dense on the curve D, for this point there exists a sequence of the points $\{d(n)\}_{n \in N}$, $d(n) \in D_\infty$, of the partition tending to the point $\tilde{d}$. But if we prove the inclusion $d(n) \in \underline{D}_x(p')$, than we obtain that $\tilde{d} \in \underline{D}_x(p')$. This follows from the closedness of the left boundary lower degree set.

Let us now choose an arbitrary point $d = (d_1, d_2) \in D_\infty$ of the partition and show that it belongs to the left boundary lower degree set.

We will denote by $\beta(d) \equiv \underline{\lim}_{t \to \infty} [\ln x(t) + t_1 - (d, \ln t)]/||\ln t||$ the lower limit from the condition (2_1), where $p' = (-1, 0)$ is the left boundary point of the lower characteristic set of $x(t)$. Let us show that $\beta(d) = 0$.

We first prove that $\beta(d) \leq 0$. Since the point $d \in D_\infty$ belongs to the denumerable partition of the curve D and each new finite partition contains all points of the previous finite partition, it follows that there exists a number $l(d) \in N$ such that $d \in D_l$, $\forall l > l(d)$, and $d \notin D_l$, $\forall l \leq l(d)$. If $d = \Delta(0,0)$ (in the case c)), we set $l(d) = 0$. We suppose that the point d is the i_1th point of the $(l(d)+1)$th partition, the i_2th point of the $(l(d)+2)$th partition and, finally, the i_mth point of the $(l(d)+m)$th partition. Note that we construct the same tangent at the point d with the slope $k(d)$ for all finite partitions of the curve D, namely $\Theta_{i_1,l(d)+1} = \Theta_{i_2,l(d)+2} = \cdots = 1/|k(d)|$. In each strip $\Pi(i_m, l(d)+m)$, $m \in N$, where the function $\ln \psi(t)$ is defined on the basis of the point $d \in D$, we choose a point $\tau(m)$ on the segment $\ln t_2 / \ln t_1 = 1/|k(d)|$. We thus get the sequence $\{\tau(m)\} \uparrow +\infty$ such that $\ln \psi(\tau(m)) = (d, \ln \tau(m))$ and $\lim_{m\to\infty} [\ln x(\tau(m)) + \tau_1(m) - (d, \ln \tau(m))]/\|\ln \tau(m)\| = 0$. From this we deduce that $\beta(d) \leq 0$.

Let us denote by $\{t(m)\} \uparrow \infty$ the sequence realizing the lower limit $\beta(d)$. Without loss of generality, let all terms of this sequence belong to the strips of $R^2_{>1}$ with the different numbers l_m, $l_m > 1$, $l_{m+1} > l_m \to +\infty$ as $m \to +\infty$. We can certainly assume that

$$d \in D(l_m), \quad m \in N. \tag{6}$$

Let us prove that $\beta(d) \geq 0$. If the sequence $\{t(m)\}$ has an infinite subsequence $\{t(m_j)\}$ such that $\ln \psi(t(m_j)) - (d, \ln t(m_j)) \geq 0$, $\forall t(m_j)$, then the desired inequality holds. Therefore, without loss of generality, we can assume that $\ln \psi(t(m)) - (d, \ln t(m)) < 0$, $\forall m \in N$.

Let us consider the cases a) and b) and regard $m \in N$ as fixed. If $t(m) \in \Pi(i_m, l_m)$ (the "basic" strip), we will have the inequality $\ln \psi_{i_m,l_m}(t(m)) - (d, \ln t(m)) < 0$. This, together with (5) and (6), implies the inclusion $t(m) \in S\Pi(i_m, l_m)$. By the definition of the sector $S\Pi(i_m, l_m)$, we conclude the estimates

$$|\ln t_2(m)/\ln t_1(m) - \Theta_{i_m,l_m}| \leq \tau_{i_m,l_m} \leq 2^{-l_m} \|\Delta(l_m 2^{l_m}, l_m) - \Delta(0, l_m)\|^{-1}. \tag{7}$$

Let us now write the equation of the tangent to the curve D at the point $\delta = \Delta(i_m, l_m)$: $\delta_2 - \Delta_2(i_m, l_m) = k(i_m, l_m)(\delta_1 - \Delta_1(i_m, l_m))$. From the concavity of the curve D, it follows that the point $d \in D$ lies not above the tangent and hence,

$$\Delta_1(i_m, l_m) - d_1 + \Theta_{i_m,l_m}(\Delta_2(i_m, l_m) - d_2) \geq 0. \tag{8}$$

We estimate the difference $R(m,d) \equiv \ln \psi(t(m)) - (d, \ln t(m))$ from below. From the inclusion $t(m) \in S\Pi(i_m, l_m)$ and (3),(7),(8) we have the estimates: $R(m,d) = \ln \psi_{i_m,l_m}(t(m)) - (d, \ln t(m)) \geq (\Delta(i_m, l_m) - d, \ln t(m)) = (\Delta_1(i_m, l_m) - d_1) \ln t_1(m) + (\Delta_2(i_m, l_m) - d_2) \ln t_2(m) = \ln t_1(m)[\{(\Delta_1(i_m, l_m) - d_1) + \Theta_{i_m,l_m}(\Delta_2(i_m, l_m) - d_2)\} + (\Delta_2(i_m, l_m) - d_2)(\ln t_2(m)/\ln t_1(m) - \Theta_{i_m,l_m})] \geq -|\Delta_2(i_m, l_m) - d_2| \cdot |\ln t_2(m)/\ln t_1(m) - \Theta_{i_m,l_m}| \cdot \ln t_1(m) \geq -2^{-l_m}(|\Delta_2(i_m, l_m) - d_2|/\|\Delta(l_m 2^{l_m}, l_m) - \Delta(0, l_m)\|) \ln t_1(m) \geq -2^{-l_m} \times (|\Delta_2(i_m, l_m) - d_2|/\|\Delta(i_m, l_m) - d\|) \ln t_1(m) \geq -2^{-l_m} \ln t_1(m) \geq -2^{-l_m} \|\ln t(m)\|$.

In a similar way (with the only difference that one must write the equation of the tangent to the curve D at a different point), we can show in all possible cases that $R(m,d) \geq -2^{-l_m}\|\ln t(m)\|$, $\forall m \in N$, $l_m > 1$, $l_m \to +\infty$ as $m \to \infty$. We have thereby derived the necessary property $\beta(d) = \lim_{m\to\infty} [\ln(1 + e^{-t_2(m)+t_1(m)}) + \ln \psi(t(m)) - (d, \ln t(m))]/\|\ln t(m)\| \geq \lim_{m\to\infty} -2^{-l_m} = 0$. The second determining property (2_2) is realized by the sequence $\tau(m)$ constructed above. We have thus proved the inclusion $D \subset \underline{D}_x(p')$, and by the construction of the solution, we have $\underline{D}_x(p') = D$.

5. The construction of the equation. The above-constructed function $x > 0$ is a solution of the Pfaff equation (1_1) with coefficients $a(t) = x^{-1}(t)\partial x(t)/\partial t_1 = \partial \ln x(t)/\partial t_1$ and $b(t) = x^{-1}(t)\partial x(t)/\partial t_2 = \partial \ln x(t)/\partial t_2$, $t \in R^2_{>1}$ satisfying the complete integrability condition, since $\ln x(t)$ is infinitely differentiable in $R^2_{>1}$. These coefficients are easily seen to be bounded.

The proof of the theorem is complete.

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