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ASYMPTOTIC REPRESENTATIONS FOR UNBOUNDED SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS CLOSE TO EQUATIONS OF EMDEN–FOWLER TYPE

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We will consider the differential equation

$$y'' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ is a continuous function and $\varphi : [y_0, +\infty[\to]0, +\infty[$ is a twice continuously differentiable function satisfying conditions

$$\lim_{y\to+\infty} \varphi(y) = \begin{cases} \text{either } 0, \\ \text{or } +\infty, \end{cases} \qquad \lim_{y\to+\infty} \frac{y\varphi''(y)}{\varphi'(y)} = \sigma \notin \{0, \pm\infty\}. \tag{2}$$

In view of (2)

$$\lim_{y\to+\infty} \frac{y\varphi'(y)}{\varphi(y)} = \sigma + 1, \tag{3}$$

which yields

$$\varphi(y) = y^{\sigma+1+o(1)} \quad \text{as} \quad y \to +\infty. \tag{4}$$

Further, when unbounded in any left neighbourhood of ω solutions are studied, we will take into account that the equation (1) is asymptotically close to an equation of Emden-Fowler type

$$y'' = \alpha_0 p(t) y^{\sigma+1},$$

which has been considered in detail (see, for example, [1-8]).

As solution y of the equation (1) defined on some interval $[t_y, \omega[\subset [a, \omega[$, will be called $P_\omega(\lambda_0)$-solution if it satisfies the conditions:

$$\lim_{t\uparrow\omega} y(t) = +\infty, \qquad \lim_{t\uparrow\omega} y'(t) = \begin{cases} \text{either } 0, \\ \text{or } +\infty, \end{cases} \qquad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0. \tag{5}$$

The purpose of this paper is to obtain asymptotic properties as $t \uparrow \omega$ of all $P_\omega(\lambda_0)$-solutions of the equation (1) with $\lambda_0 \notin \{0, 1, \pm\infty\}$.

Now we introduce the auxiliary notation:

$$\pi_\omega(t) = \begin{cases} t & \text{for } \omega = +\infty, \\ t - \omega & \text{for } \omega < +\infty, \end{cases} \qquad I(t) = \int_{A_\omega}^{t} p(\tau)\pi_\omega(\tau)\, d\tau, \qquad \Phi(y) = \int_{Y_0}^{y} \frac{dz}{\varphi(z)}, \tag{6}$$

where

$$A_\omega = \begin{cases} a & \text{for } \int_a^\omega p(\tau)|\pi_\omega(\tau)|\, d\tau = +\infty, \\ \omega & \text{for } \int_a^\omega p(\tau)|\pi_\omega(\tau)|\, d\tau < +\infty, \end{cases} \qquad Y_0 = \begin{cases} y_0 & \text{for } \sigma < 0, \\ +\infty & \text{for } \sigma > 0. \end{cases}$$

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It follows from (4) and (6) that for the function $\Phi(y)$ there exists an inverse Φ^{-1} defined either on the interval $[0,+\infty[$ if $\sigma<0$, and on the interval $[c_\varphi,0[$ with $c_\varphi = -\int_{y_0}^{+\infty}\frac{dz}{\varphi(z)}$ if $\sigma>0$; moreover

$$\begin{aligned} &\lim_{y\to+\infty}\Phi(y)=+\infty, && \lim_{z\to+\infty}\Phi^{-1}(z)=+\infty \quad \text{if } \sigma<0,\\ &\lim_{y\to+\infty}\Phi(y)=0, && \lim_{z\uparrow 0}\Phi^{-1}(z)=+\infty \quad \text{if } \sigma>0. \end{aligned} \tag{7}$$

The following statements are true for the equation (1).

Theorem 1. *For the existence of a $P_\omega(\lambda_0)$-solution of the equation* (1), *where $\lambda_0\notin\{0,\pm1,\pm\infty\}$, it is necessary and sufficient that the following conditions be satisfied*

$$\alpha_0\,\lambda_0>0, \qquad \frac{\lambda_0}{\lambda_0-1}\,\pi_\omega(t)>0 \quad \textit{if } t\in[a,\omega[^{*}, \qquad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\frac{\lambda_0\sigma}{1-\lambda_0}. \tag{8}$$

Moreover, each $P_\omega(\lambda_0)$-solution assumes the following asymptotic representations as $t\uparrow\omega$

$$\frac{y(t)}{\varphi(y(t))}=\alpha_0\sigma(1-\lambda_0)I(t)[1+o(1)], \qquad \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}[1+o(1)]. \tag{9}$$

Theorem 2. *For the existence of a $P_\omega(-1)$-solution of the equation* (1), *it is necessary and, in case $\sigma<0$ sufficient that*

$$\alpha_0=-1, \quad \omega=+\infty, \quad \lim_{t\to+\infty}\frac{tI'(t)}{I(t)}=-\frac{\sigma}{2}. \tag{10}$$

Each $P_{+\infty}(-1)$-solution assumes the following asymptotic representations as $t\to+\infty$

$$\frac{y(t)}{\varphi(y(t))}=-2\sigma I(t)[1+o(1)], \qquad \frac{y'(t)}{y(t)}=\frac{1+o(1)}{2t}. \tag{11}$$

Theorem 3. *If $\sigma>0$ and the following three conditions along with* (10) *are satisfied*

$$\begin{gathered} \lim_{t\to+\infty}\ln^2 t\left[\frac{\Phi^{-1}(2I(t))}{4I(t)\varphi(\Phi^{-1}(2I(t)))}-\frac{tI'(t)}{I(t)}\right]=0, \quad \lim_{t\to+\infty}\ln t\left[\frac{tI'(t)}{I(t)}+\frac{\sigma}{2}\right]=0,\\ \lim_{t\to+\infty}\ln t\,\left[4tI'(t)\varphi'(\Phi^{-1}(2I(t)))-1-\sigma\right]=0, \end{gathered} \tag{12}$$

then the equation (1) *has $P_{+\infty}(-1)$-solutions, which assume the following asymptotic representations*

$$\frac{y(t)}{\varphi(y(t))}=-2\sigma I(t)\left[1+o\left(\frac{1}{\ln t}\right)\right], \quad \frac{y'(t)}{y(t)}=\frac{1}{2t}\left[1+o\left(\frac{1}{\ln t}\right)\right] \quad \textit{as } t\to+\infty. \tag{13}$$

Remark 1. The formulae (10)–(11) coincide with (8)–(9) for $\lambda_0=-1$.

Proofs of Theorems 1 *and* 2. *Necessity.* Let $y:[t_y,\omega[\to R$ be a $P_\omega(\lambda_0)$-solution of the equation (1) with $\lambda_0\notin\{0,1,\pm\infty\}$. In view of (1) and the third condition of (5), we get

$$(y'(t))^2\sim\alpha_0\lambda_0 p(t)y(t)\varphi(y(t)) \qquad \text{as } t\uparrow\omega. \tag{14}$$

Moreover, (5) yields

$$\frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}[1+o(1)] \quad \text{as} \quad t\uparrow\omega, \tag{15}$$

so the second representation in (9) is true. Because of (14) and (15), taking into account that $\lim_{t\uparrow\omega}y(t)=+\infty$ and $p(t),\varphi(y(t))>0$ in a certain left neighbourhood ω, we get the first two inequalities in (8).

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*When $\omega=+\infty$ consider $a>0$.

Applying l'Hospital's rule and the formulae (3), (14) and (15), we have

$$\lim_{t\uparrow\omega}\frac{y(t)}{I(t)\varphi(y(t))}=\lim_{t\uparrow\omega}\frac{\left[\dfrac{y(t)}{\varphi(y(t))}\right]'}{I'(t)}=\lim_{t\uparrow\omega}\frac{\dfrac{y'(t)}{\varphi(y(t))}\left[1-\dfrac{y(t)\varphi'(y(t))}{\varphi(y(t))}\right]}{p(t)\pi_\omega(t)}=\alpha_0\sigma(1-\lambda_0).$$

It follows that

$$\frac{y(t)}{\varphi(y(t))}\sim\alpha_0\sigma(1-\lambda_0)I(t)\qquad\text{as } t\uparrow\omega,$$

so the second representation in (9) is true.

On the other hand, in view of (14) and (15) we get

$$\frac{y(t)}{\varphi(y(t))}\sim\frac{(\lambda_0-1)^2}{\lambda_0}\alpha_0\pi_\omega(t)I'(t)\qquad\text{as } t\uparrow\omega.$$

Comparing the two latter representations we see that the third condition of (8) is true.

Sufficiency. Suppose that the conditions (8) are true. Then $\sigma\pi_\omega(t)I(t)<0$ as $t>a$, and therefore A_ω may be defined in $I(t)$ as follows:

$$A_\omega=\begin{cases} a, & \text{for } \sigma<0,\\ \omega, & \text{for } \sigma>0.\end{cases}$$

With the help of the transformation

$$\begin{gathered}\Phi(y(t))=\alpha_0(\lambda_0-1)I(t)[1+z_1(x)],\\ \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}[1+z_2(x)],\quad x=\beta\ln|\pi_\omega(t)|,\end{gathered}\tag{16}$$

where

$$\beta=\begin{cases} 1 & \text{if } \omega=+\infty,\\ -1 & \text{if } \omega<+\infty,\end{cases}$$

we reduce the equation (1) to the following system

$$\begin{cases} z_1'=\beta\left[\dfrac{\alpha_0\lambda_0F(x,z_1)(1+z_2)}{(\lambda_0-1)^2}-G(x)(1+z_1)\right],\\[2ex] z_2'=\beta\left[\dfrac{\alpha_0(\lambda_0-1)G(x)}{\lambda_0F(x,z_1)}-\dfrac{1+(1+\lambda_0)z_2+\lambda_0{z_2}^2}{\lambda_0-1}\right],\end{cases}\tag{17}$$

where

$$\begin{gathered}G(x)=\frac{\pi_\omega(t(x))I'(t(x))}{I(t(x))},\quad F(x,z_1)=\frac{Y(t(x),z_1)}{I(t(x))\varphi(Y(t(x),z_1))},\\ Y(t(x),z_1)=\Phi^{-1}(\alpha_0(\lambda_0-1)I(t(x))(1+z_1)).\end{gathered}$$

We choose $t_0\in[a,\omega[$ such that $\beta\ln|\pi_\omega(t_0)|\ge 1$ and, in case $\sigma>0$, the inequality

$$2|\lambda_0-1|\int_t^\omega p(\tau)|\pi_\omega(\tau)|\,d\tau\le\int_{y_0}^{+\infty}\frac{dz}{\varphi(z)}\quad\text{as}\quad t\in[t_0,\omega[$$

be valid.

Fix a certain $\delta\in]0,1[$ and consider the obtained system of differential equations on the set

$$\Omega=[x_0,+\infty[\times D,\quad\text{where}\quad x_0=\beta\ln|\pi_\omega(t_0)|,\quad D=\{(z_1,z_2):|z_i|\le\delta,\ i=1,2\}.$$

On this set the right-hand sides of the system (17) are continuous in x and twice continuously differentiable in z_1,z_2.

For every fixed $x \in [x_0, +\infty[$, we expand the functions $F(x, z_1)$ and $\dfrac{1}{F(x,z_1)}$ by Taylor's formula with remainder in the Lagrange form in a neighbourhood of $z_1 = 0$ up to the second order derivatives and represent this system in form

$$\begin{cases} z_1' = F_1(x) + A_{11}(x)z_1 + A_{12}(x)z_2 + R_1(x, z_1, z_2), \\ z_2' = F_2(x) + A_{21}(x)z_1 + A_{22}(x)z_2 + R_2(x, z_1, z_2), \end{cases} \tag{18}$$

where

$$F_1(x) = \beta\left[\frac{\alpha_0\lambda_0}{(\lambda_0-1)^2}F(x,0) - G(x)\right], \quad F_2(x) = \beta\left[\frac{\alpha_0(\lambda_0-1)G(x)}{\lambda_0 F(x,0)} - \frac{1}{\lambda_0-1}\right],$$

$$A_{11}(x) = \beta\left[\frac{\lambda_0}{\lambda_0-1}(1 - M_1(x,0)) - G(x)\right], \quad A_{12}(x) = \beta\cdot\frac{\alpha_0\lambda_0 F(x,0)}{(\lambda_0-1)^2},$$

$$A_{21}(x) = \beta\frac{(\lambda_0-1)^2 G(x)}{\lambda_0 F^2(x,0)}\left(M_1(x,0) - 1\right), \quad A_{22}(x) = \beta\frac{\lambda_0+1}{1-\lambda_0},$$

$$R_1(x, z_1, z_2) = \beta\left[\frac{\alpha_0\lambda_0 M_1(x,\theta_1)}{2F(x,\theta_1)}\left(-1 - M_2(x,\theta_1) + M_1(x,\theta_1)\right)({z_1}^2 + {z_1}^2 z_2) + \right.$$
$$\left. + \frac{\lambda_0}{\lambda_0-1}(1 - M_1(x,0))z_1 z_2\right],$$

$$R_2(x, z_1, z_2) = \beta\left[\frac{\alpha_0(\lambda_0-1)^3}{2\lambda_0}\,\frac{G(x)}{F^3(x,\theta_2)}\big(M_2(x,\theta_2)M_1(x,\theta_2) - {M_1}^2(x,\theta_2) - \right.$$
$$\left. -3M_1(x,\theta_2) + 2\big)\,{z_1}^2 - \frac{\lambda_0}{\lambda_0-1}{z_2}^2\right]$$

with

$$M_1(x,\theta) = \frac{Y(t(x),\theta)\varphi'(Y(t(x),\theta))}{\varphi(Y(t(x),\theta))}, \quad M_2(x,\theta) = \frac{Y(t(x),\theta)\varphi''(Y(t(x),\theta))}{\varphi'(Y(t(x),\theta))},$$
$$|\theta_i| \le |z_1| \ (i = 1, 2).$$

In view of our choice of δ and the properties of the function Φ^{-1} given in (7), we get $\lim\limits_{x\to+\infty} Y(x,\theta) = +\infty$ as $|\theta| \le \delta$. Therefore, because of (3) and the second condition in (2), we have

$$\lim_{x\to+\infty} M_1(x,\theta) = \sigma + 1, \quad \lim_{x\to+\infty} M_2(x,\theta) = \sigma \quad \text{as} \quad |\theta| \le \delta.$$

In addition, using l'Hospital's rule we get

$$\lim_{x\to+\infty} F(x,0) = \lim_{t\uparrow\omega}\frac{\left[\dfrac{Y(t,0)}{\varphi(Y(t,0))}\right]'}{I'(t)} = \sigma\alpha_0(1-\lambda_0).$$

This equality and (8) imply

$$\lim_{x\to+\infty} F_i(x) = 0 \quad (i = 1, 2).$$

The limiting matrix of the coefficients of the linear part of the system (18) has the form

$$A = \lim_{x\to+\infty}\begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\beta\sigma\lambda_0}{1-\lambda_0} \\ \dfrac{\beta}{1-\lambda_0} & \dfrac{\beta(\lambda_0+1)}{1-\lambda_0} \end{pmatrix},$$

and

$$\lim_{|z_1|+|z_2|\to 0}\frac{R_i(x,z_1,z_2)}{|z_1|+|z_2|} = 0 \quad (i = 1, 2) \quad \text{evenly in} \quad x \in [x_0, +\infty[.$$

Writing out the characteristic equation for the matrix A, $\det[A-\mu E_2]=0$, where E_2 is the second order unit matrix, we obtain

$$\mu^2+\beta\frac{\lambda_0+1}{\lambda_0-1}\mu-\frac{\sigma\lambda_0}{(\lambda_0-1)^2}=0. \tag{19}$$

This equation has no solutions with zero real part if $\lambda_0\neq-1$ as well as if $\lambda_0=-1$ and $\sigma<0$. Hence, in these cases all conditions of Theorem 2.1 from [9] are fulfilled for the system of differential equations (18). According to this theorem, the system (18) has at least one solution $\{z_i\}_{i=1}^2:[x_1,+\infty[\longrightarrow\mathbb{R}^2$ $(x_1\geq x_0)$ tending to zero as $x\to+\infty$. This solution, because of (16), corresponds to a solution y of the equation (1) that admits the asymptotic representation

$$\Phi(y(t))=\alpha_0(\lambda_0-1)I(t)[1+o(1)],\qquad \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}[1+o(1)]\text{as } t\uparrow\omega.$$

Considering the form of the function Φ, the conditions (2), (3) and these representations, it is easy to see that the first representation assumes the form

$$\frac{y(t)}{\varphi(y(t))}=\alpha_0\sigma(\lambda_0-1)I(t)[1+o(1)]\quad\text{as}\quad t\uparrow\omega$$

and y is a $P_\omega(\lambda_0)$-solution. Theorems 1 and 2 are proved completely. □

Proof of Theorem 3. Let $\lambda_0=-1$, $\sigma>0$ and along with the conditions (8), having the form (10) in this case, the conditions (12) be fulfilled. Then, applying the transformation (16) to the equation (1), we obtain the system of differential equations (18) in complete analogy with the proof of Theorems 1 and 2. Since in this system $\lambda_0=-1$ and $\sigma>0$, the characteristic equation (19) has the solutions $\pm\frac{\sqrt{\sigma}}{2}i$. Moreover, in view of (8) and (12)

$$\lim_{x\to+\infty}x^2F_i(x)=0\quad(i=1,2),\qquad \lim_{x\to+\infty}xA_{ii}(x)=0\quad(i=1,2),$$

$$\lim_{x\to+\infty}x\left[A_{12}(x)+\frac{\beta\sigma}{2}\right]=0,\qquad \lim_{x\to+\infty}x\left[A_{21}(x)-\frac{\beta}{2}\right]=0$$

and

$$\lim_{|z_1|+|z_2|\to0}\frac{x^2R_i\left(x,\frac{z_1}{x},\frac{z_2}{x}\right)}{|z_1|+|z_2|}=0\quad(i=1,2)\quad\text{uniformly for}\quad x\in[x_0,+\infty[.$$

Hence, all the conditions of Theorem 2.2 from [9] are satisfied for the system (18). For this reason, it has at least one real solution of the form $z_i(x)=o\left(\frac{1}{x}\right)$ $(i=1,2)$ as $x\to+\infty$. In view of (16) and (10), this solution corresponds to a $P_{+\infty}(-1)$-solution y of the equation (1) that assumes the asymptotic representations

$$\Phi(y(t))=-2\alpha_0I(t)\left[1+o\left(\frac{1}{\ln t}\right)\right],\qquad \frac{y'(t)}{y(t)}=\frac{1}{2t}\left[1+o\left(\frac{1}{\ln t}\right)\right].$$

as $t\to+\infty$. Applying (2) and (3), it is easy to prove that the first representation above takes the form of the first of the asymptotic representation from (13). The theorem is proved completely. □

The result obtained in this paper essentially supplements the known results about equations of Emden–Fowler type because it deals with a class of equations that has not been studied so far.

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