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MONOTONE METHOD FOR FIRST ORDER PERIODIC BOUNDARY VALUE PROBLEMS AND PERIODIC SOLUTIONS OF DELAY DIFFERENCE EQUATIONS

Abstract. In this paper, we employ monotone iterative technique to study the existence of solutions for first order periodic boundary value problem and periodic solutions of delay difference equations.

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1. Introduction

For notation, given a, b be integers and $a < b$, we employ intervals to denote discrete set such as $Z[a,b] = \{a, a+1, \dots, b-1, b\}$, $Z[a,b) = \{a, a+1, \dots, b-1\}$, $Z[a,\infty) = \{a, a+1, \dots\}$, etc. Let $T \in Z[1,\infty)$ be fixed.

The method of upper and lower solutions coupled with the monotone iterative technique has been applied successfully to obtain results of existence and approximation of solutions for periodic boundary value problems and periodic solutions of first order functional differential equations (see [3-6, 9, 10] and references therein). However, as far as the author knows, the method of upper and lower solutions coupled with the monotone iterative technique has rarely been seen for PBVPs of difference equations [1, 2, 8, 11-16] and delay difference equations.

In this paper, we study first order periodic boundary value problems and periodic solutions of delay difference equations by means of the monotone iterative technique.

We consider the following periodic boundary valve problems (PBVPs):

$$\begin{cases} \Delta y(k) = f(k, y(k), y(k-\tau)), & k \in \{0, 1, \dots, T-1\} = I_1, \\ y(0) = y(T), \end{cases} \tag{1.1}$$

where $\Delta y(k) = y(k+1) - y(k)$, $\tau \in Z[0,\infty)$, and $f \in C(I_1 \times R^2, R)$ (i.e., f is continuous as a map from the topological space $I_1 \times R^2$ into the topological space R(of course the topology on I_1, will be the discrete topology)), and

$$\begin{cases} \Delta y(k-1) = f(k, y(k), y(k-\tau)), & k \in \{1, 2, \dots, T\} = I_2, \\ y(0) = y(T), \end{cases} \tag{1.2}$$

where $\Delta y(k-1) = y(k) - y(k-1)$, $f \in C(I_2 \times R^2, R)$, $\tau \in Z[0,\infty)$.

In a similar way to deal with (1.1) and (1.2), we consider the T-periodic solutions of the following delay difference equations:

$$\Delta y(k) = f(k, y(k), y(k-\tau)), \qquad k \in Z(-\infty, +\infty), \tag{1.3}$$

$$\Delta y(k-1) = f(k, y(k), y(k-\tau)), \qquad k \in Z(-\infty, +\infty), \tag{1.4}$$

where $f \in C(Z(-\infty,+\infty) \times R^2, R)$, $f(t,u,v) = f(t+T,u,v)$, $T \in Z[1,\infty)$, $\tau \in Z[0,\infty)$.

Section 2 is devoted to the maximum principle, which is the key to developing the monotone iterative technique. Section 3 is devoted to develop the monotone method for (1.1) and (1.2). Section 4 is devoted to develop the monotone method for (1.3) and (1.4).

2. Maximum Principle

To prove the validity of the monotone iterative technique, we shall use the following maximum principle.

Theorem 2.1. *Let $y \in E = C(Z[-\tau, T], R)$ and $0 < M < 1$, $0 < N$ such that*

(i) $\Delta y(k) + My(k) + Ny(k-\tau) \geq 0$, $k \in I_1$, $\tau \in Z[0, \infty)$,
(ii) $y(0) \geq y(T)$,
(iii) $y(0) = y(k)$, $k \in Z[-\tau, 0]$,
(iv) $\frac{N}{(1-M)^T(M+N)} < 1$.
Then $y(k) \geq 0$, $\forall k \in Z[-\tau, T]$.

Proof. Suppose, to the contrary, that $y(k) < 0$ for some $k \in Z[-\tau, T]$. It is enough to consider the following two cases.

Case 1 : $y(k) \leq 0$, $y(k) \not\equiv 0$ on $Z[0, T]$.

In this case, we have that $y(0) \geq y(T)$, $\Delta y(k) \geq 0$, $k \in I_1$. Thus $y(k) =$ constant $C < 0$ on $Z[0, T]$, and we obtain

$$0 \leq \Delta y(k) + My(k) + Ny(k-\tau) = (M+N)C,$$

which contradicts the fact that $C < 0$.

Case 2: There exist $k_1, k_2 \in Z[0, T]$ such that $y(k_1) > 0$ and $y(k_2) < 0$. Hence, two cases are possible.

Case 2.1: $y(T) \leq 0$. Define

$$y(\xi) = \max_{k \in Z[0,T)} y(k) > 0, \qquad \xi \in Z[0, T).$$

Since

$$\Delta y(k) + My(k) \geq -Ny(\xi), \qquad k \in I_1,$$

i.e.,

$$\Delta[(1-M)^{-k} y(k)] \geq -N(1-M)^{-(k+1)} y(\xi), \quad k \in I_1.$$

Sum the above inequality from ξ to $T-1$ to obtain

$$(1-M)^{-T} y(T) - (1-M)^{-\xi} y(\xi) \geq -Ny(\xi) \sum_{k=\xi}^{T-1} (1-M)^{-(k+1)},$$

i.e.,

$$-(1-M)^{-\xi} y(\xi) \geq -Ny(\xi) \sum_{k=\xi}^{T-1} (1-M)^{-(k+1)}.$$

Thus we obtain

$$-M(1-M)^{T-\xi} y(\xi) \geq -Ny(\xi)[1 - (1-M)^{T-\xi}],$$

i.e.,

$$(M+N)(1-M)^{T-\xi} \leq N,$$

that implies

$$1 \leq \frac{N}{(M+N)(1-M)^T},$$

and this contradicts condition (iv).

Case 2.2: $y(T) > 0$. Thus $y(0) \geq y(T) > 0$, and there exists $k_0 \in Z(0, T)$ such that

$$y(k_0) \leq 0, \quad y(k) > 0, \qquad k \in Z[0, k_0).$$

Let $\xi \in Z[0,k_0)$ such that

$$y(\xi) = \max_{k\in Z[0,k_0)} y(k) > 0.$$

Similarly, we have

$$\Delta[(1-M)^{-k}y(k)] \geq -N(1-M)^{-(k+1)}y(\xi), \quad k \in Z[0,k_0).$$

Sum the above inequality from ξ to $k_0 - 1$ to obtain

$$(1-M)^T \leq (1-M)^{k_0-\xi} \leq \frac{N}{M+N},$$

i.e.,

$$1 \leq \frac{N}{(M+N)(1-M)^T},$$

and this contradicts condition (iv) again. □

Theorem 2.2. *Let $y \in E = C(Z[-\tau,T],R)$ an $M>0$, $N>0$ such that*
(i) $\Delta y(k-1) + My(k) + Ny(k-\tau) \geq 0$, $k \in I_2$,
(ii) $y(0) \geq y(T)$,
(iii) $y(0) = y(k)$, $\in Z[-\tau,0]$,
(iv) $\frac{N(1+M)^T}{M+N} < 1$,
Then $y(k) \geq 0$, $\forall k \in Z[-\tau,T]$.

Proof. Suppose, to the contrary, that $y(k) < 0$ for some $k \in Z[-\tau,T]$. It is enough to consider the following two cases.

Case 1: $y(k) \leq 0$, $y(k) \not\equiv 0$ on $Z[0,T]$.

Similarly done as in Theorem 2.1.

Case 2: There exist $k_1,k_2 \in Z[0,T]$ such that $y(k_1) > 0$ and $y(k_2) < 0$. Hence, two cases are possible.

Case 2.1: $y(T) \leq 0$. Define

$$y(\xi) = \max_{k\in Z[0,T)} y(k) > 0, \qquad \xi \in Z[0,T).$$

Since

$$\Delta y(k-1) + My(k) \geq -Ny(\xi), \qquad k \in I_2,$$

i.e.,

$$\Delta[(1+M)^{k-1}y(k-1)] \geq -N(1+M)^{k-1}y(\xi), \qquad k \in I_2.$$

Sum the above inequality from $\xi+1$ to T to obtain

$$(1+M)^T y(T) - (1+M)^\xi y(\xi) \geq -Ny(\xi)\sum_{k=\xi+1}^{T}(1+M)^{k-1},$$

i.e.,

$$-(1+M)^\xi y(\xi) \geq -Ny(\xi)\sum_{k=\xi+1}^{T}(1+M)^{k-1}.$$

Thus we obtain

$$M(1+M)^\xi \leq N(1+M)^T - N(1+M)^\xi,$$

i.e.,

$$(M+N)(1+M)^{\xi} \le N(1+M)^T,$$

that implies

$$1 \le \frac{N(1+M)^T}{M+N},$$

and this contradicts condition (iv).

Case 2.2: $y(T) > 0$. Thus $y(0) \ge y(T) > 0$, and there exists $k_0 \in Z(0,T)$ such that

$$y(k_0) \le 0, \quad y(k) > 0, \quad k \in Z[0,k_0).$$

Let $\xi \in Z[0,k_0)$ such that

$$y(\xi) = \max_{k \in Z[0,k_0)} y(k) > 0.$$

Since

$$\Delta y(k-1) + My(k)) \ge -Ny(\xi), \quad k \in Z[1,k_0],$$

i.e.,

$$\Delta[(1+M)^{k-1}y(k-1)] \ge -N(1+M)^{k-1}y(\xi), \quad k \in Z[1,k_0].$$

Reasoning as in the previous case, we obtain

$$1 \le \frac{N(1+M)^T}{(M+N)},$$

and this contradicts condition (iv) again. □

Theorem 2.3. *Let* $y \in X = \{y \in C(Z(-\infty,+\infty),R) : \ y(k) = y(k+T)\}$ *and* $0 < M < 1, 0 < N$ *such that*

(i) $\Delta y(k) + My(k) + Ny(k-\tau) \ge 0$, $k \in Z(-\infty,+\infty)$,

(ii) $\frac{N}{(1-M)^T(M+N)} < 1$.

Then $y(k) \ge 0$, $\forall k \in Z(-\infty,+\infty)$.

Proof. Suppose, to the contrary, that $y(k) < 0$ for some $k \in Z[0,T]$. It is enough to consider the following two cases.

Case 1: $y(k) \le 0, \ y(k) \not\equiv 0$ for $k \in Z[0,T]$.

Similar to the Case 1 as in Theorem 2.1.

Case 2: There exist $k_1, k_2 \in Z[0,T]$ such that $y(k_1) > 0$ and $y(k_2) < 0$.

Let $\xi \in Z[0,T]$ such that

$$y(\xi) = \max_{k \in Z(-\infty,+\infty)} y(k) > 0,$$

then there exists $k_0 \in Z(\xi, \xi+T)$ such that

$$y(k_0) \le 0, \quad y(k) > 0, \quad \forall\ k \in Z(\xi,k_0).$$

Reasoning as in Theorem 2.1, we can obtain $1 \le \frac{N}{(1-M)^T(M+N)}$, and therefore condition (ii) is violated. □

Similar to the proof of Theorems 2.2 and 2.3, we have the following result.

Theorem 2.4. *Let* $y \in X = \{y \in C(Z(-\infty,+\infty),R):\ y(k) = y(k+T)\}$ *and* $M > 0, N > 0$ *such that*

(i) $\Delta y(k-1) + My(k) + Ny(k-\tau) \geq 0$, $k \in Z(-\infty,+\infty)$,

(ii) $\frac{N(1+M)^T}{M+N} < 1$.

Then $y(k) \geq 0$, $\forall k \in Z(-\infty,+\infty)$.

Remark 2.1. When N is suitably small, condition (iv) holds in Theorems 2.1 and 2.2, and condition (ii) holds in Theorems 2.3 and 2.4.

3. Monotone Method for First Order PBVPs of Delay Difference Equations

In order to develop the monotone iterative technique for (1.1) and (1.2), we shall first consider the following PBVPs for the linear equations of (1.1) and (1.2):

$$\begin{cases} \Delta y(k) + My(k) + Ny(k-\tau) = \sigma(k), & k \in I_1, \\ y(0) = y(T), \\ y(0) = y(k), \quad k \in Z[-\tau,0], \end{cases} \tag{3.1}$$

where $\sigma \in C(I_1,R)$, and

$$\begin{cases} \Delta y(k-1) + My(k) + Ny(k-\tau) = \sigma(k), & k \in I_2, \\ y(0) = y(T), \\ y(0) = y(k), \quad k \in Z[-\tau,0], \end{cases} \tag{3.2}$$

where $\sigma \in C(I_2,R)$.

We shall denote by

$$E^* = \{y \in E:\ y(k) = y(0), \quad \forall k \in Z[-\tau,0]\},$$

where E are defined in Section 2. Let E^* with norm

$$||y||_1 = \max_{k \in Z[-\tau,T]} |y(k)|$$

for $y \in E^*$, then E^* is a Banach space.

A function $\alpha \in E^*$ is said to be a lower solution to (3.1), if it satisfies

$$\begin{aligned} \Delta\alpha(k) + M\alpha(k) + N\alpha(k-\tau) \leq \sigma(k), \qquad k \in I_1, \\ \alpha(0) \leq \alpha(T). \end{aligned} \tag{3.3}$$

An upper solution for (3.1) is defined analogously by reversing the above inequalities.

A function $\alpha \in E^*$ is said to be a lower solution to (3.2), if it satisfies

$$\begin{aligned} \Delta\alpha(k-1) + M\alpha(k) + N\alpha(k-\tau) \leq \sigma(k), \qquad k \in I_2, \\ \alpha(0) \leq \alpha(T). \end{aligned} \tag{3.4}$$

An upper solution for (3.2) is defined analogously by reversing the above inequalities.

For $\alpha, \beta \in E^*$ we shall write $\alpha \leq \beta$ if $\alpha(k) \leq \beta(k)$ for all $k \in Z[-\tau, T]$. In such a case, we shall denote

$$[\alpha, \beta] = \{y \in E^* : \alpha \leq y \leq \beta\}.$$

Theorem 3.1. *Suppose that there exists a lower solution α and an upper solution β of* (3.1) *such that $\alpha \leq \beta$, and assume that condition* (iv) *of Theorem* 2.1 *is satisfied. Then* (3.1) *has a unique solution $y \in [\alpha, \beta]$.*

Proof. Consider now the PBVP

$$\begin{aligned} \Delta y(k) + My(k) &= -Np(k, y(k-\tau)) + \sigma(k), \quad k \in I_1, \\ y(0) &= y(T), \\ y(0) &= y(k), \quad k \in Z[-\tau, 0], \end{aligned} \tag{3.1*}$$

where

$$p(k, x) = \begin{cases} \alpha(k), & \text{if } x < \alpha(k), \\ x, & \text{if } \alpha(k) \leq x \leq \beta(k), \\ \beta(k), & \text{if } x > \beta(k). \end{cases}$$

It can be easily checked that $p : I_1 \times R \to [\alpha, \beta]$ is continuous.

Let us define an operator $\phi : E^* \to E^*$ by

$$(\phi y)(k) = \begin{cases} \sum_{j=0}^{T-1} G(k, j)[-Np(j, y(j-\tau)) + \sigma(j)], & k \in Z[0, T], \\ \sum_{j=0}^{T-1} G(0, j)[-Np(j, y(j-\tau)) + \sigma(j)], & k \in Z[-\tau, 0], \end{cases} \tag{3.5}$$

where

$$G(k, j) = \begin{cases} \dfrac{(1-M)^{k-j-1}}{1-(1-M)^T}, & j \leq k-1, \\ \dfrac{(1-M)^{T+k-j-1}}{1-(1-M)^T}, & j \geq k. \end{cases}$$

We can easily show $\phi : E^* \to E^*$ is continuous.

Since $-Np(k, y(k-\tau)) + \sigma(k)$ is bounded on I_1, then ϕ is bounded on $Z[-\tau, T]$. The existence of a fixed point y for the operator ϕ follows now from the Brouwer fixed point theorem. That means $(3.1)^*$ has a solution $y \in E^*$.

Now we will show that $y \in [\alpha, \beta]$.

First we prove that $y \geq \alpha$. Set $u(k) = y(k) - \alpha(k), k \in Z[-\tau, T]$. Since $p(k, y(k-\tau)) - \alpha(k-\tau) \leq \max_{k \in I_1}\{u(k-\tau), 0\}$. Then by the definition of the lower solution,we obtain:

(i) $\Delta u(k) + Mu(k) + N \max_{k \in I_1}\{u(k-\tau), 0\} \geq 0, \quad k \in I_1$

(ii) $u(0) \geq u(T)$,

(iii) $u(0) = u(k), \quad k \in Z[-\tau, 0]$,

(iv) $\frac{N}{(1-M)^T(M+N)} < 1$.

Suppose, to the contrary, that $y(k) < \alpha(k)$ for some $k \in Z[-\tau, T]$. It is enough to consider the following two cases:

Case 1: $u(k) \leq 0,\ u(k) \not\equiv 0$ on $Z[0,T]$.

In this case, we have that $u(0) \geq u(T),\ \Delta u(k) \geq 0,\ k \in I_1$. Thus $u(k)$ =constant $C < 0$ on $Z[0,T]$, and we obtain

$$0 \leq \Delta u(k) + Mu(k) + N \max_{k \in I_1}\{u(k-\tau), 0\} = MC,$$

which contradicts the fact that $C < 0$.

Case 2: There exist $k_1, k_2 \in Z[0,T]$ such that $u(k_1) > 0$ and $u(k_2) < 0$. Hence, two cases are possible.

Case 2.1: $u(T) \leq 0$. Define

$$u(\xi) = \max_{k \in Z[0,T)} u(k) > 0, \qquad \xi \in Z[0,T).$$

Since $\max_{k \in I_1}\{u(k-\tau), 0\} \leq u(\xi)$, then

$$\Delta u(k) + Mu(k) \geq -Nu(\xi), \qquad k \in Z[0,T).$$

Case 2.2: $u(T) > 0$. Thus $u(0) \geq u(T) > 0$ and there exists $k_0 \in Z(0,T)$ such that

$$u(k_0) \leq 0, \quad u(k) > 0, \quad k \in Z[0,k_0).$$

Let $\xi \in Z[0,k_0)$ such that $u(\xi) = \max_{k \in Z[0,k_0)} u(k) > 0$. Then $\max_{k \in Z[0,k_0)}\{u(k-\tau), 0\} \leq u(\xi)$, and

$$\Delta u(k) + Mu(k) \geq -Nu(\xi), \quad k \in Z[0,k_0).$$

In both cases 2.1 and 2.2, similar to the proof of Theorem 2.1, we obtain $1 \leq \frac{N}{(1-M)^T(M+N)}$, and therefore condition (iv) isviolated. This implies $y \geq \alpha$.

Similarly, we can prove $y \leq \beta$.

Since $y \in [\alpha, \beta]$, this implies that y is also a solution of (3.1).

Finally, suppose that there exist two solutions y_1 and y_2 of (3.1) on $[\alpha, \beta]$. Applying Theorem 2.1 again one can prove $\nu = y_1 - y_2 \geq 0$ on $Z[-\tau, T]$. As the same argument is valid for $y_2 - y_1$, then $y_2 - y_1 \geq 0$. So we have $y_1 = y_2$. □

Theorem 3.2. *Suppose that there exists a lower solution α and an upper solution β of* (3.2) *such that $\alpha \leq \beta$, and assume that condition* (iv) *of Theorem* 2.2 *is satisfied. Then* (3.2) *has a unique solution $y \in [\alpha, \beta]$.*

Proof. Consider now the PBVP

$$\Delta y(k-1) + My(k) = -Np(k, y(k-\tau)) + \sigma(k), \quad k \in I_2, \tag{3.1}$$
$$y(0) = y(T),$$
$$y(0) = y(k), \qquad k \in Z[-\tau, 0]. \tag{3.2*}$$

Let us define an operator $\phi : E^* \to E^*$ by

$$(\phi y)(k) = \begin{cases} \sum_{j=1}^{T} G(k,j)[-Np(j, y(j-\tau)) + \sigma(j)], & k \in Z[0,T], \\ \sum_{j=1}^{T} G(0,j)[-Np(j, y(j-\tau)) + \sigma(j)], & k \in Z[-\tau, 0], \end{cases} \tag{3.6}$$

where

$$G(k,j) = \begin{cases} \dfrac{(1+M)^{T+j-1-k}}{(1+M)^T - 1}, & j \le k, \\ \dfrac{(1+M)^{j-1-k}}{(1+M)^T - 1}, & j \ge k+1. \end{cases}$$

We can easily show $\phi : E^* \to E^*$ is continuous.

Since $-Np(k, y(k-\tau)) + \sigma(k)$ is bounded on I_2, then ϕ is bounded on $Z[-\tau, T]$. The existence of a fixed point y for the operator ϕ follows now from the Brouwer fixed point theorem. That means $(3.2)^*$ has a solution $y \in E^*$.

Similar to the proof of Theorem 3.1, we can prove that (3.2) has a unique solution y, and $y \in [\alpha, \beta]$.

Now we are in a position to prove the validity of the monotone method for (1.1) and (1.2). First we shall introduce the concepts of lower and upper solutions for these problems.

A function $\alpha \in E^*$ is said to be a lower solution to (1.1), if it satisfies

$$\Delta\alpha(k) \le f(k, \alpha(k), \alpha(k-\tau)), \quad k \in I_1,$$
$$\alpha(0) \le \alpha(T).$$

An upper solution for (1.1) is defined analogously by reversing the above inequalities.

A function $\alpha \in E^*$ is said to be a lower solution to (1.2), if it satisfies

$$\Delta\alpha(k-1) \le f(k, \alpha(k), \alpha(k-\tau)), \quad k \in I_2,$$
$$\alpha(0) \le \alpha(T).$$

An upper solution for (1.2) is defined analogously by reversing the above inequalities. □

Theorem 3.3. *Suppose that there exists a lower solution α and an upper solution β of* (1.1) *such that $\alpha \le \beta$ on $Z[-\tau, T]$.*

Assume that there exist $0 < M < 1, N > 0$ satisfying: (H_1) $f(k, u_2, v_2) - f(k, u_1, v_1) \ge -M(u_2 - u_1) - N(v_2 - v_1)$, for $k \in I_1$,

whenever $\alpha(k) \le u_1 \le u_2 \le \beta(k)$, and $\alpha(k-\tau) \le v_1 \le v_2 \le \beta(k-\tau)$.

(H_2) $\frac{N}{(1-M)^T(M+N)} < 1$.

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal solution to the problem (1.1) in the segment $[\alpha, \beta]$.

Proof. For each given $\eta \in [\alpha, \beta]$, we consider the PBVP (3.1) with

$$\sigma(k) = \sigma_\eta(k) = f(k, \eta(k), \eta(k-\tau)) + M\eta(k) + N\eta(k-\tau).$$

We shall refer to this problem as $(PL)_\eta$.

Since $\eta \in [\alpha, \beta]$ we have by (H_1) and the definitions of lower and upper solutions, that

$$\Delta\alpha(k) + M\alpha(k) + N\alpha(k-\tau) \leq$$
$$\leq f(k, \alpha(k), \alpha(k-\tau)) + M\alpha(k) + N\alpha(k-\tau) \leq$$
$$\leq f(k, \eta(k), \eta(k-\tau)) + M\eta(k) + N\eta(k-\tau) = \sigma_\eta(k)$$

and

$$\Delta\beta(k) + M\beta(k) + N\beta(k-\tau) \geq \sigma_\eta(k), \quad k \in I_1.$$

As a consequence α and β are, respectively, a lower and an upper solutions for $(PL)_\eta$, and Theorem 3.1 permits us to define the operator $A : [\alpha, \beta] \to [\alpha, \beta]$, where A$\eta$ is the unique solutions of $(PL)_\eta$ on $[\alpha, \beta]$.

Concerning the mapping A, by applying Theorem 2.1, it is easy to prove that

Claim 3.1: A is monotone increasing mapping on the segment $[\alpha, \beta]$, namely, $A\eta_1 \leq A\eta_2$ when $\eta_1, \eta_2 \in [\alpha, \beta]$ and $\eta_1 \leq \eta_2$.

Thus we may define the sequences $\{\alpha_n\}, \{\beta_n\}$ by $\alpha_{n+1} = A\alpha_n,\ \beta_{n+1} = A\beta_n,\ \alpha_0 = \alpha,\ \beta_0 = \beta$.

Using Claim 3.1, it is easy to verify that

$$\alpha_0 = \alpha \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_0 = \beta.$$

Since $\{\alpha_n\}$ is nondecreasing, $\{\beta_n\}$ is nonincreasing, $\{\alpha_n\}$ and $\{\beta_n\}$ is bo-unded, we have that

$$\lim_{n\to\infty} \alpha_n(k) := \alpha^*(k) \quad \text{and} \quad \lim_{n\to\infty} \beta_n(k) := \beta^*(k)$$

uniformly and monotonically on $Z[-\tau, T]$. Using the definition of $(PL)_\eta$ and passing the limit when n tends to ∞, we conclude that $\alpha^*(k)$ and $\beta^*(k)$ are both solutions to problem (1.1).

Furthermore, if $y \in [\alpha, \beta]$ is a solution to problem (1.1), then, by induction, $\alpha_n(k) \leq y(k) \leq \beta_n(k)$ on $Z[-\tau, T]$, $n = 0, 1, 2, \ldots$ and hence, $y \in [\alpha^*, \beta^*]$. This shows that $\alpha^*(k)$ and $\beta^*(k)$ are, respectively, minimal and maximal solutions to problem (1.1) in the segment $[\alpha, \beta]$. □

Similar to the proof of Theorem 3.3, we have the following theorem.

Theorem 3.4. *Suppose that there exists a lower solution α and an upper solution β of* (1.2) *such that $\alpha \leq \beta$ on $Z[-\tau, T]$.*

Assume that there exist $M > 0, N > 0$ satisfying:

(H_1) $f(k, u_2, v_2) - f(k, u_1, v_1) \geq -M(u_2 - u_1) - N(v_2 - v_1)$, *for* $k \in I_2$, *whenever* $\alpha(k) \leq u_1 \leq u_2 \leq \beta(k)$ *and* $\alpha(k-\tau) \leq v_1 \leq v_2 \leq \beta(k-\tau)$.

(H_2) $\frac{N(1+M)^T}{M+N} < 1$.

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal solution to the problem (1.2) in the segment $[\alpha, \beta]$.

4. Monotone Method for Periodic Solutions of Delay Difference Equations

In this section, we are in a position to prove the validity of monotone method for (1.3) and (1.4). First, we shall introduce the concepts of lower and upper solutions for these problems.

Let X be defined as in Section 2, and let X with the norm

$$||y||_2 = \max_{k \in Z[0,T]} |y(k)|$$

for $y \in X$, then X is a Banach space.

A function $\alpha \in X$ is said to be a lower solution to (1.3), if it satisfies

$$\Delta\alpha(k) \le f(k, \alpha(k), \alpha(k-\tau)), \qquad k \in Z(-\infty, +\infty). \tag{4.1}$$

An upper solution for (1.3) is defined analogously by reversing the above inequalities.

A function $\alpha \in X$ is said to be a lower solution to (1.4), if it satisfies

$$\Delta\alpha(k-1) \le f(k, \alpha(k), \alpha(k-\tau)), \qquad k \in Z(-\infty, +\infty). \tag{4.2}$$

An upper solution for (1.4) is defined analogously by reversing the above inequalities.

By the same arguments as in Section 3, we have the following results:

Theorem 4.1. *Suppose that there exists a lower solution α and an upper solution β of* (1.3) *such that $\alpha \le \beta$ on $Z(-\infty, +\infty)$.*

Assume that there exist $0 < M < 1$, $N > 0$ satisfying:

(H_1) $f(k, u_2, v_2) - f(k, u_1, v_1) \ge -M(u_2 - u_1) - N(v_2 - v_1)$, *for* $k \in Z(-\infty, +\infty)$, *whenever* $\alpha(k) \le u_1 \le u_2 \le \beta(k)$,

and $\alpha(k-\tau) \le v_1 \le v_2 \le \beta(k-\tau)$.

(H_2) $\frac{N}{(1-M)^T(M+N)} < 1$.

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal T-periodic solution to the problem (1.3) *in the segment* $[\alpha, \beta]$.

Theorem 4.2. *Suppose that there exists a lower solution α and an upper solution β of* (1.4) *such that $\alpha \le \beta$ on $Z(-\infty, +\infty)$.*

Assume that there exist $M > 0, N > 0$, satisfying:

(H_1) $f(k, u_2, v_2) - f(k, u_1, v_1) \ge -M(u_2 - u_1) - N(v_2 - v_1)$, for $k \in Z(-\infty, +\infty)$,

whenever $\alpha(k) \le u_1 \le u_2 \le \beta(k)$,

and $\alpha(k-\tau) \le v_1 \le v_2 \le \beta(k-\tau)$. (H_2) $\frac{N(1+M)^T}{M+N} < 1$.

Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and nonincreasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly and monotonically to the extremal T-periodic solution to the problem (1.4) in the segment $[\alpha, \beta]$.

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