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OPTIMAL PROCESSES IN THE SPECIFIC CONTROL SYSTEMS

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The well-known methods from [1], [2] allow receiving of necessary conditions of optimality in Pontryagin's maximum principle form for major problems of optimal control. Below the specific case of smooth-convex problem of optimization is considered, in which using these methods is difficult in principle. The smooth-convex problem of minimization (see [2]) has the form:

$$f_0(x,w)\to \inf \mid F(x,w)=0,\quad f_i(x,w)\le 0\quad (i=\overline{1,n}),\quad w\in W,$$

where $f_i : X\times W\to R$, $i=\overline{0,n}$, $F : X\times W\to Y$ are given mappings,X, Y are Banach spaces, R is the set of all real numbers, W is an arbitrary set. In the case where f_i and F are independent of x, the extremal principle from [2] is not valid. Just in this case we consider the problem

$$f_0(w)\to \inf \mid F(w)=0,\quad f_i(w)\le 0\quad (i=\overline{1,n}),\quad w\in W,\tag{1}$$

where W is a Banach space.

Theorem 1. *Let for the problem* (1) *the following assumptions be fulfilled:*
I) *for* $\forall w_1\in W, w_2\in W$ *and* $\alpha\in[0,1],\exists w\in W$ *such that*

$$F(w)=\alpha F(w_1)+(1-\alpha)F(w_2),$$
$$f_i(w)\le \alpha f_i(w_1)+(1-\alpha)f_i(w_2),\quad i=\overline{0,n};$$

II) *the functions* $f_i, i=\overline{0,n}$ *are Fréchet differentiable at* $\widehat{w}$ *when* $F(\widehat{w})=0$, *and the mapping* F *is continuously differentiable and regular at* $\widehat{w}$.

Then for any solution $\widehat{w}$ *of the problem* (1), *there exist numbers* $\lambda_i\ge 0, i=\overline{0,n}$, *and an element* y^* *of the conjugate space* Y^* *such that the conditions*

a) $(\lambda_0,\lambda_1,\dots,\lambda_n,y^*)\ne(0,\dots,0)$;

b) $\lambda_i f_i(\widehat{w})=0$, $i=\overline{1,n}$;

c) $L(\widehat{w},\lambda_0,\lambda_1,\dots,\lambda_n,y^*) = \min\limits_{w\in W} L(w,\lambda_0,\lambda_1,\dots,\lambda_n,y^*)$, *where* $L(w,\lambda_0,\lambda_1,\dots,\lambda_n,y^*)=\sum\limits_{i=0}^{n}\lambda_i f_i(w)+\langle y^*,F(w)\rangle$;

d) $\sum\limits_{i=0}^{n}\lambda_i\frac{\partial f_i(\widehat{w})}{\partial w}+(F'(\widehat{w}))^*y^*=0$;

e) *If there exists* $w_0\in W$ *such that* $F(w_0)=0$ *is fulfilled and* $f_i(w_0)<0$ *for all* $i=\overline{1,n}$ *for which* $f_i(\widehat{w})=0$, *then* $\lambda_0\ne 0$ *and the conditions* a)–d) *are sufficient for optimality of the admissible element* $\widehat{w}$;

f) *If among the Lagrange multipliers satisfying condition* d) *there are no multipliers of the form* $(0,\lambda_1,\dots,\lambda_n,y^*)$, *then the system of normal multipliers* $(1,\lambda_1,\dots,\lambda_n,y^*)$ *is uniquely defined,*
are fulfilled.

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Proof. First of all we note that the condition d) is a corollary of condition c). Indeed, from the condition c) it follows $L_w(\widehat{w}, \lambda_1, \dots, \lambda_n, y^*) = 0$, from which we have:

$$L_w(\widehat{w}, \lambda_1, \dots, \lambda_n, y^*) = \sum_{i=0}^{n} \lambda_i \frac{\partial f_i(\widehat{w})}{\partial w} + \langle y^*, F(w)\rangle \circ F'(\widehat{w}) =$$

$$= \sum_{i=0}^{n} \lambda_i \frac{\partial f_i(\widehat{w})}{\partial w} + \langle y^*, F'(\widehat{w})\rangle = \sum_{i=0}^{n} \lambda_i \frac{\partial f_i(\widehat{w})}{\partial w} + (F'(\widehat{w}))^* y^* = 0.$$

Further, since the mapping F is continuously differentiable and regular at $\widehat{w}$, then (see $[3], p.314$) for any neighborhood $U(\widehat{w})$ of the point $\widehat{w}$ the set $F(U(\widehat{w}))$ contains a neighborhood of zero of the space Y. But then using the Lagrange principle of taking restrictions off (see $[4], p.107$), we have conditions a),b) and c).

Let $\lambda_0 = 0$. Then in case where $w = w_0$, from c) we have

$$\sum_{i=1}^{n} \lambda_i f_i(w_0) \geq \sum_{i=1}^{n} \lambda_i f_i(\widehat{w}). \tag{2}$$

Since $\lambda_i \geq 0$ $(i = \overline{1,n})$, using the condition b), from (2) we have $\lambda_i = 0, i = \overline{1,n}$. If in this case $y^* \neq 0$, then in any neighborhood of zero of the space Y there exists a point y for which $\langle y^*, y\rangle < 0$. Hence $\exists w \in W \mid \langle y^*, F(w)\rangle < 0$ and this contradicts c). So, if $\lambda_0 = 0$, then $(\lambda_0, \lambda_1, \dots, \lambda_n, y^*) = (0, \dots, 0)$, and this contradicts a); i.e., $\lambda_0 \neq 0$, and $\lambda_0 = 1$. In this case we have

$$f_0(\widehat{w}) = f_0(\widehat{w}) + \sum_{i=0}^{n} \lambda_i f_i(\widehat{w}) + \langle y^*, F(\widehat{w})\rangle) \leq$$

$$\leq f_0(w) + \sum_{i=0}^{n} \lambda_i f_i(w) + \langle y^*, F(w)\rangle) \leq f_0(w),$$

$\forall w \in W \mid F(w) = 0,\ f_i(w) \leq 0,\ i = \overline{1,n}$, i.e., $\widehat{w}$ is e solution of the problem (1).

Let now $(1, \overline{\lambda_1}, \dots, \overline{\lambda_n}, \overline{y^*}) \neq (1, \lambda_1, \dots, \lambda_n, y^*)$ be two normal systems of Lagrange multipliers. Then

$$\frac{\partial f_0(\widehat{w})}{\partial w} + \sum_{i=1}^{n} \overline{\lambda_i} \frac{\partial f_i(\widehat{w})}{\partial w} + (F'(\widehat{w}))^* \overline{y^*} = 0,$$

$$\frac{\partial f_0(\widehat{w})}{\partial w} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i(\widehat{w})}{\partial w} + (F'(\widehat{w}))^* y^* = 0,$$

and we have

$$\sum_{i=1}^{n} \mu_i \frac{\partial f_i(\widehat{w})}{\partial w} + (F'(\widehat{w}))^* z^* = 0,$$

where

$$\mu_i = \lambda_i - \overline{\lambda_i}, \quad z^* = y^* - \overline{y^*}.$$

From this equation we have $\mu_0 = 0$, $\mu_1, \dots, \mu_n$, z^* is a nontrivial system of Lagrange multipliers and this contradicts the normality of the problem. □

In the case where the mapping F has the form

$$F(w) = \begin{cases} \dot{x} - f(x, u), \\ y^2 + g(x, u), \end{cases}$$

where $w = (x, y, u)$, $x \in W_{1,1}^n[t_0, t_1]$, $y \in L_2[t_0, t_1]$, $u \in L_1[t_0, t_1]$,

$$y^2 = \begin{pmatrix} y_1^2 \\ \vdots \\ y_m^2 \end{pmatrix}, \quad f = \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix}, \quad g = \begin{pmatrix} g^1 \\ \vdots \\ g^m \end{pmatrix}$$

and

$$f_0 = \int_{t_0}^{t_1} f^0(x(t), u(t))dt, \quad f_i(w) = q_i(x(t_0), x(t_1)), \quad i = \overline{1, s} \quad (s \leq 2n),$$

we consider the problem

$$I = \int_{t_0}^{t_1} f^0(x(t), u(t))dt \to \inf \tag{3}$$

under the restrictions:

$$\dot{x} = f(x(t), u(t)), \tag{4}$$

$$g(x(t), u(t)) \leq 0, \tag{5}$$

$$q(x(t_0), x(t_1)) \leq 0. \tag{6}$$

If the vector functions f, g, q are linear with respect to all their arguments, the restrictions (4),(5) are fulfilled almost everywhere on $[t_0, t_1]$ and the restriction (4) satisfies the conditions: for any (x, u) satisfying (4), the system of vectors $\text{grad}_u g^j(x, u)$, $j \in J(x, u)$, is linearly independent (here by $J(x, u)$ we denote the set of such indices $j \in \{1, 2, \ldots, m\}$ for which $g^j(x, u) = 0$), then the assumptions I), II) theorem 1 are fulfilled and using this theorem we have the following necessary conditions of optimality for the problem (3)–(6):

Theorem 2 (necessary conditions of optimality). *Let $(x(t), u(t))$ be a solution of the problem (3)–(6). Then there exist multipliers $\psi_0 \geq 0$, $\lambda \in R^s$, $\psi(t) \in W^n_{1,1}[t_0, t_1]$ and $\mu(t) \in L^m_\infty[t_0, t_1]$ such that almost everywhere on $[t_0, t_1]$ the following conditions are fulfilled*

$$\mu_j(t) \geq 0, \tag{7}$$

$$\mu_j(t) g^j(x(t), u(t)) = 0, \quad j = \overline{1, m}, \tag{8}$$

$$H(x(t), u(t), \psi_0, \psi(t)) = \min_{u \in \{u | g(x(t), u) \leq 0\}} H(x(t), u, \psi_0, \psi(t)), \tag{9}$$

$$\frac{d\psi}{dt} = \frac{\partial \mathcal{R}(x(t), u(t), \psi_0, \psi(t), \mu(t))}{\partial x}, \tag{10}$$

$$\frac{\partial \mathcal{R}(x(t), u(t), \psi_0, \psi(t), \mu(t))}{\partial u} = 0, \tag{11}$$

where

$$H(x(t), u(t), \psi_0, \psi(t)) = \psi_0 f_0(x(t), u(t)) - \sum_{i=1}^{n} \psi_i(t) f^i(x(t), u(t)),$$

$$\mathcal{R}(x(t), u(t), \psi_0, \psi(t), \mu(t)) = H(x(t), u(t), \psi_0, \psi(t)) + \sum_{i=1}^{m} \mu_i(t) g^i(x(t), u(t))$$

and

$$(\psi_0, \psi(t)) \neq (0, 0), \psi(t_0) = \sum_{i=1}^{s} \lambda_i \frac{\partial q^i}{\partial x(t_0)}, \psi(t_1) = -\sum_{i=1}^{s} \lambda_i \frac{\partial q^i}{\partial x(t_1)}. \tag{12}$$

The conditions (7)–(12) allow to solve some linear control problems, in particular, the problem

$$I = \int_{0}^{T} -u(t)dt \to \inf$$

under the restrictions

$$\dot{x} = ax(t) - u(t),$$
$$0 \le u(t) \le ax(t),$$
$$x(0) = x_0, \ \ x(T) = x_1,$$

where $a = \text{const} > 0$, $x_1 > x_0 > 0$. Using the conditions (7)–(12), we have the following optimal solution

$$(x(t), u(t)) = \begin{cases} (e^{at}, 0), & t \in [0, t^*], \\ (e^{at^*}, ae^{at^*}), & t \in [t^*, T], \end{cases}$$

where t^* is defined from the condition $x(t^*) = x_1$.

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