Memoirs on Differential Equations and Mathematical Physics
Volume 34, 2005, 115–134

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CORRECT BOUNDARY VALUE PROBLEMS FOR SOME CLASSES OF SINGULAR ELLIPTIC DIFFERENTIAL EQUATIONS ON A PLANE

Abstract. The investigation of differential equations of the type

$$\frac{\partial^n \omega}{\partial \overline{z}^n} + a_{n-1}\frac{\partial^{n-1}\omega}{\partial \overline{z}^{n-1}} + a_{n-2}\frac{\partial^{n-2}\omega}{\partial \overline{z}^{n-2}} + \cdots + a_0\omega = 0$$

with sufficiently smooth coefficients $a_0, a_1, \ldots, a_{n-1}$ (the theory of meta-analytic functions) traces back to the work of G. Kolosov [6]. Subsequently, a vast number of papers in this direction were published by many authors. The present work deals with some singular cases of the above-given equation. Correct boundary value problems are pointed out, and their in a sense complete analysis is given.

2000 Mathematics Subject Classification. 30E25, 30G20.

Key words and phrases. Meta-analytic functions, singular, elliptic, differential, equation, correct, boundary, problems.

რეზიუმე. საკმარისად გლუვკოეფიციენტებიანი

$$\frac{\partial^n \omega}{\partial \overline{z}^n} + a_{n-1}\frac{\partial^{n-1}\omega}{\partial \overline{z}^{n-1}} + a_{n-2}\frac{\partial^{n-2}\omega}{\partial \overline{z}^{n-2}} + \cdots + a_0\omega = 0$$

დიფერენციალურ განტოლებათა კვლევა (მეტაჰარმონიულ ფუნქციათა თეორია) სათავეს იღებს გ. კოლოსოვის [6] ნაშრომში. შემდგომ ამ მიმართულებით გამოქვეყნდა სხვადასხვა ავტორთა მიერ შესრულებულ გამოკვლევათა დიდი რაოდენობა.

წინამდებარე ნაშრომში შეისწავლება აღნიშნული განტოლების ზოგიერთი სინგულარული შემთხვევა; ამ განტოლებათათვის მითითებულია კორექტული სასაზღვრო ამოცანები და მოცემულია მათი გარკვეული აზრით სრული ანალიზი.

In Memory of Professor G. Manjavidze

1^0. In the domain G containing the origin of the plane of a complex variable $z = x + iy$ we consider a differential equation of the type

$$E_\nu \omega \equiv z^{2\nu} \frac{\partial^2 \omega}{\partial \overline{z}^2} + A z^\nu \frac{\partial \omega}{\partial \overline{z}} + B\omega = 0, \tag{1}$$

where A and B are given complex numbers, $\nu \geq 2$ is a given natural number and, as usual, $\frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$. To avoid a more simple case we assume that

$$B \neq 0. \tag{2}$$

The function $\omega(z)$ is said to be a solution of the equation (1), if it belongs to the class $C^2(G\backslash\{0\})$ and satisfies (1) at every point of the domain $G\backslash\{0\}$. We denote by $\mathbb{K}$ the set of such functions; it should be noted that it is wide enough.

Every non-trivial (not identically equal to zero) function from the set $\mathbb{K}$, being a classical solution of an elliptic differential equation in the neighborhood of any non-zero point of the domain G, has an isolated singularity at the point $z = 0$. The analysis of the structure of the functions $\omega \in \mathbb{K}$ shows highly complicated nature of their behaviour (in the vicinity of the singular point $z = 0$) and, undoubtedly, is of independent interest because it allows one to obtain a priori estimates of solutions and of their derivatives which in turn are necessary for the correct statement and for the investigation of boundary value problems. A highly complicated nature of behaviour of solutions in the vicinity of the origin can be explained first by the fact that the equation (1) at the point $z = 0$ degenerates up to the zero order.

For every function $\omega(z) \in \mathbb{K}$ we introduce into consideration the following natural characteristic, i.e., the function of the real argument $\rho > 0$,

$$T_\omega(\rho) \equiv \max_{0 \leq \varphi \leq 2\pi} \Big\{ \Big|\omega(\rho e^{i\varphi})\Big| + \Big|\frac{\partial \omega}{\partial \overline{z}}(\rho e^{i\varphi})\Big| \Big\}. \tag{3}$$

According to Theorem 1 proven below, we in particular conclude that for every non-trivial solution $\omega(z)$ the function (3) increases more rapidly not only than an arbitrary power of $\frac{1}{\rho}$ as $\rho \to 0$, but more rapidly than the function $\exp\{\frac{\delta}{\rho^{\nu-1}}\}$ for certain positive numbers δ.

With the equation (1) is tightly connected the characteristic equation

$$\lambda^2 + A\lambda + B = 0,$$

where λ is an unknown complex number, which, by (2), has two non-zero, possibly coinciding, roots; we denote them by λ_1 and λ_2, and in what follows it will be assumed that

$$|\lambda_1| \leq |\lambda_2|. \tag{4}$$

Having in hand the roots λ_1 and λ_2, we can factorize the operator E_ν in the form

$$E_\nu = \Big(z^\nu \frac{\partial}{\partial \overline{z}} - \lambda_1 I\Big) \circ \Big(z^\nu \frac{\partial}{\partial \overline{z}} - \lambda_2 I\Big),$$

and immediately obtain that every function $\omega(z) \in \mathbb{K}$ under the condition

$$\lambda_1 \neq \lambda_2$$

is representable as

$$\omega(z) = \phi(z) \exp\Big\{\frac{\lambda_1 \overline{z}}{z^\nu}\Big\} + \psi(z) \exp\Big\{\frac{\lambda_2 \overline{z}}{z^\nu}\Big\}, \tag{5}$$

and under the condition

$$\lambda_0 \equiv \lambda_1 = \lambda_2$$

as

$$\omega(z) = [\phi(z)\overline{z} + \psi(z)] \exp\Big\{\frac{\lambda_0 \overline{z}}{z^\nu}\Big\}, \tag{6}$$

where $\phi(z)$ and $\psi(z)$ are arbitrary holomorphic functions in the domain $G\backslash\{0\}$; $z = 0$ is an isolated singular point for $\phi(z)$ and $\psi(z)$.

2^0. Below we will need the following two statements whose proof is based on the well-known Fragman–Lindelöf principle (see, e.g., [1], [2], and also [3]).

Lemma 1. *Let $\phi(z)$ be a function holomorphic in the deleted neighborhood of the point $z = 0$ and such that*

$$\phi(z) = 0\ (\exp\{g(z)\}), \quad z \to 0, \tag{7}$$

where

$$g(z) = \frac{1}{|z|^{k-2}}\{\delta + a\cos(k \arg z) + b \sin(k \arg z)\},$$

$k \geq 3$ is natural, δ, a, b are real numbers, and

$$\delta = \sqrt{a^2 + b^2} \cos \pi\beta, \quad \beta = \max\Big\{0, \frac{k-4}{2k-4}\Big\}.$$

Then the function $\phi(z)$ is identically equal to zero.

Lemma 2. *Let ϕ a function holomorphic in the deleted neighborhood of the point $z = 0$ and such that the condition (7) is fulfilled with*

$$g(z) = \frac{1}{|z|}\{\sqrt{a^2 + b^2} + a\cos(3\arg z) + b\sin(3\arg z)\}$$

and a, b real numbers. Then the function $\phi(z)$ has the removable singularity at the point $z = 0$.

3^0. The following theorem holds (cf. [3]).

Theorem 1. *Let δ be a real number such that $\delta < |\lambda_1| \cos \pi\beta$, where*

$$\beta = \max\Big\{0, \frac{\nu - 3}{2\nu - 2}\Big\}. \tag{8}$$

Then for every non-trivial solution $\omega(z) \in \mathbb{K}$

$$\overline{\lim_{\rho \to 0+}} \frac{T_\omega(\rho)}{\exp\Big\{\frac{\delta}{\rho^{\nu-1}}\Big\}} = +\infty. \tag{9}$$

Proof. First, let $\lambda_1 \neq \lambda_2$. Then differentiating the general solution (5) with respect to $\overline{z}$, we have

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{\lambda_1}{z^\nu}\phi(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^\nu}\Big\} + \frac{\lambda_2}{z^\nu}\psi(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^\nu}\Big\},$$

which together with (5) provides us with

$$\begin{aligned}\phi(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^\nu}\Big\} &= \frac{1}{\lambda_1-\lambda_2}\Big(\lambda_1\omega - z^\nu\frac{\partial\omega}{\partial\overline{z}}\Big),\\ \psi(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^\nu}\Big\} &= \frac{1}{\lambda_1-\lambda_2}\Big(\lambda_2\omega - z^\nu\frac{\partial\omega}{\partial\overline{z}}\Big).\end{aligned} \tag{10}$$

Let for some solution $\omega(z) \in \mathbb{K}$ the condition (9) be violated; this means that there exist positive numbers M and ρ_0 such that

$$T_\omega(\rho) \leq M\exp\Big\{\frac{\delta}{\rho^{\nu-1}}\Big\}, \quad 0<\rho<\rho_0,$$

whence, with regard for (3), we obtain

$$\begin{gathered}|w(\rho e^{i\varphi})| \leq M\cdot\exp\{\frac{\delta}{\rho^{\nu-1}}\},\\ \Big\|\frac{\partial\omega}{\partial\overline{z}}(\rho e^{i\varphi})\Big\| \leq M\cdot\exp\Big\{\frac{\delta}{\rho^{\nu-1}}\Big\}, \quad 0<\rho<\rho_0, \quad 0\leq\varphi\leq 2\pi.\end{gathered} \tag{11}$$

In its turn, from (11) and (10) it follows the existence of a positive number M_0 such that

$$\begin{gathered}|\phi(z)| \leq M_0\exp\Big\{\frac{1}{|z|^{\nu-1}}[\delta - |\lambda_1|\cos(\psi_1-(\nu+1)\varphi)]\Big\},\\ |\psi(z)| \leq M_0\exp\Big\{\frac{1}{|z|^{\nu-1}}[\delta - |\lambda_2|\cos(\psi_2-(\nu+1)\varphi)]\Big\},\\ 0<|z|<\rho_0, \quad 0\leq\varphi\leq 2\pi,\end{gathered} \tag{12}$$

where $\varphi = \arg z$, $\psi_k = \arg\lambda_k$, $k=1,2$.

From the inequalities (12) by virtue of Lemma 1 we find that $\phi(z) \equiv \psi(z) \equiv 0$, i.e., the solution $\omega(z)$ is trivial.

Let now $\lambda_0 \equiv \lambda_1 = \lambda_2$. Then differentiating the general solution (6) with respect to $\overline{z}$, we have

$$\frac{\partial\omega}{\partial\overline{z}} = \Big[\phi(z)\Big(1+\frac{\lambda_0\overline{z}}{z^\nu}\Big) + \frac{\lambda_0}{z^\nu}\psi(z)\Big]\exp\Big\{\frac{\lambda_0\overline{z}}{z^\nu}\Big\},$$

which together with (6) provides us with

$$\begin{aligned}z^\nu\phi(z)\exp\Big\{\frac{\lambda_0\overline{z}}{z^\nu}\Big\} &= z^\nu\frac{\partial\omega}{\partial\overline{z}} - \lambda_0\omega,\\ z^\nu\psi(z)\exp\Big\{\frac{\lambda_0\overline{z}}{z^\nu}\Big\} &= (z^\nu+\lambda_0\overline{z})\omega - \overline{z}z^\nu\frac{\partial\omega}{\partial\overline{z}}.\end{aligned} \tag{13}$$

The formulas (13) obtained above are analogous to the formulas (10) which makes it possible to repeat our reasoning and conclude that the non-trivial solutions $\omega(z) \in \mathbb{K}$ are unable to violate the condition (9). □

From the above-proven theorem it immediately follows that for every non-trivial solution $\omega(z) \in \mathbb{K}$

$$\overline{\lim_{\rho\to 0+}} \frac{T_\omega(\rho)}{\exp\left\{\frac{\delta}{\rho\sigma}\right\}} = +\infty,$$

where δ is any real number, and the real number $\sigma < \nu - 1$.

4^0. Theorem 1 admits generalizations to more general systems of differential equations of the type

$$\sum_{k=0}^{m} z^{\nu k} A_k \frac{\partial^k \omega}{\partial \overline{z}^k} = 0, \tag{14}$$

where $\nu \geq 2$, $m \geq 1$ are given natural numbers, A_k, $k = 0, 1, \ldots, m$, are given complex square matrices of dimension $n \times n$, and

$$\det A_0 \neq 0, \quad \det A_m \neq 0, \tag{15}$$

$$A_k A_j = A_j A_k, \quad j, k = 0, 1, \ldots, m. \tag{16}$$

Under a solution of the system (14) we mean the vector function $\omega(z) = (\omega_1(z), \omega_2(z), \ldots, \omega_n(z))$ belonging to the class $C^m(G\backslash\{0\})$ and satisfying (14) at every non-zero point of the domain G.

By Λ we denote the set of all possible complex roots of the polynomial

$$\sum_{k=0}^{m} \tau_k \lambda^k = 0,$$

where the coefficient τ_k is some eigenvalue of the matrix A_k, $k = 0, 1, \ldots, m$. Introduce the number

$$\delta_0 \equiv \min_{\lambda\in\Lambda} |\lambda|,$$

which by (15) satisfies the inequality $\delta_0 > 0$.

The following theorem holds.

Theorem 1^***.** *Let $\psi(z)$ be a function analytic in some deleted neighborhood of the point $z = 0$ and having possibly arbitrary isolated singularities (concentration of singularities of the function $\psi(z)$ at the point $z = 0$ is not excluded). Further, let δ, σ be real numbers such that either $\sigma < \nu - 1$ (σ is arbitrary) or $\sigma = \nu - 1$, $\delta < \delta_0 \cos \pi\beta$ where the number β is given by the formula* (8)*. Then there are no non-trivial solutions of the system* (14) *satisfying the asymptotic condition*

$$\widetilde{T}_\omega(|z|) = 0\Big(|\psi(z)| \exp\left\{\frac{\delta}{|z|^\sigma}\right\}\Big), \quad z \to 0,$$

where

$$\widetilde{T}_\omega(\rho) \equiv \max_{0\leq\varphi\leq 2\pi} \sum_{k=1}^{n} \sum_{p=0}^{m-1} \Big|\frac{\partial^p \omega_k}{\partial \overline{z}^p}(\rho e^{i\varphi})\Big|, \quad \rho > 0.$$

5^0. Everywhere below G will denote a finite domain (containing the origin of coordinates of the complex plane) with the boundary Γ consisting of a finite number of simple, closed, non-intersecting Lyapunov contours. In the sequel, we will consider a special case of the equation (1), when $\nu = 2$, i.e., we consider the equation

$$z^4 \frac{\partial^2 \omega}{\partial \bar{z}^2} + A z^2 \frac{\partial \omega}{\partial \bar{z}} + B\omega = 0, \tag{17}$$

and study the following two boundary value problems.

Problem $R(\delta, \sigma)$. On the contour Γ there are prescribed Hölder continuous functions $a(t)$, $\gamma(t)$ where the function $\gamma(t)$ is real and $a(t) \neq 0$, $t \in \Gamma$. Real positive numbers δ, σ are also given. It is required to find a continuously extendable to $\overline{G} \backslash \{0\}$ solution of the equation (17) satisfying both the asymptotic condition

$$\overline{\lim_{\rho \to 0}} \frac{T_\omega(\rho)}{\exp\left\{\frac{\delta}{\rho^\sigma}\right\}} < +\infty \tag{18}$$

and the boundary condition

$$\operatorname{Re}\{a(t)\omega(t)\} = \gamma(t), \quad t \in \Gamma. \tag{19}$$

Problem $Q(\delta, \sigma)$**.** On the contour Γ there are prescribed Hölder continuous functions $\gamma_k(t)$, $a_{k,m}(t)$, $k, m = 1, 2$, where $\gamma_1(t)$, $\gamma_2(t)$ are real and

$$\det \|a_{k,m}(t)\| \neq 0, \quad t \in \Gamma.$$

Real positive numbers δ, σ are also given. It is required to find a continuously extendable (together with its derivative $\frac{\partial \omega}{\partial \bar{z}}$) to $\overline{G} \backslash \{0\}$ solution of the equation (17) satisfying both the condition (18) and the boundary condition

$$\operatorname{Re}\{a_{k,1}(t)\omega(t) + a_{k,2}(t)\frac{\partial \omega}{\partial \bar{z}}(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2. \tag{20}$$

Along with the problems formulated above, let us consider the following boundary value problems.

Problem $R_0(\overline{\overline{p}})$. Given an integer p, it is required to find a function $\phi_0(z)$ holomorphic in the domain G, continuously extendable to $\overline{G}$ and satisfying the boundary condition

$$\operatorname{Re}\{\alpha(t)\phi_0(t)\} = \gamma(t), \quad t \in \Gamma, \tag{21}$$

where $\alpha(t) = a(t)t^{2-p} \exp\{\frac{\lambda_1 \bar{t}}{t^2}\}$.

Problem $Q_0'(p)$. Given an integer p, it is required to find a vector function $(\phi_0(z), \psi_0(z))$ holomorphic in the domain G, continuously extendable to $\overline{G}$ and satisfying the boundary condition

$$\operatorname{Re}\{\alpha_{k,1}(t)\phi_0(t) + \alpha_{k,2}(t)\psi_0(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2, \tag{22}$$

where

$$\alpha_{k,m}(t) = \left[a_{k,1}(t)t^{2-p} + \frac{\lambda_m a_{k,2}(t)}{t^p}\right] \exp\left\{\frac{\lambda_m \bar{t}}{t^2}\right\}, \quad k, m = 1, 2.$$

Problem $Q_0''(p)$. Given an integer p, it is required to find a vector function $(\phi_0(z))$, $\psi_0(z)$ holomorphic in the domain G, continuously extendable to $\overline{G}$ and satisfying the boundary condition

$$\mathrm{Re}\{\beta_{k,1}(t)\phi_0(t)+\beta_{k,2}(t)\psi_0(t)\}=\gamma_k(t), \quad t\in\Gamma, \quad k=1,2, \tag{23}$$

where

$$\beta_{k,1}(t)=\Big[\frac{a_{k,1}(t)}{t^p}|t^2|+a_{k,2}(t)\Big(t^{1-p}+\frac{\lambda_0\overline{t}}{t^{2+p}}\Big)\Big]\exp\Big\{\frac{\lambda_0\overline{t}}{t^2}\Big\}$$

$$\beta_{k,2}(t)=\Big[a_{k,1}(t)t^{2-p}+\frac{\lambda_0}{t^p}a_{k,2}(t)\Big]\exp\Big\{\frac{\lambda_0\overline{t}}{t^2}\Big\}.$$

On the basis of the following obvious relations

$$\alpha(t)\neq 0, \quad t\in\Gamma,$$

$$\det\|\beta_{k,m}(t)\|=-t^{3-2p}\det\|a_{k,m}(t)\|e^{\frac{2\lambda_0\overline{t}}{t^2}}=0, \quad t\in\Gamma,$$

$$\det\|\alpha_{k,m}(t)\|=(\lambda_2-\lambda_1)t^{2-2p}\det\|a_{k,m}(t)\|e^{\frac{\lambda_1+\lambda_2}{t^2}\overline{t}}\neq 0, \quad t\in\Gamma,$$

if only $\lambda_1\neq\lambda_2$, we conclude that for every integer p the problems $R_0(p)$, $Q_0'(p)$, $Q_0''(p)$ refer to those boundary value problems which are well-studied (see, e.g., [4], [5]). In particular, it is known that the corresponding homogeneous problems ($\gamma(t)\equiv\gamma_1(t)\equiv\gamma_2(t)\equiv 0$) have finite numbers of linearly independent solutions[1] (and, as it is not difficult to see, these numbers become arbitrarily large as $p\to+\infty$). Also formulas for index calculation and criteria for the solvability of the problems are available.

6^0. We have the following

Theorem 2. *Let $|\lambda_1|<|\lambda_2|$. Then the boundary value problems $R(|\lambda_1|,1)$ and $R_0(0)$ are simultaneously solvable (unsolvable), and in case of their solvability the relation*

$$\omega(z)=z^2\phi_0(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}, \quad z\in G\backslash\{0\}, \tag{24}$$

establishes a bijective relation between the solutions of these problems.

Proof. First we have to find a general representation of solutions of the equation (17) which are continuously extendable to $\overline{G}\backslash\{0\}$ and satisfy the condition (18), where $\delta=|\lambda_1|$, $\sigma=1$. Towards this end, we use the equalities (10) and find that the functions $\phi(z)$ and $\psi(z)$, holomorphic in the domain $G\backslash\{0\}$, satisfy the conditions

$$\phi(z)=0\Big(\exp\Big\{\frac{|\lambda_1|}{|z|}\Big[1-\cos(\psi_1-3\arg z)\Big]\Big\}\Big), \quad z\to 0,$$

$$\psi(z)=0\Big(\exp\Big\{\frac{|\lambda_1|}{|z|}\Big[1-\frac{|\lambda_2|}{|\lambda_1|}\cos(\psi_2-3\arg z)\Big]\Big\}\Big), \quad z\to 0,$$

$$\psi_k=\arg\lambda_k, \quad k=1,2.$$

[1] Here and everywhere below, the linear independence is understood over the field of real numbers.

The first of the above conditions on the basis of Lemma 2 shows that $z = 0$ is a removable singular point for the function $\phi(z)$. Next, if we take into account the inequality $\left|\frac{\lambda_2}{\lambda_1}\right| > 1$, then by virtue of Lemma 1 the second condition shows that the function $\psi(z) \equiv 0$. This immediately implies that the relation

$$\frac{\partial \omega}{\partial \overline{z}} = \frac{\lambda_1}{z^2}\phi(z)\exp\left\{\frac{\lambda_1 \overline{z}}{z^2}\right\}$$

is valid. Consequently,

$$\left|\frac{\lambda_1}{z^2}\right| |\phi(z)| = 0\Big(\exp\left\{\frac{|\lambda_1|}{|z|}(1 - \cos(\psi_1 - 3\arg z))\right\}\Big), \quad z \to 0. \tag{25}$$

In turn, (25) yields

$$\left|\frac{\lambda_1}{z^2}\right| |\phi(z)| = 0(1), \quad z \to 0, \quad \arg z = \frac{\psi_1}{3}. \tag{26}$$

Considering the Taylor series expansion of the holomorphic function $\lambda_1 \phi(z)$

$$\lambda_1 \phi(z) = a_0 + a_1 z + a_2 z^2 + \cdots ,$$

and substituting this expansion in (26), we obtain

$$\left|\frac{a_0 + a_1 z}{z^2}\right| = 0(1), \quad z \to 0, \quad \arg z = \frac{\psi_1}{3},$$

and hence $a_0 = a_1 = 0$. From the above–said it follows that

$$\omega(z) = z^2 \phi_0(z) \exp\left\{\frac{\lambda_1 \overline{z}}{z^2}\right\}, \quad z \in G \backslash \{0\}, \tag{27}$$

where $\phi_0(z)$ is a function holomorphic in the domain G. Further, if the solution $\omega(z)$ is continuously extendable to $\overline{G}\backslash\{0\}$, then the function $\phi_0(z)$ is likewise continuously extendable to $\overline{G}$.

Conversely, it is obvious that any function of the type (27) provides us with a solution of the equation (17), which is continuously extendable to $\overline{G}\backslash\{0\}$ and satisfies the condition (18), where $\delta = |\lambda_1|$, $\sigma = 1$.

It remains to take into account the boundary conditions (19) and (21) (where $p = 0$) which immediately leads us to the validity of the theorem. □

Since any linearly independent system of functions $\phi_0(z)$ by means of the relation (24) transforms into that of the functions $\omega(z)$ (and conversely), on the basis of the above proven Theorem 2 it is possible to carry out the complete investigation of the boundary value problem $R(|\lambda_1|, 1)$ under the assumption $|\lambda_1| < |\lambda_2|$.

We have the following

Theorem 3. *Let at least one of the relations*

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \tag{28}$$

be violated. Then either the homogeneous problem $R(\delta, \sigma)$ has an infinite set of linearly independent solutions, or the inhomogeneous problem is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$.

Proof. By the inequality (4), violation at least of one of the relations (28) means the fulfilment of one of the following conditions:

$$\delta \neq |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \tag{29}$$

or

$$\sigma \neq 1 \quad (\sigma \quad \text{is arbitrary}), \quad |\lambda_1| < |\lambda_2|, \tag{30}$$

or

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \tag{31}$$

or

$$\delta \neq |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \tag{32}$$

or

$$\sigma \neq 1 \quad (\delta \quad \text{is arbitrary}) \quad |\lambda_1| = |\lambda_2|. \tag{33}$$

We consider these cases separately. Let (29) be fulfilled. In its turn, this case splits into the following two cases: either

$$\delta < |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \tag{29*}$$

or

$$\delta > |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|. \tag{29**}$$

Let the case (29^*) be fulfilled, and let $\omega(z)$ be a solution of the equation (17) satisfying the condition (18). Since $\nu = 2$, the number β given by the formula (8) is equal to zero. On the basis of Theorem 1, this implies that the solution $\omega(z) \equiv 0$, and hence the inhomogeneous boundary value problem $R(\delta, 1)$ is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$.

Let now the condition (29^{**}) be fulfilled. We call an arbitrary real number N and prove that the number of linearly independent solutions of the homogeneous boundary value problem $R(\delta, 1)$ is greater than N. Indeed, we select a natural number p so large that the number of linearly independent solutions of the homogeneous boundary value problem $R_0(p)$ be greater than N. Denote these solutions by $\phi_0^{(1)}(z), \phi_0^{(2)}(z) \cdots, \phi_0^{(m)}(z)$, $(m > N)$ and introduce the functions

$$\omega_k(z) = z^{2-p} \phi_0^{(k)} \exp\left\{\frac{\lambda_1 \overline{z}}{z^2}\right\}, \quad k = 1, 2, \ldots, m. \tag{34}$$

It is clear that the system of functions (34) is likewise independent.

By the representation (5), every function from (34) is a continuously extendable to $\overline{G} \backslash \{0\}$ solution of the equation (17) which by virtue of (21) satisfies the homogeneous boundary condition (19). Further, since the condition (29^{**}) is fulfilled, on the basis of the obvious relation

$$\frac{\partial \omega_k}{\partial \overline{z}} = \frac{\lambda_1}{z^p} \phi_0^{(k)}(z) \exp\left\{\frac{\lambda_1 \overline{z}}{z^2}\right\} = 0\Big(\exp\left\{\frac{\delta}{|z|}\right\}\Big), \quad z \to 0,$$

we immediately can conclude that every function of the system (34) satisfies the asymptotic condition (18), and hence the homogeneous boundary value problem $R(\delta, 1)$ has infinitely many linearly independent solutions.

Let now the condition (30) be fulfilled. This case in its turn falls into two cases: either

$$\sigma < 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| < |\lambda_2|, \tag{30*}$$

or

$$\sigma > 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| < |\lambda_2|. \tag{30**}$$

It is evident that in the case (30*) (analogously to the case (29*)) the inhomogeneous boundary value problem $R(\delta, \sigma)$ is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$, and in the case (30**) (analogously to the case (29**) the homogeneous boundary value problem $R(\delta, \sigma)$ has infinitely many linearly independent solutions.

Let now the condition (31) be fulfilled. This case in its turn splits into two cases: either

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \quad \lambda_1 \neq \lambda_2, \tag{31*}$$

or

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad \lambda_1 = \lambda_2. \tag{31**}$$

Let us prove that in both cases (31*) and (31**) the homogeneous boundary value problem $R(\delta, 1)$ has infinitely many linearly independent solutions. We start with the case (31*). Evidently, every function of the type

$$\omega(z) = z^2 \phi_0(z) e^{\frac{\lambda_1 \bar{z}}{z^2}} + z^2 \psi_0(z) e^{\frac{\lambda_2 \bar{z}}{z^2}}, \quad z \in G \backslash \{0\} \tag{35}$$

(where $\phi_0(z)$, $\psi_0(z)$ are holomorphic in the domain G functions) is a solution of the equation (17) satisfying the condition (18), where $\sigma = |\lambda_1|$, $\sigma = 1$ (in proving Theorem 5 below we will show that the converse statement is valid, i.e., every solution of the equation (17) satisfying the condition (18) with $\delta = |\lambda_1|$, $\sigma = 1$ has the form (35)). Next, if the holomorphic functions $\phi_0(z)$ and $\psi_0(z)$ are continuously extendable to $\overline{G}$, then the solution $\omega(z)$ is likewise continuously extendable to $\overline{G} \backslash \{0\}$. Consider the following problem: find two functions $\phi_0(z)$ and $\psi_0(z)$, holomorphic in the domain G and continuously extendable to G by the boundary condition

$$\operatorname{Re}\left\{a(t) t^2 \phi_0(t) e^{\frac{\lambda_1 \bar{t}}{t^2}} + a(t) t^2 \psi_0(t) e^{\frac{\lambda_2 \bar{t}}{t^2}}\right\} = 0, \quad t \in \Gamma. \tag{36}$$

It follows from the above said that every solution of the problem (36) provides us by the formula (35) with a solution of the homogeneous boundary value problem $R(|\lambda_1|, 1)$.

On the other hand, the problem (36) has infinitely many linearly independent solutions. Indeed, let

$$\phi_1^*(z), \phi_2^*(z), \ldots, \phi_l^*(z)$$

be a complete system of solutions of the conjugate boundary value problem: given a real Hölder continuous function $\beta(t)$, find the function $\phi_0(z)$ holomorphic in the domain G and continuously extendable to $\overline{G}$ by the boundary condition

$$\operatorname{Re}\left[\alpha(t) \phi_0(t)\right] = \beta(t), \quad t \in \Gamma, \tag{37}$$

where

$$\alpha(t)=a(t)t^2\exp\Big\{\frac{\lambda_1\overline{t}}{t^2}\Big\}.$$

Take an arbitrary natural number N_0 and consider a natural number N such that

$$N+1-2l>N_0.$$

Introduce now the polynomial

$$\psi_0(z)=C_0+C_1z+\cdots+C_nz^N, \tag{38}$$

where $C_j,\ j=0,1,\dots,N$, are yet undefined real coefficients. Further, taking the right-hand side of the problem (37) in the form

$$\beta(t)=-\operatorname{Re}\Big[a(t)t^2\exp\Big\{\frac{\lambda_2\overline{t}}{t^2}\Big\}\psi_0(t)\Big],\quad t\in\Gamma,$$

we obtain a boundary value problem which will certainly be solvable if

$$\int\limits_{\Gamma}\alpha(t)\beta(t)\phi_k^*(t)dt=0,\quad 1\le k\le l.$$

Thus if real constants C_j are chosen such that

$$\sum_{j=0}^{N}D_{kj}C_j=0,\quad k=1,2,\dots,l, \tag{39}$$

where

$$D_{kj}=\int\limits_{\Gamma}\alpha(t)\phi_k^*(t)\operatorname{Re}\Big[a(t)t^{2+j}e^{\frac{\lambda_2\overline{t}}{t^2}}\Big]dt,$$

then the problem (37) is solvable. In turn, the conditions (39) form a system consisting of $2l$ linear algebraic homogeneous equations with $N+1$ real unknowns, of which at least $N+1-2l$ we can take arbitrarily. This means that in the decomposition (38) we can take $N+1-2l$ real coefficients. Substituting this decomposition in the boundary condition (36), we can find the function $\phi_0(z)$. It is obvious that the problem (36) has an infinite number of linearly independent solutions.

If the condition (31^{**}) is fulfilled, then any function of the type

$$\omega(z)=(z\overline{z}\phi_0(z)+z^2\psi_0(z))e^{\frac{\lambda_1\overline{z}}{z^2}},\quad z\in G\backslash\{0\} \tag{40}$$

(where $\phi_0(z)$ and $\psi_0(z)$ are functions holomorphic in G), is a solution of the equation (17) satisfying the condition (18), where $\delta=|\lambda_1|$, $\sigma=1$ (in proving Theorem 6 below, we will establish the validity of the converse statement, i.e., any solution of the equation (17) satisfying the condition (18), where $\delta=|\lambda_1|$, $\sigma=1$, has the form (40)). Moreover, if the holomorphic functions $\phi_0(z)$ and $\psi_0(z)$ are continuously extendable to $\overline{G}$, then the solution $\omega(z)$ is likewise continuously extendable to $\overline{G}\backslash\{0\}$.

Let us consider the following boundary value problem. Find two functions $\phi_0(z)$ and $\psi_0(z)$, holomorphic in the domain G and continuously extendable to $\overline{G}$ by the boundary condition

$$\operatorname{Re}\left[a(t)(t\bar{t}\phi_0(t)+t^2\psi_0(t))e^{\frac{\lambda_1\bar{t}}{t^2}}\right]=0, \quad t\in\Gamma. \tag{41}$$

Any solution of the problem (41) provides us by the formula (40) with a solution of the boundary value problem $R(|\lambda_1|,1)$. But the problem (41), just as the problem (36), has an infinite number of linearly independent solutions. Hence the homogeneous problem $R(|\lambda_1|,1)$ has an infinite number of linearly independent solutions.

The case (32) splits into the following two cases: either

$$\delta<|\lambda_1|, \quad \sigma=1, \quad |\lambda_1|=|\lambda_2|, \tag{32*}$$

or

$$\delta>|\lambda_1|, \quad \sigma=1, \quad |\lambda_1|=|\lambda_2|. \tag{32**}$$

In the case (32*), just as in the case (29*), on the basis of Theorem 1 we immediately find that the equation (17) has no non-trivial solution satisfying the condition (18), and hence the inhomogeneous boundary value problem $R(\delta,1)$ is unsolvable for any right-hand side $\gamma(t)\not\equiv 0$.

In the case (32*) it is obvious that any solution of the boundary value problem $R(|\lambda_1|,1)$ is also a solution of the problem $R(\delta,1)$. But the homogeneous boundary value problem $R(|\lambda_1|,1)$ has an infinite number of linearly independent solutions (see the case (31) above), consequently the homogeneous problem $R(\delta,1)$ has an infinite number of linearly independent solutions, as well.

The case (33) splits into the following two cases: either

$$\sigma<1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1|=|\lambda_2| \tag{33*}$$

or

$$\sigma>1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1|=|\lambda_2|. \tag{33**}$$

In the case (33*), just as in the case (32*), on the basis of Theorem 1 we immediately find that the inhomogeneous boundary value problem $R(\delta,\sigma)$ is unsolvable for any right-hand side $\gamma(t)\not\equiv 0$, $t\in\Gamma$, and in the case (33**) (just as in the case (32**)) the homogeneous boundary value problem $R(\delta,\sigma)$ has an infinite number of linearly independent solutions. □

On the basis of the above proven Theorems 2 and 3 we have

Theorem 4. *The boundary value problem $R(\delta,\sigma)$ is Noetherian if and only if the relations* (28) *are fulfilled.*

7^0. In the foregoing section we have investigated the boundary value problem $R(\delta,\sigma)$. As we have found out, this problem is correct only under the condition (28). The last of those relations allows one to exclude from the consideration a wide class of equations of the type (17).

In the present section, not mentioning it specially, we assume that

$$|\lambda_1| = |\lambda_2|,$$

and for equations of the type (17) we give the correct statement and investigation of the boundary value problems.

Everywhere below, by δ_0 we denote the number $\delta_0 = |\lambda_1|$. We have the following

Theorem 5. *If*

$$\arg\lambda_1 \neq \arg\lambda_2,$$

then the boundary value problems $Q(\delta_0, 1)$ *and* $Q_0'(0)$ *are simultaneously solvable (unsolvable), and in case they are solvable, the relation* (35) *allows us to establish a bijective correspondence between the solutions of these problems.*

Proof. First we have to find a general representation of those solutions of the equation (17) which (together with its derivative with respect to $\overline{z}$) are continuously extendable to $G\backslash\{0\}$ and satisfy the condition (18), where $\delta = \delta_0$, $\sigma = 1$. To this end, we again use the equalities (10) and find that the functions $\phi(z)$ and $\psi(z)$, holomorphic in the domain $G\backslash\{0\}$, satisfy the conditions

$$\phi(z) = 0\Big(\exp\Big\{\frac{\delta_0}{|z|}[1-\cos(\psi_1 - 3\arg z)]\Big\}\Big), \quad z\to 0,$$
$$\psi(z) = 0\Big(\exp\Big\{\frac{\delta_0}{|z|}[1-\cos(\psi_2 - 3\arg z)]\Big\}\Big), \quad z\to 0,$$
$$\psi_k = \arg\lambda_k, \quad k=1,2.$$

Thus on the basis of Lemma 2 we conclude that $z=0$ is a removable singular point for the functions $\phi(z)$ and $\psi(z)$. Further, it is obvious that

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{\lambda_1\phi(z)}{z^2}\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\} + \frac{\lambda_2\psi(z)}{z^2}\exp\Big\{\frac{\lambda_2\overline{z}}{z^2}\Big\} =$$
$$= 0\Big(\exp\Big\{\frac{\delta_0}{|z|}\Big\}\Big), \quad z\to 0.$$

Hence we obtain the following two relations:

$$\frac{\delta_0}{r^2}\Big|\phi\Big(r\exp\Big\{\frac{i\psi_1}{3}\Big\}\Big)\Big| \leq \text{const} +$$
$$+\frac{\delta_0}{r^2}\Big|\psi\Big(r\exp\Big\{\frac{i\psi_1}{3}\Big\}\Big)\Big|\exp\Big\{\frac{\delta_0}{r}[\cos(\psi_2-\psi_1)-1]\Big\},$$
$$\frac{\delta_0}{r^2}\Big|\psi\Big(r\exp\Big\{\frac{i\psi_2}{3}\Big\}\Big)\Big| \leq \text{const} +$$
$$+\frac{\delta_0}{r^2}\Big|\phi\Big(r\exp\Big\{\frac{i\psi_1}{3}\Big\}\Big)\Big|\exp\Big\{\frac{\delta_0}{r}[\cos(\psi_2-\psi_1)-1]\Big\},$$

whence it respectively follow

$$\Big|\frac{\phi(z)}{z^2}\Big| = 0(1), \quad z\to 0, \quad \arg z = \frac{\psi_1}{3},$$

and

$$\left|\frac{\psi(z)}{z^2}\right| = 0(1), \quad z \to 0, \quad \arg z = \frac{\psi_2}{3}.$$

This implies that the functions $\phi(z)$ and $\psi(z)$ admit the representations

$$\phi(z) = z^2\phi_0(z), \quad \psi(z) = z^2\psi_0(z),$$

where $\phi_0(z)$ and $\psi_0(z)$ are functions holomorphic in the domain G.

Consequently, any solution of the equation (17) satisfying the condition (18) ($\delta = \delta_0$, $\sigma = 1$) is representable in the form

$$\omega(z) = z^2\phi_0(z)\exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\} + z^2\psi_0(z)\exp\left\{\frac{\lambda_2\overline{z}}{z^2}\right\}, \tag{42}$$

and hence

$$\frac{\partial\omega}{\partial\overline{z}} = \lambda_1\phi_0(z)\exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\} + \lambda_2\psi_0(z)\exp\left\{\frac{\lambda_2\overline{z}}{z^2}\right\}. \tag{43}$$

Next, if the solution (42) (together with its derivative (43)) is continuously extendable to $\overline{G}\backslash\{0\}$, then we find that the functions $\phi_0(z)$ and $\psi_0(z)$ are likewise continuously extendable to $\overline{G}$.

Conversely, it is evident that any function of the type (42) provides us with a continuously extendable (together with its derivative $\frac{\partial\omega}{\partial\overline{z}}$) solution of the equation (17), satisfying the condition (18), where $\delta = \delta_0$, $\sigma = 1$. It remains to take into account the boundary conditions (20) and (22) (where $p = 0$) which directly leads to the conclusion of our theorem.

On the basis of the above proven Theorem 5 in particular it follows that the number of linearly independent solutions of the homogeneous boundary value problem $Q(\sigma_0, 1)$ is finite. This number coincides with that of the linearly independent solutions of the homogeneous boundary value problem $Q_0'(0)$, because any linearly independent system of holomorphic vector functions

$$(\phi_k(z), \psi_k(z)), \quad 1 \le k \le m, \tag{44}$$

transforms by the relation

$$\omega_k(z) = \phi_k(z)\exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\} + \psi_k(z)\exp\left\{\frac{\lambda_2\overline{z}}{z^2}\right\}, \quad k = 1, 2, \ldots, m, \tag{45}$$

into a linearly independent system of functions $\omega_k(z)$, $k = 1, 2, \ldots, m$, and vice versa. Indeed, let the system of holomorphic vector functions (44) be independent, and

$$\sum_{k=1}^{m} C_k\omega_k(z) \equiv 0,$$

where C_k are complex (in particular, real) coefficients. Then

$$\sum_{k=1}^{m} C_k\phi_k(z) \equiv -e^{\frac{\lambda_2-\lambda_1}{z^2}\overline{z}} \sum_{k=1}^{m} C_k\psi_k(z). \tag{46}$$

Differentiating both parts of the equality (46) with respect to $\overline{z}$, we obtain

$$\frac{\lambda_2-\lambda_1}{z^2}\, e^{\frac{\lambda_2-\lambda_1}{z^2}\overline{z}} \sum_{k=1}^{m} C_k\psi_k(z) \equiv 0.$$

Hence (since $\lambda_2 \neq \lambda_1$)

$$\sum_{k=1}^{m} C_k\psi_k(z) \equiv 0. \tag{47}$$

It follows from (46) and (47) that

$$\sum_{k=1}^{m} C_k\phi_k(z) \equiv 0, \tag{48}$$

while (48) and (47), by virtue of the fact that the system (44) is linearly independent, yield $C_k = 0$, $k = 1, 2, \ldots, m$. □

The converse statement is obvious because the linear dependence of the system of vector functions (44) immediately implies that of the system of functions (45).

We have the following

Theorem 6. *If*

$$\psi_1 \equiv \arg\lambda_1 = \arg\lambda_2,$$

then the boundary value problems $Q(\delta_0, 1)$ and $Q_0''(0)$ are simultaneously solvable (unsolvable), and if they are solvable, then the relation (40) *allows us to establish the bijective correspondence between the solutions of these problems.*

Proof. First of all, just as in the proof of Theorems 2 and 5, we have to find a general representation of those solutions of the equation (17) which (together with the derivative $\frac{\partial\omega}{\partial\overline{z}}$) are continuously extendable to $\overline{G}\backslash\{0\}$ and satisfy the condition (18), where $\delta = \delta_0$, $\sigma = 1$. Towards this end, we use the equalities (13) and find that the functions $\phi(z)$ and $\psi(z)$, holomorphic in $G\backslash\{0\}$, satisfy the conditions

$$z^2\phi(z) = 0(g(z)), \quad z^2\psi(z) = 0(g(z)), \quad z \to 0, \tag{49}$$

where

$$g(z) = \exp\Big\{\frac{\delta_0}{|z|}(1-\cos(\psi_1 - 3\arg z))\Big\}.$$

By virtue of the relations (49) and Lemma 2, we obtain that $z = 0$ is a removable singular point for the functions $z^2\phi$ and $z^2\psi$, i.e., the solution ω is representable in the form

$$\omega(z) = H(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}, \quad z \in G\backslash\{0\}, \tag{50}$$

where

$$H(z) = \overline{z}\frac{\widetilde{\phi}(z)}{z^2} + \frac{\widetilde{\psi}(z)}{z^2},$$

and $\widetilde{\phi}$ and $\widetilde{\psi}$ are functions holomorphic in G. In turn, from the representation (50) it follows

$$\frac{\partial\omega}{\partial\overline{z}} = H_1(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}, \quad z\in G\backslash\{0\},$$

where

$$H_1(z) = \frac{\widetilde{\phi}(z)}{z^2}\Big(1+\frac{\lambda_1\overline{z}}{z^2}\Big) + \frac{\lambda_1}{z^4}\widetilde{\psi}(z).$$

Further, taking into account the condition (18), we get

$$H(z) = 0(1), \quad z\to 0, \arg z = \frac{1}{3}(\psi_1+2\pi k), \tag{51}$$

$$H_1(z) = 0(1), \quad z\to 0, \arg z = \frac{1}{3}(\psi_1+2\pi k), \tag{52}$$

$$k=0,1,2,\ldots.$$

Expanding the holomorphic functions $\widetilde{\phi}$ and $\widetilde{\psi}$ into their Taylor series

$$\begin{aligned}\widetilde{\phi}(z) &= a_0+a_1z+a_2z^2+\cdots, \\ \widetilde{\psi}(z) &= b_0+b_1z+b_2z^2+\cdots,\end{aligned} \tag{53}$$

and substituting them in (51), we have

$$\frac{a_0\overline{z}+b_1z+b_0}{z^2} = 0(1), \quad \arg z = \frac{\psi_1+2\pi k}{3}, \tag{54}$$

where the coefficient $b_0=0$. Taking this into account and using the relation (54) for the coefficients a_0 and b_1, we obtain the following equalities

$$a_0e^{-2i\varphi_0}+b_1=0, \quad \varphi_0=\frac{\psi_1}{3},$$

$$a_0e^{-2i\varphi_0}+b_1=0, \quad \varphi_1=\frac{\psi_1+2\pi}{3},$$

which (with regard for $e^{-2i\varphi_0}-e^{-2i\varphi_1}\neq 0$) show that the coefficients $a_0 = b_1 = 0$.

Substituting now the expansions (53) and (52), we have

$$\frac{1}{r^3}[\lambda_1a_1e^{-4i\varphi_k}+\lambda_1b_2re^{-i\varphi_k}+r^2(a_1+\lambda_1a_2e^{-2i\varphi_k}+$$

$$+\lambda_1b_3)] = 0(1), \quad r\to 0, \quad \varphi_k=\frac{\psi_1+2\pi k}{3}, \quad k=0,1,2,\ldots,$$

which immediately give us $a_1=0$. Taking this fact into account, we obtain

$$\frac{1}{r^2}[\lambda_1b_2e^{-i\varphi_k}+\lambda_1r(a_2e^{-2i\varphi_k}+b_3)] = O(1), \quad r\to 0,$$

and therefore $b_2=0$. In its turn, we have

$$a_2e^{-2i\varphi_0}+b_3=0, \quad \varphi_0=\frac{\psi_1}{3},$$

$$a_2e^{-2i\varphi_1}+b_3=0, \quad \varphi_1=\frac{\psi_1+2\pi}{3},$$

by virtue of which $a_2 = b_3 = 0$.

Thus the holomorphic functions $\widetilde{\phi}$ and $\widetilde{\psi}$ have the form

$$\widetilde{\phi}(z) = z^3\phi_0(z), \quad \widetilde{\psi}(z) = z^4\psi_0(z), \tag{55}$$

where the functions ϕ_0 and ψ_0 are holomorphic in the domain G. Substituting (55) and (50), we obtain the representation (40). Next, if the solution (40) together with its derivative

$$\frac{\partial\omega}{\partial\overline{z}} = \Big[\phi_0(z)\Big(z + \frac{\lambda_1\overline{z}}{z^2}\Big) + \lambda_1\psi_0(z)\Big]e^{\frac{\lambda_1\overline{z}}{z^2}} \tag{56}$$

is continuously extendable to $\overline{G}\backslash\{0\}$, we will find that the holomorphic functions ϕ_0 and ψ_0 are continuously extendable to $\overline{G}$.

Conversely, any function of the type (40) provides us with a continuously extendable (together with its derivative (56)) to $\overline{G}\backslash\{0\}$ solution of the equation (17), satisfying the condition (18) with $\delta = \delta_0$, $\sigma = 1$. It remains to take into account the boundary conditions (20) and (23) (with $p = 0$) which immediately leads us to the conclusion of our theorem. □

It is not difficult to see that any linearly independent system of holomorphic vector functions (44) transforms by the relation

$$\omega_k(z) = \Big(z\overline{z}\phi_k(z) + z^2\psi_k(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}\Big), \quad z \in G\backslash\{0\}$$

(analogously to the relation (45)), into a linearly independent system of functions $\omega_k(z)$, $k = 1, 2, \ldots, m$, and vice versa. Therefore the numbers of linearly independent solutions of homogeneous boundary problems $Q(\delta_0, 1)$ and $Q_0''(0)$ coincide.

We have the following

Theorem 7. *Let at least one of the equalities*

$$\delta = \delta_0, \quad \sigma = 1, \tag{57}$$

be violated. Then either the homogeneous boundary value problem $Q(\delta, \sigma)$ *has an infinite number of linearly independent solutions, or the inhomogeneous problem is unsolvable for any right-hand side* $(\gamma_1(t), \gamma_2(t)) \not\equiv 0$.

Proof. The violation of at least of one of the equalities (57) implies that one of the following conditions is fulfilled:

$$\delta < \delta_0, \quad \sigma = 1, \tag{58}$$

or

$$\delta > \delta_0, \quad \sigma = 1, \tag{59}$$

or

$$\sigma < 1 \quad (\sigma \quad \text{is arbitrary}), \tag{60}$$

or

$$\sigma > 1 \quad (\sigma \quad \text{is arbitrary}). \tag{61}$$

Under the condition (58) (and under the condition (60)), on the basis of Theorem 1 it immediately follows that the equation (17) has no non-trivial solution satisfying the condition (18), and hence the inhomogeneous boundary value problem $Q(\delta, \sigma)$ is unsolvable for any right-hand side $(\gamma_1(t), \gamma_2(t)) \not\equiv 0$.

Let us prove that under the condition (59) the homogeneous boundary value problem $Q(\delta, 1)$ has an infinite number of linearly independent solutions. Indeed, let the condition (59) be fulfilled and, moreover, $\arg\lambda_1 \neq \arg\lambda_2$. We take an arbitrary natural number N and choose a natural number p so large that the number of linearly independent solutions of the homogeneous boundary value problem $Q_0'(p)$ be greater than N. We denote these solutions by

$$(\phi_0^{(k)}(z), \psi_0^{(k)}(z)), \quad k = 1, 2, \dots, m, \quad m > N. \tag{62}$$

It is not difficult to see that the system of functions (62) transforms by the relation

$$\omega_k(z) = z^2\phi_0^{(k)}(z) \exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\} + \\ + z^2\psi_0^{(k)}(z) \exp\left\{\frac{\lambda_2\overline{z}}{z^2}\right\}, \quad z \in G\backslash\{0\},$$

into a linearly independent system of solutions of the homogeneous boundary value problem $Q(\delta, \sigma)$. Therefore this problem has an infinite number of linearly independent solutions.

Let now the condition (59) be fulfilled, and $\arg\lambda_1 = \arg\lambda_2$. We take an arbitrary natural number N and choose a natural number p so large that the number of linearly independent solutions of the homogeneous boundary value problem $Q_0''(p)$ be greater than N. We denote again these solutions by (62). It is not difficult to see that the system of functions (62) transforms by the relation

$$\omega_k(z) = (z\overline{z}\phi_0^{(k)}(z) + z^2\psi_0^{(k)}(z)) \exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\}, \\ z \in G\backslash\{0\}, \quad k = 1, 2, \dots, m,$$

into a linearly independent system of solutions of the homogeneous boundary value problem $Q(\delta, \sigma)$. Therefore this problem has an infinite number of linearly independent solutions.

It remains to consider the case (61). But any solution of the homogeneous boundary value problem $Q(\delta, 1)$ (for $\delta > \delta_0$) is likewise a solution of the homogeneous boundary value problem $Q(\delta, \sigma)$ (for $\sigma > 1$). Therefore the latter problem has an infinite number of linearly independent solutions. □

On the basis of the above-proven Theorems 6 and 7 we have the following

Theorem 8. *The boundary value problem* $Q(\delta, \sigma)$ *is Noetherian if and only if the condition* (57) *is fulfilled.*

References

1. E. Titchmarsch, The theory of functions. (Translated from the English into Russian) *Nauka, Moscow,* 1980.
2. A. I. Markushevich, The theory of analytic functions. (Russian) *GITTL, Moscow–Leningrad,* 1950.
3. G. T. Makatsaria, Carleman–Vekua equations with polar singularities. *Dissertation for the degree of Cand. Sci. in Phys.-Math. Tbilisi,* 1986.
4. N. P. Vekua, Systems of singular integral equations and certain boundary value problems. (Russian) *Nauka, Moscow,* 1970.
5. B. V. Bojarski, Analysis of the solvability of boundary value problems of the theory of functions. (Russian) *Investigations dealing with the current problems of the theory of functions of a complex variable,* 57–79, *GIFML, Moscow,* 1961.
6. G. B. Kolosov, On conjugate partial differential equations. (Russian) *Trudy Electrotekhnich. Inst. Petrograd* **2** (1914), 179–199.

(Received 12.10.2004)

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