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ON OSCILLATORY PROPERTIES OF ORDINARY DIFFERENTIAL EQUATIONS OF GENERALIZED EMDEN–FOWLER TYPE

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1. Introduction

Consider the ordinary differential equation

$$u^{(n)}(t)+p(t)|u(t)|^{\mu(t)}\operatorname{sign}u(t)=0, \tag{1.1}$$

where $p\in L_{\rm loc}(R_+;R), \mu\in C(R_+;(0,+\infty)))$. The equation (1.1) is a generalization of the ordinary nonlinear differential equation of Emden–Fowler type.

Oscillatory properties of Emden–Fowler type equations are studied well enough [1-3]. As to the equation (1.1), analogous issues for it have not been considered earlier. Below optimal conditions will be given for the equation (1.1) to have either of the properties **A** and **B** (the definitions see below).

Let $t_0\in R_+$. A function $u:[t_0,+\infty)\to R$ is said to be a proper solution of the equation (1.1) if it is locally absolutely continuous along with its derivatives up to the order $n-1$ inclusively, $\sup\{|u(s)|: s\in[t,+\infty)\}>0$ for $t\geq t_0$ and the equality (1.1) holds almost everywhere on $[t_0,+\infty)$. A proper solution $u:[t_0,+\infty)\to R$ of the equation (1.1) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution u is said to be nonoscillatory.

Definition 1.1. We say that the equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even, and either is oscillatory or satisfies

$$|u^{(i)}(t)|\downarrow 0 \quad \text{as} \quad t\uparrow+\infty \tag{1.2}$$

when n is odd.

Definition 1.2. We say that the equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies either (1.2) or

$$|u^{(i)}(t)|\uparrow+\infty \quad \text{as} \quad t\uparrow+\infty \tag{1.3}$$

when n is even, and either is oscillatory or satisfies (1.3) when n is odd.

2. Differential Equations with Property A

Theorem 2.1. *Let $p\in L_{\rm loc}(R_+;R_+)$,*

$$\mu: R_+\to(0,1], \quad \limsup_{t\to\infty} t^{1-\mu(t)}<+\infty, \tag{2.1}$$

and for some $\varepsilon>0$ and any $\lambda\in[n-2,n-1]$

$$\liminf_{t\to\infty} t^{n-2-\lambda}\int_0^t s^{n-l+\lambda\mu(s)}p(s)ds>\prod_{i=0}^{n-2}|\lambda-i|+\varepsilon. \tag{2.2}$$

Then the equation (1.1) *has Property* **A**.

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Theorem 2.2. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, the condition* (2.1) *be fulfilled and there exist $\varepsilon > 0$ such that for any $\lambda \in [n-2, n-1]$*

$$\liminf_{t\to\infty} t^{-1} \int_0^t s^{n-\lambda+\lambda\mu(s)} p(s)ds > \prod_{i=0}^{n-1} |\lambda - i| + \varepsilon. \tag{2.3}$$

Then the equation (1.1) *has Property* **A**.

Theorem 2.3. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, the condition* (2.1) *be fulfilled and*

$$\liminf_{t\to\infty} t^{-1} \int_0^t s^n p(s)ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - i| : \lambda \in [n-2, n-1]\}, \tag{2.4}$$

where $\gamma = \liminf_{t\to\infty} t^{\mu(t)-1}$. Then the equation (1.1) *has Property* **A**.

Theorem 2.4. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$,*

$$\mu : R_+ \to [1, +\infty), \quad \limsup_{t\to\infty} t^{\mu(t)-1} < +\infty \tag{2.5}$$

and there exist $\varepsilon > 0$ such that when n is even for any $\lambda \in [0, 1]$

$$\liminf_{t\to+\infty} t^{-\lambda} \int_0^t s^{n-1+\lambda\mu(s)} p(s)ds > \prod_{i=0; i\neq 1}^{n-1} |\lambda - i| + \varepsilon,$$

while when n is odd the condition (2.2) *is fulfilled for any $\lambda \in [n-2, n-1]$ and the condition*

$$\liminf_{t\to+\infty} t^{1-\lambda} \int_0^t s^{n-2+\lambda\mu(s)} p(s)ds > \prod_{i=0; i\neq 2}^{n-1} |\lambda - i| + \varepsilon,$$

for any $\lambda \in [1, 2]$. Then the equation (1.1) *has Property* **A**.

Theorem 2.5. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, the condition* (2.5) *be fulfilled and there exist $\varepsilon > 0$ such that when n is even the condition* (2.3) *is fulfilled for any $\lambda \in [0, 1]$, while when n is odd the condition* (2.3) *is fulfilled for any $\lambda \in [1, 2] \cup [n-2, n-1]$. Then the equation* (1.1) *has Property* **A**.

Theorem 2.6. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, the condition* (2.5) *be fulfilled, and when n is even*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^n p(s)ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - i| : \lambda \in [0, 1]\}, \tag{2.6}$$

while when n is odd

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^n p(s)ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - i|,\ \lambda \in [1, 2] \cup [n-2, n-1]\}, \tag{2.7}$$

where $\gamma = \liminf_{t\to+\infty} t^{\mu(t)-1}$. Then the equation (1.1) *has Property* **A**.

3. Differential Equations with Property B

Theorem 3.1. *Let $p \in L_{\mathrm{loc}}(R_+; R_-)$, the condition* (2.1) *be fulfilled and there exist $\varepsilon > 0$ such that when n is even*

$$\liminf_{t\to+\infty} t^{n-3-\lambda} \int_0^t s^{2+\lambda\mu(s)} |p(s)|ds > \prod_{i=0; i\neq n-2}^{n-1} |\lambda - i| + \varepsilon \tag{3.1}$$

for any $\lambda \in [n-3, n-2]$, *while when* n *is odd the condition* (3.1) *is fulfilled for any* $\lambda \in [0,1]$ *and the condition*

$$\liminf_{t\to+\infty} t^{-\lambda} \int_0^t s^{n-1+\lambda\mu(s)}|p(s)|ds > \prod_{i=0;i=1}^{n-1} |\lambda - i| + \varepsilon$$

for any $\lambda \in [n-3, n-2]$. *Then the equation* (1.1) *has Property* **B**.

Theorem 3.2. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, *the condition* (2.1) *be fulfilled and there exist* $\varepsilon > 0$ *such that when* n *is even*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^{n-\lambda+\lambda\mu(s)}|p(s)|ds > \prod_{i=0}^{n-1} |\lambda - i| + \varepsilon \tag{3.2}$$

for any $\lambda \in [n-3, n-2]$, *while when* n *is odd the condition* (3.2) *is fulfilled for any* $\lambda \in [0,1] \cup [n-3, n-2]$. *Then the equation* (1.1) *has Property* **B**.

Theorem 3.3. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, *the condition* (2.1) *be fulfilled and when* n *is even*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^n |p(s)|ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - 1| : \lambda \in [n-3, n-2]\}, \tag{3.3}$$

while when n *is odd*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^n |p(s)|ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - 1| : \lambda \in [0,1] \cup [n-3, n-2]\}, \tag{3.4}$$

where $\gamma = \liminf_{t\to+\infty} t^{\mu(t)-1}$. *Then the equation* (1.1) *has Property* **B**.

Theorem 3.4. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, *the condition* (2.5) *be fulfilled and there exist* $\varepsilon > 0$ *such that when* n *is even*

$$\liminf_{t\to+\infty} t^{1-\lambda} \int_0^t s^{n-2+\lambda\mu(s)}|p(s)|ds > \prod_{i=0;i\neq 2}^{n-1} |\lambda - 1| + \varepsilon$$

for any $\lambda \in [1,2]$, *while when* n *is odd*

$$\liminf_{t\to+\infty} t^{-\lambda} \int_0^t s^{n-1+\lambda\mu(s)}|p(s)|ds > \prod_{i=0;i\neq 1}^{n-1} |\lambda - i| + \varepsilon$$

for any $\lambda \in [0,1]$. *Then the equation* (1.1) *has Property* **B**.

Theorem 3.5. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, *the condition* (2.5) *be fulfilled and there exist* $\varepsilon > 0$ *such that when* n *is even*

$$\liminf_{t\to\infty} t^{-1} \int_0^t s^{n-\lambda+\lambda\mu(s)}|p(s)|ds > \prod_{i=0}^{n-1} |\lambda - i| + \varepsilon \tag{3.5}$$

for any $\lambda \in [0,1]$, *while when* n *is odd the condition* (3.5) *is fulfilled for any* $\lambda \in [0,1]$. *Then the equation* (1.1) *has Property* **B**.

Theorem 3.6. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, *the condition* (2.5) *be fulfilled and when* n *is even*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^n |p(s)|ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - 1| : \lambda \in [1,2]\}, \tag{3.6}$$

while when n is odd

$$\liminf_{t\to+\infty} {}^{-1} \int\limits_0^t s^n |p(s)| ds > \max\{\gamma^{-\lambda} \prod_{i=0}^{n-1} |\lambda - 1| : \lambda \in [0,1]\}, \tag{3.7}$$

where $\gamma = \liminf\limits_{t\to+\infty} t^{\mu(t)-1}$. *Then the equation* (1.1) *has Property* **B**.

It should be noted that the strict inequalities (2.4) in Theorem 2.3, (2.6)–(2.7) in Theorem 2.6, (3.3)–(3.4) in Theorem 3.3, and (3.6)–(3.7) in Theorem 3.6 can not be replaced by the non-strict ones.

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