

T. Tadumadze and L. Alkhazishvili

**FORMULAS OF VARIATION  
OF SOLUTION FOR NON-LINEAR  
CONTROLLED DELAY DIFFERENTIAL  
EQUATIONS WITH CONTINUOUS  
INITIAL CONDITION**

**Abstract.** Formulas of variation of solution for controlled differential equations with variable delays are proved. The continuous initial condition means that at the initial moment the values of the initial function and the trajectory coincide.

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## INTRODUCTION

In the present work the formulas of variation of solution for controlled differential equations with variable delays are proved. These formulas play an important role in proving necessary conditions of optimality for optimal problems. The formulas of variation of solution for various classes of delay differential equations are given in [1]–[4].

## 1. FORMULATION OF MAIN RESULTS

Let  $\mathcal{J} = [a, b]$  be a finite interval;  $\mathcal{O} \subset R^n$ ,  $G \subset R^r$  be open sets. Let the function  $f : \mathcal{J} \times \mathcal{O}^s \times G \rightarrow R^n$  satisfy the following conditions: for almost all  $t \in \mathcal{J}$ , the function  $f(t, \cdot) : \mathcal{O}^s \times G \rightarrow R^n$  is continuously differentiable; for any  $(x_1, \dots, x_s, u) \in \mathcal{O}^s \times G$ , the functions  $f(t, x_1, \dots, x_s, u)$ ,  $f_{x_i}(t, x_1, \dots, x_s, u)$ ,  $i = 1, \dots, s$ ,  $f_u(t, x_1, \dots, x_s, u)$  are measurable on  $\mathcal{J}$ ; for arbitrary compacts  $K \subset \mathcal{O}$ ,  $M \subset G$  there exists a function  $m_{K,M}(\cdot) \in L(\mathcal{J}, R_+)$ ,  $R_+ = [0, +\infty)$ , such that for any  $(x_1, \dots, x_s, u) \in K^s \times M$  and for almost all  $t \in \mathcal{J}$ , the following inequality is fulfilled

$$|f(t, x_1, \dots, x_s, u)| + \sum_{i=1}^s |f_{x_i}(\cdot)| + |f_u(\cdot)| \leq m_{K,M}(t).$$

Let the scalar functions  $\tau_i(t)$ ,  $i = 1, \dots, s$ ,  $t \in R$ , be absolutely continuous and satisfy the conditions:  $\tau_i(t) \leq t$ ,  $\dot{\tau}_i(t) > 0$ ,  $i = 1, \dots, s$ . Let  $\Phi$  be the set of piecewise continuous functions  $\varphi : \mathcal{J}_1 = [\tau, b] \rightarrow \mathcal{O}$  with a finite number of discontinuity points of the first kind, satisfying the conditions  $\text{cl } \varphi(\mathcal{J}_1) \subset \mathcal{O}$ ,  $\tau = \min\{\tau_1(a), \dots, \tau_s(a)\}$ ,  $\|\varphi\| = \sup\{|\varphi(t)|, t \in \mathcal{J}_1\}$ ;  $\Omega$  be the set of measurable functions  $u : \mathcal{J} \rightarrow G$  satisfying the condition:  $\text{cl}\{u(t) : t \in \mathcal{J}\}$  is a compact lying in  $G$ ,  $\|u\| = \sup\{|u(t)| : t \in \mathcal{J}\}$ .

To every element  $\varphi = (t_0, \varphi, u) \in A = \mathcal{J} \times \Phi \times \Omega$ , let us correspond the differential equation

$$\dot{x}(t) = f(t, x(\tau_1(t)), \dots, x(\tau_s(t)), u(t)) \quad (1.1)$$

with the continuous initial condition

$$x(t) = \varphi(t), \quad t \in [\tau, t_0]. \quad (1.2)$$

**Definition 1.1.** Let  $\varphi = (t_0, \varphi, u) \in A$ ,  $t_0 < b$ . A function  $x(t) = x(t; \varphi) \in \mathcal{O}$ ,  $t \in [\tau, t_1]$ ,  $t_1 \in (t_0, b]$ , is said to be a solution of the equation (1.1) with the initial condition (1.2), or a solution corresponding to the element  $\varphi \in A$ , defined on the interval  $[\tau, t_1]$ , if on the interval  $[\tau, t_0]$  the function  $x(t)$  satisfies the condition (1.2), while on the interval  $[t_0, t_1]$  it is absolutely continuous and almost everywhere satisfies the equation (1.1).

Let us introduce the set of variation

$$V = \{\delta\varphi = (\delta t_0, \delta\varphi, \delta u) : |\delta t_0| \leq \alpha = \text{const}, \delta\varphi = \sum_{i=1}^k \lambda_i \delta\varphi_i, \quad (1.3)$$

$$|\lambda_i| \leq \alpha, \quad i = 1, \dots, k, \quad \|\delta u\| \leq \alpha, \quad \delta u \in \Omega - \tilde{u}\},$$

where  $\delta\varphi_i \in \Phi - \tilde{\varphi}$ ,  $i = 1, \dots, k$ ,  $\tilde{\varphi} \in \Phi$ ,  $\tilde{u} \in \Omega$  are fixed functions,  $\alpha > 0$  is a fixed number.

Let  $\tilde{x}(t)$  be a solution corresponding to the element  $\tilde{\varphi} = (\tilde{t}_0, \tilde{\varphi}, \tilde{u}) \in A$ , defined on the interval  $[\tau, \tilde{t}_1]$ ,  $\tilde{t}_i \in (a, b)$ ,  $i = 0, 1$ . There exist numbers  $\varepsilon_1 > 0$ ,  $\delta_1 > 0$  such that for an arbitrary  $(\varepsilon, \delta\varphi) \in [0, \varepsilon_1] \times V$ , to the element  $\tilde{\varphi} + \varepsilon\delta\varphi \in A$  there corresponds a solution  $x(t; \tilde{\varphi} + \varepsilon\delta\varphi)$  defined on  $[\tau, \tilde{t}_1 + \delta_1] \subset J_1$  (see Lemma 2.2).

Due to uniqueness, the solution  $x(t, \tilde{\varphi})$  is a continuation of the solution  $\tilde{x}(t)$  on the interval  $[\tau, \tilde{t}_1 + \delta_1]$ . Therefore the solution  $\tilde{x}(t)$  in the sequel is assumed to be defined on the interval  $[\tau, \tilde{t}_1 + \delta_1]$ .

Let us define the increment of the solution  $\tilde{x}(t) = x(t; \tilde{\varphi})$

$$\begin{aligned} \Delta x(t) &= \Delta x(t; \varepsilon\delta\varphi) = x(t; \tilde{\varphi} + \varepsilon\delta\varphi) - \tilde{x}(t), \\ (t, \varepsilon, \delta\varphi) &\in [\tau, \tilde{t}_1 + \delta_1] \times [0, \varepsilon_1] \times V. \end{aligned} \quad (1.4)$$

**Theorem 1.1.** *Let the function  $\tilde{\varphi}(t)$  be absolutely continuous in some left semi-neighborhood of the point  $\tilde{t}_0$  and let there exist the finite limits*

$$\begin{aligned} \dot{\varphi}^- &= \dot{\tilde{\varphi}}(\tilde{t}_0^-), \\ \lim_{\omega \rightarrow \omega_0^-} \tilde{f}[\omega] &= f^-, \quad \omega = (t, x_1, \dots, x_s) \in (a, \tilde{t}_0] \times \mathcal{O}^s, \\ \tilde{f}[\omega] &= f(\omega, \tilde{u}(t)), \quad \omega_0^- = (\tilde{t}_0, \tilde{\varphi}(\tau_1(\tilde{t}_0^-)), \dots, \tilde{\varphi}(\tau_s(\tilde{t}_0^-))). \end{aligned}$$

Then there exist numbers  $\varepsilon_2 > 0$ ,  $\delta_2 > 0$  such that for an arbitrary  $(t, \varepsilon, \delta\varphi) \in [\tilde{t}_0, \tilde{t}_1 + \delta_2] \times [0, \varepsilon_2] \times V^-$ ,  $V^- = \{\delta\varphi \in V : \delta t_0 \leq 0\}$ , the formula

$$\Delta x(t; \varepsilon\delta\varphi) = \varepsilon\delta x(t; \delta\varphi) + o(t; \varepsilon\delta\varphi)^1 \quad (1.5)$$

is valid, where

$$\delta x(t; \delta\varphi) = Y(\tilde{t}_0; t) [(\dot{\varphi}^- - f^-)\delta t_0 + \delta\varphi(\tilde{t}_0^-)] + \beta(t; \delta\varphi), \quad (1.6)$$

$$\begin{aligned} \beta(t; \delta\varphi) &= \sum_{i=1}^s \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta\varphi(\xi) d\xi + \\ &+ \int_{\tilde{t}_0}^t Y(\xi; t) \tilde{f}_u[\xi] \delta u(\xi) d\xi, \end{aligned}$$

$$\gamma_i(t) = \tau_i^{-1}(t), \quad \tilde{f}_{x_i}[t] = \tilde{f}_{x_i}(t, \tilde{x}(\tau_1(t)), \dots, \tilde{x}(\tau_s(t))),$$

$Y(\xi; t)$  is the matrix-function satisfying the equation

$$\frac{\partial Y(\xi; t)}{\partial \xi} = - \sum_{i=1}^s Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi), \quad \xi \in [\tilde{t}_0, t], \quad (1.7)$$

<sup>1</sup>Here and in the sequel, the values (scalar or vector) which have the corresponding order of smallness uniformly for  $(t, \delta\varphi)$ , will be denoted by  $O(t; \varepsilon\delta\varphi)$ ,  $o(t; \varepsilon\delta\varphi)$ .

and the condition

$$Y(\xi; t) = \begin{cases} I, & \xi = t, \\ \Theta, & \xi > t. \end{cases} \quad (1.8)$$

Here  $I$  is the identity matrix,  $\Theta$  is the zero matrix.

**Theorem 1.2.** *Let the function  $\tilde{\varphi}(t)$  be absolutely continuous in some right semi-neighborhood of the point  $\tilde{t}_0$  and let there exist the finite limits*

$$\begin{aligned} \dot{\varphi}^+ &= \dot{\tilde{\varphi}}(\tilde{t}_0+), \\ \lim_{\omega \rightarrow \omega_0^+} \tilde{f}[\omega] &= f^+, \quad \omega = (t, x_1, \dots, x_s) \in [\tilde{t}_0, b) \times \mathcal{O}^s, \\ \omega_0^+ &= (\tilde{t}_0, \tilde{\varphi}(\tau_1(\tilde{t}_0+)), \dots, \tilde{\varphi}(\tau_s(\tilde{t}_0+))). \end{aligned}$$

Then for any  $s_0 \in (\tilde{t}_0, \tilde{t}_1)$  there exist numbers  $\varepsilon_2 > 0$ ,  $\delta_2 > 0$  such that for an arbitrary  $(t, \varepsilon, \delta\varphi) \in [s_0, \tilde{t}_1 + \delta_2] \times [0, \varepsilon_2] \times V^+$ ,  $V^+ = \{\delta\varphi \in V : \delta t_0 \geq 0\}$ , the formula (1.5) is valid, where  $\delta x(t; \delta\varphi)$  has the form

$$\delta x(t; \delta\varphi) = Y(\tilde{t}_0; t)[(\dot{\varphi}^+ - f^+)\delta t_0 + \delta\varphi(\tilde{t}_0+)] + \beta(t; \delta\varphi). \quad (1.9)$$

The following theorem follows from Theorems 1.1 and 1.2.

**Theorem 1.3.** *Let the conditions of Theorems 1.1 and 1.2 be fulfilled. Moreover, let*

$$\dot{\varphi}^- - f^- = \dot{\varphi}^+ - f^+ = f_0,$$

and the functions  $\delta\varphi_i(t)$ ,  $i = 1, \dots, k$  be continuous at the point  $\tilde{t}_0$  (see (1.3)). Then for any  $s_0 \in (\tilde{t}_0, \tilde{t}_1)$  there exist numbers  $\varepsilon_2 > 0$ ,  $\delta_2 > 0$  such that for an arbitrary  $(t, \varepsilon, \delta\varphi) \in [s_0, \tilde{t}_1 + \delta_2] \times [0, \varepsilon_2] \times V$  the formula (1.5) is valid, where  $\delta x(t; \varepsilon\delta\varphi)$  has the form

$$\delta x(t; \delta\varphi) = Y(\tilde{t}_0, t)[f_0\delta t_0 + \delta\varphi(\tilde{t}_0)] + \beta(t; \delta\varphi).$$

## 2. AUXILIARY LEMMAS

To every element  $\varphi = (t_0, \varphi, u) \in A$ , let us correspond the functional-differential equation

$$\begin{aligned} \dot{y}(t) &= f(t_0, \varphi, u, y)(t) = \\ &= f(t, h(t_0, \varphi, y)(\tau_1(t)), \dots, h(t_0, \varphi, y)(\tau_s(t)), u(t)) \end{aligned} \quad (2.1)$$

with the initial condition

$$y(t_0) = \varphi(t_0), \quad (2.2)$$

where the operator  $h(\cdot)$  is defined by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t), & t \in [\tau, t_0], \\ y(t), & t \in [t_0, b]. \end{cases} \quad (2.3)$$

**Definition 2.1.** Let  $\wp = (t_0, \varphi, u) \in A$ . An absolutely continuous function  $y(t) = y(t; \wp) \in \mathcal{O}$ ,  $t \in [r_1, r_2] \subset \mathcal{J}$ , is said to be a solution of the equation (2) with the initial condition (2.2), or a solution corresponding to the element  $\wp \in A$ , defined on the interval  $[r_1, r_2]$ , if  $t_0 \in [r_1, r_2]$ ,  $y(t_0) = \varphi(t_0)$  and the function  $y(t)$  satisfies the equation (2) almost everywhere on  $[r_1, r_2]$ .

*Remark 2.1.* Let  $y(t; \wp)$ ,  $t \in [r_1, r_2]$ ,  $\wp \in A$ , be a solution of the equation (2) with the initial condition (2.2). Then the function

$$x(t; \wp) = h(t_0, \varphi, y(\cdot; \wp))(t), \quad t \in [r_1, r_2], \quad (2.4)$$

is a solution of the equation (1.1) with the initial condition (1.2) (see Definition 1.1, (2.3)).

**Lemma 2.1.** Let  $\tilde{y}(t)$ ,  $t \in [r_1, r_2] \subset (a, b)$ , be a solution corresponding to the element  $\tilde{\wp} \in A$ ; let  $K_1 \subset \mathcal{O}$  be a compact which contains some neighborhood of the set  $\text{cl } \tilde{\varphi}(\mathcal{J}_1) \cup \tilde{y}([r_1, r_2])$  and let  $M_1 \subset G$  be a compact which contains some neighborhood of the set  $\text{cl } \tilde{u}(\mathcal{J})$ . Then there exist numbers  $\varepsilon_1 > 0$ ,  $\delta_1 > 0$  such that for an arbitrary  $(\varepsilon, \delta\varphi) \in [0, \varepsilon_1] \times V$ , to the element  $\tilde{\wp} + \varepsilon\delta\varphi \in A$  there corresponds a solution  $y(t; \tilde{\wp} + \varepsilon\delta\varphi)$  defined on  $[r_1 - \delta_1, r_2 + \delta_1] \subset \mathcal{J}$ . Moreover,

$$\begin{aligned} \varphi(t) &= \tilde{\varphi}(t) + \varepsilon\delta\varphi(t) \in K_1, \quad t \in \mathcal{J}_1, \\ u(t) &= \tilde{u}(t) + \varepsilon\delta u(t) \in M_1, \quad t \in \mathcal{J}, \\ y(t; \tilde{\wp} + \varepsilon\delta\varphi) &\in K_1, \quad t \in [r_1 - \delta_1, r_2 + \delta_1], \\ \lim_{\varepsilon \rightarrow 0} y(t; \tilde{\wp} + \varepsilon\delta\varphi) &= y(t; \tilde{\wp}) \\ &\text{uniformly for } (t, \delta\varphi) \in [r_1 - \delta_1, r_2 + \delta_1] \times V. \end{aligned} \quad (2.5)$$

This lemma is analogous to Lemma 2.1 in [1, p. 21] and it is proved analogously.

**Lemma 2.2.** Let  $\tilde{x}(t)$ ,  $t \in [\tau, \tilde{t}_1]$  be a solution corresponding to the element  $\tilde{\wp} \in A$ ,  $\tilde{t}_i \in (a, b)$ ,  $i = 0, 1$ ; let  $K_1 \subset \mathcal{O}$  be a compact which contains some neighborhood of the set  $\text{cl } \tilde{\varphi}(\mathcal{J}_1) \cup \tilde{x}([\tilde{t}_0, \tilde{t}_1])$  and let  $M_1 \subset G$  be a compact which contains some neighborhood of the set  $\text{cl } \tilde{u}(\mathcal{J})$ . Then there exist numbers  $\varepsilon_1 > 0$ ,  $\delta_1 > 0$  such that for any  $(\varepsilon, \delta\varphi) \in [0, \varepsilon_1] \times V$ , to the element  $\tilde{\wp} + \varepsilon\delta\varphi \in A$  there corresponds the solution  $x(t; \tilde{\wp} + \varepsilon\delta\varphi)$ ,  $t \in [\tau, \tilde{t}_1 + \delta_1] \subset \mathcal{J}_1$ . Moreover,

$$\begin{aligned} x(t; \tilde{\wp} + \varepsilon\delta\varphi) &\in K_1, \quad t \in [\tau, \tilde{t}_1 + \delta_1], \\ u(t) &= \tilde{u}(t) + \varepsilon\delta u(t) \in M_1, \quad t \in \mathcal{J}. \end{aligned} \quad (2.6)$$

It is easy to see that if in Lemma 2.1  $r_1 = \tilde{t}_0$ ,  $r_2 = \tilde{t}_1$ , then  $\tilde{y}(t) = \tilde{x}(t)$ ,  $t \in [\tilde{t}_0, \tilde{t}_1]$ ;  $x(t; \tilde{\wp} + \varepsilon\delta\varphi) = h(t_0, \varphi, y(\cdot; \tilde{\wp} + \varepsilon\delta\varphi))(t)$ ,  $(t, \varepsilon, \delta\varphi) \in [\tau, \tilde{t}_1 + \delta_1] \times [0, \varepsilon_1] \times V$  (see (2.4)).

Thus Lemma 2.2 is a simple corollary (see (2.5)) of Lemma 2.1.

Due to uniqueness, the solution  $y(t; \tilde{\varphi})$  on the interval  $[r_1 - \delta_1, r_2 + \delta_1]$  is a continuation of the solution  $\tilde{y}(t)$ ; therefore the solution  $\tilde{y}(t)$  in the sequel is assumed to be defined on the whole interval  $[r_1 - \delta_1, r_2 + \delta_1]$ .

Let us define the increment of the solution  $\tilde{y}(t) = y(t; \tilde{\varphi})$ ,

$$\begin{aligned} \Delta y(t) &= \Delta y(t; \varepsilon \delta \varphi) = y(t; \tilde{\varphi} + \varepsilon \delta \varphi) - \tilde{y}(t), \\ (t, \varepsilon, \delta \varphi) &\in [r_1 - \delta_1, r_2 + \delta_1] \times [0, \varepsilon_1] \times V. \end{aligned} \quad (2.7)$$

It is obvious (see Lemma 2.1) that

$$\lim_{\varepsilon \rightarrow 0} \Delta y(t; \varepsilon \delta \varphi) = 0 \quad \text{uniformly for } (t, \delta \varphi) \in [r_1 - \delta_1, r_2 + \delta_1] \times V. \quad (2.8)$$

**Lemma 2.3** ([1, p. 35]). *For arbitrary compacts  $K \subset \mathcal{O}$ ,  $M \subset G$  there exists a function  $L_{K,M}(\cdot) \in L(\mathcal{J}, R_+)$  such that for an arbitrary  $x'_i, x''_i \in K$ ,  $i = 1, \dots, s$ ,  $u', u'' \in M$  and for almost all  $t \in \mathcal{J}$ , the inequality*

$$\begin{aligned} |f(t, x'_1, \dots, x'_s, u') - f(t, x''_1, \dots, x''_s, u'')| &\leq \\ &\leq L_{K,M}(t) \left( \sum_{i=1}^s |x'_i - x''_i| + |u' - u''| \right) \end{aligned} \quad (2.9)$$

is valid.

**Lemma 2.4.** *Let the conditions of Theorem 1.1 be fulfilled. Then there exist numbers  $\varepsilon_2 > 0, \delta_2 > 0$  such that for any  $(\varepsilon, \delta \varphi) \in [0, \varepsilon_2] \times V^-$  the inequality*

$$\max_{t \in [t_0, r_2 + \delta_2]} |\Delta y(t)| \leq O(\varepsilon \delta \varphi) \quad (2.10)$$

is valid. Moreover,

$$\Delta y(\tilde{t}_0) = \varepsilon [\delta \varphi(\tilde{t}_0^-) + (\tilde{\varphi}^- - f^-) \delta t_0] + o(\varepsilon \delta \varphi). \quad (2.11)$$

*Proof.* Let us define the following sets

$$\begin{aligned} I_1 &= \{i \in \{1, \dots, s\} : \tau_i(\tilde{t}_0) < \tilde{t}_0, \gamma_i(\tilde{t}_0) > r_2\}, \\ I_2 &= \{i \in \{1, \dots, s\} : \tau_i(\tilde{t}_0) < \tilde{t}_0, \gamma_i(\tilde{t}_0) \leq r_2\}, \\ I_3 &= \{i \in \{1, \dots, s\} : \tau_i(\tilde{t}_0) = \tilde{t}_0\}. \end{aligned}$$

Let  $\varepsilon_2 \in (0, \varepsilon_1]$ ,  $\delta_2 \in (0, \delta_1]$  be so small that for any  $(\varepsilon, \delta \varphi) \in [0, \varepsilon_2] \times V^-$  the following relations are fulfilled:

$$\begin{aligned} \gamma_i(t_0) &> r_2 + \delta_2, \quad i \in I_1, \\ \tau_i(\tilde{t}_0) &< t_0, \quad i \in I_2. \end{aligned}$$

The function  $\Delta y(t)$  on the interval  $[\tilde{t}_0, r_2 + \delta_1]$  satisfies the equation

$$\dot{\Delta y}(t) = a(t; \varepsilon \delta \varphi), \quad (2.12)$$

where

$$a(t; \varepsilon \delta \varphi) = f(t_0, \varphi, u, \tilde{y} + \Delta y)(t) - f(\tilde{t}_0, \tilde{\varphi}, \tilde{u}, \tilde{y})(t). \quad (2.13)$$

Now let us rewrite the equation (2.13) in the integral form

$$\Delta y(t) = \Delta y(\tilde{t}_0) + \int_{\tilde{t}_0}^t a(\xi, \varepsilon \delta \varphi) d\xi, \quad t \in [\tilde{t}_0, r_2 + \delta_1].$$

Hence

$$|\Delta y(t)| \leq |\Delta y(\tilde{t}_0)| + \int_{\tilde{t}_0}^t |a(\xi; \varepsilon \delta \varphi)| d\xi = \Delta y(\tilde{t}_0) + a_1(t; \varepsilon \delta \varphi). \quad (2.14)$$

Now let us prove the equality (2.11). It is easy to see that

$$\begin{aligned} \Delta y(\tilde{t}_0) &= y(\tilde{t}_0; \tilde{\varphi} + \varepsilon \delta \varphi) - \tilde{y}(\tilde{t}_0) = \varphi(t_0) + \\ &+ \int_{\tilde{t}_0}^{\tilde{t}_0} f(t_0, \varphi, u, \tilde{y} + \Delta y)(t) dt - \tilde{\varphi}(\tilde{t}_0). \end{aligned} \quad (2.15)$$

Next,

$$\lim_{\varepsilon \rightarrow 0} \delta \varphi(t_0) = \delta \varphi(\tilde{t}_0-), \quad \text{uniformly for } \delta \varphi \in V^-.$$

So,

$$\begin{aligned} \varphi(t_0) - \tilde{\varphi}(\tilde{t}_0) &= \int_{\tilde{t}_0}^{t_0} \tilde{\varphi}(t) dt + \varepsilon \delta \varphi(\tilde{t}_0) + \varepsilon (\delta \varphi(t_0) - \delta \varphi(\tilde{t}_0-)) = \\ &= \varepsilon [\dot{\varphi}^- \delta t_0 + \delta \varphi(\tilde{t}_0-)] + o(\varepsilon \delta \varphi). \end{aligned} \quad (2.16)$$

It is clear that

$$\lim_{(\varepsilon, t) \rightarrow (0, \tilde{t}_0-)} h(t_0, \varphi, \tilde{y} + \Delta y)(\tau_i(t)) = \tilde{\varphi}(\tau_i(\tilde{t}_0-)), \quad i = 1, \dots, s,$$

(see (2.3), (2.8)). Consequently,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [t_0, \tilde{t}_0]} |\tilde{f}(t_0, \varphi, \tilde{y} + \Delta y)(t) - f^-| = 0 \quad (2.17)$$

From (2.17) it follows that

$$\begin{aligned} \int_{t_0}^{\tilde{t}_0} \tilde{f}(t_0, \varphi, u, \tilde{y} + \Delta y)(t) dt &= -\varepsilon f^- \delta t_0 + \\ &+ \int_{t_0}^{\tilde{t}_0} [\tilde{f}(t_0, \varphi, u, \tilde{y} + \Delta y)(t) - f^-] dt = -\varepsilon f^- \delta t_0 + o(\varepsilon \delta \varphi). \end{aligned} \quad (2.18)$$

From (2.15), by virtue of (2.16), (2.18), we obtain (2.11). Now, to prove the inequality (2.10), let us estimate the function  $a_1(t, \varepsilon \delta \varphi)$ ,  $t \in [\tilde{t}_0, r_2 + \delta_2]$ . We



have

$$\begin{aligned}
a_1(t; \varepsilon \delta \varphi) &= \sum_{i=1}^s \int_{\tilde{t}_0}^t L_{K_1, M_1}(\xi) \left( h(t_0, \varphi, \tilde{y} + \Delta y)(\tau_i(\xi)) - h(\tilde{t}_0, \tilde{\varphi}, \tilde{y})(\tau_i(\xi)) \right) + \\
&\quad + \varepsilon |\delta u(\xi)| d\xi \leq \sum_{i \in I_1} \int_{\tilde{t}_0}^{r_2 + \delta_2} L_{K_1, M_1}(\xi) \delta \varphi(\tau_i(\xi)) d\xi + \\
&\quad + \sum_{i \in I_2} \int_{\tau_i(\tilde{t}_0)}^t L_{K_1, M_1}(\gamma_i(\xi)) |h(t_0, \varphi, \tilde{y} + \Delta y)(\xi) - h(\tilde{t}_0, \tilde{\varphi}, \tilde{y})(\xi)| \dot{\gamma}_i(\xi) d\xi + \\
&\quad + \sum_{i \in I_3} \int_{\tilde{t}_0}^t L_{K_1, M_1}(\xi) |\Delta y(\xi)| \dot{\gamma}_i(\xi) d\xi + O(\varepsilon \delta \varphi) \leq O(\varepsilon \delta \varphi) + \\
&\quad + \sum_{i \in I_2} \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} L_{K_1, M_1}(\gamma_i(t)) |h(t_0, \varphi, \tilde{y} + \Delta y)(t) - \tilde{\varphi}(t)| \dot{\gamma}_i(t) dt + \\
&\quad + \int_{\tilde{t}_0}^t L(\xi) |\Delta y(\xi)| d\xi,
\end{aligned}$$

where

$$L(\xi) = \sum_{i=1}^s \chi(\gamma_i(\xi)) L_{K_1, M_1}(\gamma_i(\xi)) \dot{\gamma}_i(\xi),$$

and  $\chi(t)$  is the characteristic function of the interval  $\mathcal{J}$ .

If  $I_k = \emptyset$ , we assume that  $\sum_{i \in I_k} \alpha_i = 0$ . Let  $I_2 \neq \emptyset$ , and  $i \in I_2$ ,

$$\begin{aligned}
\beta_i &= \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} L_{K_1, M_1}(\gamma_i(t)) |h(t_0, \varphi, \tilde{y} + \Delta y)(t) - \tilde{\varphi}(t)| \dot{\gamma}_i(t) dt = \\
&= \varepsilon \int_{\tau_i(\tilde{t}_0)}^{t_0} L_{K_1, M_1}(\gamma_i(t)) |\delta \varphi(t)| \dot{\gamma}_i(t) dt + \\
&\quad + \int_{t_0}^{\tilde{t}_0} L_{K_1, M_1}(\gamma_i(t)) |y(t, \tilde{\varphi} + \varepsilon \delta \varphi) - \tilde{\varphi}(t)| \dot{\gamma}_i(t) dt.
\end{aligned}$$

It is obvious that when  $t \in [t_0, \tilde{t}_0]$ ,

$$\begin{aligned} |y(t, \tilde{\varphi} + \varepsilon\delta\varphi) - \tilde{\varphi}(t)| &= |\varphi(t_0) + \int_{t_0}^t f(t_0, \varphi, u, \tilde{y} + \Delta y)(\xi) d\xi - \tilde{\varphi}(t)| \leq \\ &\leq O(\varepsilon\delta\varphi) + \int_{t_0}^{\tilde{t}_0} |\dot{\tilde{\varphi}}(t)| dt + \int_{t_0}^{\tilde{t}_0} |f(t_0, \varphi, u, \tilde{y} + \Delta y)(t)| dt. \end{aligned} \quad (2.19)$$

For sufficiently small  $\varepsilon_2$  the integrands are bounded, hence  $\beta_i = O(\varepsilon\delta\varphi)$  (see (2.18)). Thus,

$$a_1(t; \varepsilon\delta\varphi) \leq O(\varepsilon\delta\varphi) + \int_{\tilde{t}_0}^t L(\xi) |\Delta y(\xi)| d\xi. \quad (2.20)$$

From (2.14), on account of (2.11), (2.20), on the interval  $t \in [\tilde{t}_0, r_2 + \delta_2]$  we have

$$|\Delta y(t)| \leq O(\varepsilon\delta\varphi) + \int_{\tilde{t}_0}^t L(\xi) |\Delta y(\xi)| d\xi.$$

From which, by virtue of Gronwall's lemma, we obtain (2.10).  $\square$

**Lemma 2.5.** *Let the conditions of Theorem 2 be fulfilled. Then there exist numbers  $\varepsilon_2 > 0$ ,  $\delta_2 > 0$  such that for any  $(\varepsilon, \delta\varphi) \in [0, \varepsilon_2] \times V^+$  the inequality*

$$\max_{t \in [t_0, r_2 + \delta_2]} |\Delta y(t)| \leq O(\varepsilon\delta\varphi) \quad (2.21)$$

is valid. Moreover,

$$\Delta y(t_0) = \varepsilon \left[ \delta\varphi(\tilde{t}_0+) + (\dot{\varphi}^+ - f^+) \delta t_0 \right] + o(\varepsilon\delta\varphi). \quad (2.22)$$

This lemma is proved analogously to lemma 2.4.

### 3. PROOF OF THEOREM 1.1

Let  $r_1 = \tilde{t}_0$ ,  $r_2 = \tilde{t}_1$ ,  $\varphi \in A$ , then

$$\tilde{y}(t) = \tilde{x}(t), \quad t \in [\tilde{t}_0, \tilde{t}_1 + \delta_1],$$

$$y(t; \tilde{\varphi} + \varepsilon\delta\varphi) = x(t; \tilde{\varphi} + \varepsilon\delta\varphi), \quad t \in [t_0, \tilde{t}_1 + \delta_1], \quad (\varepsilon\delta\varphi) \in [0, \varepsilon_1] \times V^-$$

(see Lemmas 2.1 and 2.2). Thus

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t), & t \in [\tau, t_0), \\ y(t; \tilde{\varphi} + \varepsilon\delta\varphi) - \tilde{\varphi}(t), & t \in [t_0, \tilde{t}_0], \\ \Delta y(t), & t \in [\tilde{t}_0, \tilde{t}_1 + \delta_1] \end{cases}$$

(see (1.4), (2.7)).

By virtue of Lemma 2.4 and (2.19), there exist numbers  $\varepsilon_2 \in (0, \varepsilon_1]$ ,  $\delta_2 \in (0, \delta_1]$  such that

$$|\Delta x(t)| \leq O(\varepsilon \delta \wp) \quad \forall (t, \varepsilon, \delta \wp) \in [\tau, \tilde{t}_1 + \delta_2] \times [0, \varepsilon_2] \times V^-, \quad (3.1)$$

$$\Delta x(\tilde{t}_0) = \varepsilon \left\{ \delta \wp(\tilde{t}_0^-) + (\dot{\varphi}^- - f^-) \delta t_0 \right\} + o(\varepsilon \delta \wp). \quad (3.2)$$

The function  $\Delta x(t)$  on the interval  $[\tilde{t}_0, t_1 + \delta_2]$  satisfies the following equation

$$\dot{\Delta} x(t) = \sum_{i=1}^s \tilde{f}_{x_i}[t] \Delta x(\tau_i(t)) + \varepsilon \tilde{f}_u[t] \delta u(t) + R(t; \varepsilon \delta \wp), \quad (3.3)$$

where

$$\begin{aligned} R(t; \varepsilon \delta \wp) = & f(t, \tilde{x}(\tau_1(t)) + \Delta x(\tau_1(t)), \dots, \tilde{x}(\tau_s(t)) + \Delta x(\tau_s(t)), \tilde{u}(t) + \varepsilon \delta u(t)) - \\ & - \tilde{f}[t] - \sum_{i=1}^s \tilde{f}_{x_i}[t] \Delta x(\tau_i(t)) - \varepsilon \tilde{f}_u[t] \delta u(t). \end{aligned} \quad (3.4)$$

By means of the Cauchy formula, the solution of the equation (3.3) can be represented in the form

$$\begin{aligned} \Delta x(t) = & Y(\tilde{t}_0; t) \Delta x(\tilde{t}_0) + \varepsilon \int_{\tilde{t}_0}^t Y(\xi; t) \tilde{f}_u[\xi] \delta u(\xi) d\xi + \\ & + \sum_{i=0}^1 h_i(t; \tilde{t}_0, \varepsilon \delta \wp), \quad t \in [\tilde{t}_0, \tilde{t}_1 + \delta_1], \end{aligned} \quad (3.5)$$

where

$$\begin{cases} h_0(t; \tilde{t}_0, \varepsilon \delta \wp) = \sum_{i=1}^s \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \Delta x(\xi) d\xi, \\ h_1(t; \tilde{t}_0, \varepsilon \delta \wp) = \int_{\tilde{t}_0}^t Y(\xi; t) R(\xi; \varepsilon \delta \wp) d\xi. \end{cases} \quad (3.6)$$

The matrix function  $Y(\xi; t)$  satisfies the equation (1.7) and the condition (1.8).

By virtue of Lemma 3.4 [1, p. 37], the function  $Y(\xi; t)$  is continuous on the set  $\Pi = \{(\xi, t) : a \leq \xi \leq t, t \in \mathcal{J}\}$ . Hence

$$Y(\tilde{t}_0; t) \Delta x(\tilde{t}_0) = \varepsilon Y(\tilde{t}_0; t) \left\{ \delta \wp(\tilde{t}_0^-) + (\dot{\varphi}^- - f^-) \delta t_0 \right\} + o(t; \varepsilon \delta \wp) \quad (3.7)$$

(see (3.3)).

Now we transform  $h_0(t; \tilde{t}_0, \varepsilon \delta \wp)$ . We have

$$\begin{aligned} h_0(t; \tilde{t}_0, \varepsilon \delta \wp) &= \sum_{i \in I_1 \cup I_2} \left[ \varepsilon \int_{\tau_i(\tilde{t}_0)}^{t_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta \varphi(\xi) d\xi + \right. \\ &\left. + \int_{t_0}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \Delta x(\xi) d\xi \right] = \sum_{i \in I_1 \cup I_2} \left[ \varepsilon \alpha_i(t) + \beta_i(t) \right]. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \alpha_i(t) &= \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta \varphi(\xi) d\xi - \\ &\quad - \int_{t_0}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta \varphi(\xi) d\xi, \\ \beta_i(t) &= o(t; \varepsilon \delta \wp) \end{aligned}$$

(see (2.19)). Thus,

$$\begin{aligned} h_0(t; \tilde{t}_0, \varepsilon \delta \wp) &= \\ &= \varepsilon \sum_{i=1}^s \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta \varphi(\xi) d\xi + o(t; \varepsilon \delta \wp). \end{aligned} \quad (3.8)$$

Finally we estimate  $h_1(t; \tilde{t}_0, \varepsilon \delta \wp)$ . We have:

$$\begin{aligned} |h_1(t; \tilde{t}_0, \varepsilon \delta \wp)| &\leq \|Y\| \int_{\tilde{t}_0}^{\tilde{t}_1 + \delta_2} \left| \int_0^1 \left[ \frac{d}{d\xi} f(t, \tilde{x}(\tau_1(t)) + \xi \Delta x(\tau_1(t)), \dots, \right. \right. \\ &\quad \left. \left. \tilde{x}(\tau_s(t)) + \xi \Delta x(\tau_s(t)), u(t) + \xi \varepsilon \delta u(t) \right) - \right. \\ &\quad \left. - \sum_{i=1}^s \tilde{f}_{x_i}[t] \Delta x(\tau_i(t)) - \varepsilon \tilde{f}_u[t] \delta u(t) \right] d\xi \Big| dt \leq \\ &\leq \|Y\| \int_{\tilde{t}_0}^{\tilde{t}_1 + \delta_2} \left\{ \int_0^1 \left[ \sum_{i=1}^s |f_{x_i}(t, \tilde{x}(\tau_1(t)) + \xi \Delta x(\tau_1(t)), \dots, \right. \right. \\ &\quad \left. \left. \tilde{x}(\tau_s(t)) + \xi \Delta x(\tau_s(t)), u(t) + \xi \varepsilon \delta u(t)) - \tilde{f}_{x_i}[t] \right| \times \right. \\ &\quad \left. \times |\Delta x(\tau_i(t))| + \right. \\ &\quad \left. + \varepsilon |f_u(t, \tilde{x}(\tau_1(t)) + \xi \Delta x(\tau_1(t)), \dots, \right. \\ &\quad \left. \tilde{x}(\tau_s(t)) + \xi \Delta x(\tau_s(t)), u(t) + \xi \varepsilon \delta u(t)) - \tilde{f}_u[t]| \cdot |\delta u(t)| \right] d\xi \Big\} dt \leq \\ &\leq \|Y\| \left( O(\varepsilon \delta \wp) \sum_{i=1}^s \sigma_i(\varepsilon \delta \wp) + \varepsilon \alpha \sigma_0(\varepsilon \delta \wp) \right), \end{aligned} \quad (3.9)$$

where

$$\|Y\| = \sup_{(\xi, t) \in \Pi} |Y(\xi; t)|,$$

$$\sigma_i(\varepsilon\delta\varphi) = \int_{\tilde{t}_0}^{\tilde{t}_1 + \delta_2} \left[ \int_0^1 |f_{x_i}(t, \tilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots) - \tilde{f}_{x_i}[t]| d\xi \right] dt,$$

$$\sigma_0(\varepsilon\delta\varphi) = \int_{\tilde{t}_0}^{\tilde{t}_1 + \delta_2} \left[ \int_0^1 |f_u(t, \tilde{x}(\tau_1(t)) + \xi\Delta x(\tau_1(t)), \dots) - \tilde{f}_u[t]| d\xi \right] dt.$$

By Lebesgue's theorem

$$\lim_{\varepsilon \rightarrow 0} \sigma_i(\varepsilon\delta\varphi) = 0, \quad i = 0, \dots, s, \quad \text{uniformly for } \delta\varphi \in V^-.$$

Thus

$$|h_1(t; \tilde{t}_0, \varepsilon\delta\varphi)| \leq o(t; \varepsilon\delta\varphi). \quad (3.10)$$

From (3.5), taking into account (3.7), (3), (3.10), we obtain (1.5), where  $\delta x(t; \delta\varphi)$  has the form (1.6).

#### 4. PROOF OF THEOREM 1.2

Analogously to the proof of Theorem 1.1, by virtue of Lemma 2.5 we obtain

$$|\Delta x(t)| \leq O(\varepsilon\delta\varphi) \quad \forall (t, \varepsilon\delta\varphi) \in [\tau, \tilde{t}_1 + \delta_2] \times (0, \varepsilon_2] \times V^+, \quad (4.1)$$

$$\Delta x(t_0) = \varepsilon[\delta\varphi(\tilde{t}_0+) + (\dot{\varphi}^+ - f^+)\delta t_0] + o(\varepsilon\delta\varphi). \quad (4.2)$$

Let  $s_0 \in (\tilde{t}_0, \tilde{t}_1)$  be the fixed point and let a number  $\varepsilon_2 \in (0, \varepsilon_1]$  be so small that for an arbitrary  $(\varepsilon, \delta\varphi) \in [0, \varepsilon_2] \times V^+$  the inequalities

$$t_0 < s_0, \quad \tau_i(t_0) < \tilde{t}_0, \quad i \in I_1 \cup I_2$$

are valid. The function  $\Delta x(t)$  satisfies the equation (3.3) on the interval  $[t_0, \tilde{t}_1 + \delta_2]$ , which by means of the Cauchy formula can be represented in the form

$$\Delta x(t) = Y(t_0; t)\Delta x(t_0) + \varepsilon \int_{t_0}^t Y(\xi, t)\tilde{f}_u[\xi]\delta u(\xi) d\xi +$$

$$+ \sum_{i=0}^1 h_i(t; t_0, \varepsilon\delta\varphi), \quad t \in [t_0, \tilde{t}_1 + \delta_2], \quad (4.3)$$

where the functions  $h_i(t; t_0, \varepsilon\delta\varphi)$ ,  $i = 0, 1$ , have the form (3.6)

By virtue of Lemma 3.4 [1, p. 37], the function  $Y(\xi; t)$  is continuous on the set  $[\tilde{t}_0, s_0] \times [s_0, \tilde{t}_1 + \delta_2]$ . Hence

$$Y(t_0; t)\Delta x(t_0) = \varepsilon Y(\tilde{t}_0; t)[\delta\varphi(\tilde{t}_0+) + (\dot{\varphi}^+ - f^+)\delta t_0] + o(t; \varepsilon\delta\varphi). \quad (4.4)$$

Now we transform  $h_0(t; t_0, \varepsilon\delta\varphi)$ :

$$\begin{aligned} h_0(t; t_0, \varepsilon\delta\varphi) &= \sum_{i \in I_1 \cup I_2} \left[ \varepsilon \int_{\tau_i(t_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta\varphi(\xi) d\xi + \right. \\ &\quad \left. + \int_{\tilde{t}_0}^{t_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \Delta x(\xi) d\xi \right] + \\ &\quad + \sum_{i \in I_3} \int_{\tau_i(t_0)}^{t_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \Delta x(\xi) d\xi = \\ &= \sum_{i \in I_1 \cup I_2} (\varepsilon\alpha_i(t) + \beta_i(t)) + \sum_{i \in I_3} \sigma_i(t). \end{aligned}$$

It is obvious that  $\beta_i(t) = o(t; \varepsilon\delta\varphi)$ ,  $\sigma_i(t) = o(t; \varepsilon\delta\varphi)$ .

Next,

$$\begin{aligned} \alpha_i(t) &= \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta\varphi(\xi) d\xi - \\ &\quad - \int_{\tau_i(\tilde{t}_0)}^{\tau_i(t_0)} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta\varphi(\xi) d\xi. \end{aligned}$$

Consequently

$$\begin{aligned} h_0(t; t_0, \varepsilon\delta\varphi) &= \\ &= \varepsilon \sum_{i=1}^s \int_{\tau_i(\tilde{t}_0)}^{\tilde{t}_0} Y(\gamma_i(\xi); t) \tilde{f}_{x_i}[\gamma_i(\xi)] \dot{\gamma}_i(\xi) \delta\varphi(\xi) d\xi + o(t; \varepsilon\delta\varphi). \end{aligned} \quad (4.5)$$

The following estimation is proved analogously

$$|h_1(t; t_0, \varepsilon\delta\varphi)| \leq o(t; \varepsilon\delta\varphi) \quad (4.6)$$

(see (3.9)). Finally we note that

$$\varepsilon \int_{t_0}^t Y(\xi; t) \tilde{f}_u[\xi] \delta u(\xi) d\xi = \varepsilon \int_{\tilde{t}_0}^t Y(\xi; t) \tilde{f}_u[\xi] \delta u(\xi) d\xi + o(t; \varepsilon\delta\varphi) \quad (4.7)$$

From (4.3), taking into account (4.4)–(4.7), we obtain (1.5), where  $\delta x(t; \delta\varphi)$  has the form (1.9).

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Authors' addresses:

T. Tadumadze  
I. Vekua Institute of Applied Mathematics  
I. Javakishvili Tbilisi State University  
2, University St., Tbilisi 0143  
Georgia

L. Alkhazishvili  
Department of Applied Mathematics  
and Computer Sciences  
I. Javakishvili Tbilisi State University  
2, University St., Tbilisi 0143  
Georgia