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ON THE CAUCHY TYPE PROBLEM FOR TWO-DIMENSIONAL FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. In this paper, we establish new efficient conditions sufficient for the solvability as well as unique solvability of the Cauchy type problem for two-dimensional functional differential systems in both linear and nonlinear cases. The main results are applied in the case where the system considered is the differential system with argument deviations.

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1. Introduction

On the interval $[a, b]$, we consider the two-dimensional differential system

$$x_1'(t) = F_1(x_1, x_2)(t), \quad x_2'(t) = F_2(x_1, x_2)(t), \tag{1.1}$$

where F_1, $F_2 : C([a, b]; \mathbb{R}) \times C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ are continuous operators. By a solution to the system (1.1) we understand a pair (x_1, x_2) of absolutely continuous on $[a, b]$ functions satisfying (1.1) almost everywhere on $[a, b]$.

Various initial and boundary value problems are studied in the literature. We are interested in the Cauchy type problem

$$x_1(a) = \varphi_1(x_1, x_2), \quad x_2(a) = \varphi_2(x_1, x_2) \tag{1.2}$$

for the system (1.1), where φ_1, $\varphi_2 : C([a, b]; \mathbb{R}) \times C([a, b]; \mathbb{R}) \to \mathbb{R}$ are continuous functionals. Along with the problem (1.1), (1.2), we consider the linear problem

$$x_1'(t) = \ell_1(x_2)(t) + q_1(t), \quad x_2'(t) = \ell_2(x_1)(t) + q_2(t), \tag{1.3}$$

$$x_1(a) = c_1, \quad x_2(a) = c_2, \tag{1.4}$$

where $\ell_1, \ell_2 : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ are linear bounded operators, q_1, $q_2 \in L([a, b]; \mathbb{R})$, and c_1, $c_2 \in \mathbb{R}$.

The Cauchy problem and other types of boundary value problems for the ordinary differential equations and their systems have been studied in detail (see, e.g., [2], [4], [11]– [13], [27] and references therein). As for functional differential equations, the foundations of the theory of boundary value problems for a large class of such equations were constructed in monographs [1], [10], [21], [23] (see also references therein).

The results known for ordinary differential systems were extended and generalized for functional differential systems with the so-called Volterra right-hand sides in the works of Kiguradze and Sokhadze (see, e.g., [17], [18]). Efficient conditions sufficient for the solvability as well as unique solvability of various boundary value problems for n-dimensional functional differential systems of non-Volterra type were established, e.g., in [5], [14]–[16], [19], [20], [22]. Note also that the Cauchy problem for scalar functional differential equations was investigated in [3], [6], [7].

We have studied the Cauchy type problem for n-dimensional functional differential systems in [25]. In this paper, new results are established in this line for the two–dimensional system (1.1) in both linear and nonlinear cases. Differential systems with argument deviations are considered in more detail, in which case further results are obtained.

The paper is organized as follows. In Section 2, auxiliary definitions and remarks are given. Section 3 deals with the linear problem (1.3), (1.4). The nonlinear problem (1.1), (1.2) is studied in Section 4. By means of comparison of the nonlinear problem with a suitable linear one, the solvability of the problem (1.1), (1.2) can be proved under one-sided restrictions imposed on the right-hand side of the system (1.1). Some of the results given in

Sections 3 and 4 are proved using the so-called weak theorem on differential inequalities stated in [26]. Therefore, for the sake of completeness, the main results of [26] are discussed in Section 5. Theorems presented in this paper are unimprovable in a certain sense, which is shown by counter-examples constructed in Section 6.

2. Notation and Definitions

The following notation is used throughout the paper.

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$;

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\big\{|u(t)| : \ t \in [a,b]\big\};$$

$C([a,b];\mathbb{R}_+) = \big\{u \in C([a,b];\mathbb{R}) : \ u(t) \geq 0 \text{ for } t \in [a,b]\big\}$;

$\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$;

$\widetilde{C}_{loc}([a,b[\,;\mathbb{R})$ is the set of functions $u : [a,b[\to\mathbb{R}$ such that $u \in \widetilde{C}([a,\beta];\mathbb{R})$ for every $\beta \in\,]a,b[$;

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h : [a,b] \to \mathbb{R}$ equipped with the norm

$$\|h\|_L = \int_a^b |h(s)|\, ds;$$

$L([a,b];\mathbb{R}_+) = \big\{h \in L([a,b];\mathbb{R}) : \ h(t) \geq 0 \text{ for a.e. } t \in [a,b]\big\}$;

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

$\widetilde{\mathcal{L}}_{ab}$ is the set of operators $\ell \in \mathcal{L}_{ab}$ which are strongly bounded, i.e., such that

$$|\ell(u)(t)| \leq \eta(t)\|u\|_C \ \text{ for a.e. } \ t \in [a,b] \ \text{ and all } \ u \in C([a,b];\mathbb{R})$$

with $\eta \in L([a,b];\mathbb{R}_+)$.

$K([a,b] \times A; B)$, where $A \subseteq \mathbb{R}^m$ $(m \in \mathbb{N})$ and $B \subseteq \mathbb{R}$, is the set of functions $f : [a,b] \times A \to B$ satisfying the Carathéodory conditions, i.e., such that

(i) $f(\cdot, x) : [a,b] \to B$ is a measurable function for all $x \in A$,
(ii) $f(t, \cdot) : A \to B$ is a continuous function for almost every $t \in [a,b]$,
(iii) for every $r > 0$ there exists a function $q_r \in L([a,b];\mathbb{R}_+)$ such that

$$|f(t,x)| \leq q_r(t) \ \text{ for a.e. } \ t \in [a,b] \ \text{ and all } \ x \in A, \ \ \|x\| \leq r.$$

Definition 2.1. An operator $\ell \in \mathcal{L}_{ab}$ is said to be *nondecreasing* if it maps the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$. We denote by $\mathcal{P}_{ab}$ the class of linear nondecreasing operators. We say that an operator $\ell \in \mathcal{L}_{ab}$ is *nonincreasing* if $-\ell \in \mathcal{P}_{ab}$.

Example 2.1. Let $\ell \in \mathcal{L}_{ab}$ be defined by

$$\ell(z)(t) \overset{\text{def}}{=} h(t)z(\tau(t)) \ \text{ for a.e. } \ t \in [a,b] \ \text{ and all } \ z \in C([a,b];\mathbb{R}), \tag{2.1}$$

where $h \in L([a,b];\mathbb{R})$ and $\tau : [a,b] \to [a,b]$ is a measurable function. Then $\ell \in \mathcal{P}_{ab}$ if and only if

$$h(t) \geq 0 \text{ for a.e. } t \in [a,b].$$

Definition 2.2. We say that $\ell \in \mathcal{L}_{ab}$ is an a-Volterra operator if for every $b_0 \in]a,b]$ and $z \in C([a,b];\mathbb{R})$ satisfying

$$z(t) = 0 \text{ for } t \in [a,b_0]$$

we have

$$\ell(z)(t) = 0 \text{ for a.e. } t \in [a,b_0].$$

Example 2.2. The operator $\ell \in \mathcal{L}_{ab}$ defined by (2.1) is an a-Volterra one if and only if

$$|h(t)|\Big(\tau(t) - t\Big) \leq 0 \text{ for a.e. } t \in [a,b].$$

Definition 2.3. Let $\ell \in \mathcal{L}_{ab}$ and $b_0 \in]a,b[$. The operator $\ell^{ab_0} : C([a,b_0];\mathbb{R}) \to L([a,b_0];\mathbb{R})$ defined by

$$\ell^{ab_0}(z)(t) \overset{\text{def}}{=} \ell(\widetilde{z})(t) \text{ for a.e. } t \in [a,b_0] \text{ and all } z \in C([a,b_0];\mathbb{R}),$$

where

$$\widetilde{z}(t) = \begin{cases} z(t) & \text{for } t \in [a,b_0[\\ z(b_0) & \text{for } t \in [b_0,b] \end{cases},$$

is called the restriction of the operator ℓ to the space $C([a,b_0];\mathbb{R})$.

If $b_0 < b_1 \leq b$ and $z \in C([a,b_1];\mathbb{R})$, then we write $\ell^{ab_0}(z)$ instead of $\ell^{ab_0}(z|_{[a,b_0]})$.

Remark 2.1. If ℓ is an a-Volterra operator, then it is clear that for every $b_0 \in]a,b[$ and $z \in C([a,b];\mathbb{R})$ the condition

$$\ell^{ab_0}(z)(t) = \ell(z)(t) \text{ for a.e. } t \in [a,b_0]$$

holds.

3. Linear Problem

In this section, we establish new efficient conditions sufficient for the unique solvability of the linear problem (1.3), (1.4). Differential systems with argument deviations are considered in more detail, in which case further results are obtained. Note also that the second order functional differential equation

$$u''(t) = \ell(u)(t) + q(t),$$

where $\ell \in \mathcal{L}_{ab}$ and $q \in L([a,b];\mathbb{R})$, can be regarded as a particular case of the system (1.3). A statement concerning this equation is given at the end of the next section (see Corollary 3.2 below).

3.1. **Formulation of Results.** We first formulate the main results, the proofs being given later in Subsection 3.3.

Theorem 3.1. *Let $k \in \{1,2\}$, $m \in \{0,1\}$, and $\ell_i = \ell_{i,0} - \ell_{i,1}$ with $\ell_{i,j} \in \mathcal{P}_{ab}$ $(i = 1,2,\ j = 0,1)$. Assume that there exist functions $\beta_1, \beta_2 \in \widetilde{C}([a,b];\mathbb{R})$ such that*

$$\beta_i(t) > 0 \ \text{for} \ t \in [a,b], \ \ i = 1,2, \tag{3.1}$$

$$\beta_1'(t) \geq \ell_{k,0}(\beta_2)(t) + \ell_{k,1}(\beta_2)(t) \ \text{for a.e.} \ t \in [a,b], \tag{3.2}$$

$$\beta_2'(t) \leq -\ell_{3-k,0}(\beta_1)(t) - \ell_{3-k,1}(\beta_1)(t) \ \text{for a.e.} \ t \in [a,b], \tag{3.3}$$

$$\int_a^b \ell_{k,1-m}(\beta_2)(s)ds \leq \beta_1(a), \tag{3.4}$$

and

$$\int_a^b \ell_{3-k,m}(\beta_1)(s)\,ds + \int_a^b \ell_{3-k,1-m}\big(\chi(\ell_{k,1-m}(\beta_2))\big)(s)\,ds \leq \beta_2(b), \tag{3.5}$$

where the inequality (3.5) *is supposed to be strict if $\ell_{3-k,m} = 0$. Here*

$$\chi(h)(t) \overset{\text{def}}{=} \int_a^t h(s)\,ds \ \text{for} \ t \in [a,b], \ \ h \in L([a,b];\mathbb{R}). \tag{3.6}$$

Then the problem (1.3), (1.4) *has a unique solution.*

If the operators ℓ_1, ℓ_2 are monotone and one of them is an a-Volterra operator, then the assumption $\beta_1 \in \widetilde{C}([a,b];\mathbb{R})$ in the previous theorem can be weakened (see Theorem 3.2). On the other hand, if both operators ℓ_1, ℓ_2 are a-Volterra ones, then the problem (1.3), (1.4) is uniquely solvable without any additional assumption (see, e.g., [14, § 1.2.3]).

Theorem 3.2. *Let $k \in \{1,2\}$, $m \in \{0,1\}$, $(-1)^m\ell_k, (-1)^{1-m}\ell_{3-k} \in \mathcal{P}_{ab}$, and let the operator ℓ_{3-k} be an a-Volterra one. Assume that there exist $\gamma_1 \in \widetilde{C}_{loc}([a,b[\,;\mathbb{R})$ and $\gamma_2 \in \widetilde{C}([a,b];\mathbb{R})$ such that*

$$\gamma_1(t) > 0 \ \text{for} \ t \in [a,b[\,, \ \ \gamma_2(t) > 0 \ \text{for} \ t \in [a,b], \tag{3.7}$$

$$\gamma_1'(t) \geq (-1)^m\ell_k(\gamma_2)(t) \ \text{for a.e.} \ t \in [a,b], \tag{3.8}$$

and

$$\gamma_2'(t) \leq (-1)^m\ell_{3-k}(\gamma_1)(t) \ \text{for a.e.} \ t \in [a,b].^1 \tag{3.9}$$

Then the problem (1.3), (1.4) *has a unique solution.*

Remark 3.1. Since possibly $\gamma_1(t) \to +\infty$ as $t \to b-$, the condition (3.9) of the previous theorem is understood in the sense that for any $b_0 \in\,]a,b[$ the relation

$$\gamma_2'(t) \leq (-1)^m\ell_{3-k}^{ab_0}(\gamma_1)(t) \ \text{for a.e.} \ t \in [a,b_0] \tag{3.10}$$

[1] See Remark 3.1.

holds, where $\ell_{3-k}^{ab_0}$ is the restriction of the operator ℓ_{3-k} to the space $C([a,b_0];\mathbb{R})$.

In the next statement, the solvability conditions are given in terms of norms of the operators appearing on the right-hand side of the system (1.3).

Theorem 3.3. *Let $k \in \{1,2\}$, $m \in \{0,1\}$, $(-1)^m \ell_{3-k} \in \mathcal{P}_{ab}$, and $\ell_k = \ell_{k,0} - \ell_{k,1}$ with $\ell_{k,0}, \ell_{k,1} \in \mathcal{P}_{ab}$. Assume that*

$$A_{3-k} A_{k,m} < 1 \tag{3.11}$$

and

$$A_{3-k} A_{k,1-m} < 4 + 4\sqrt{1 - A_{3-k} A_{k,m}}\,, \tag{3.12}$$

where

$$A_{3-k} = \int\limits_a^b \big|\ell_{3-k}(1)(s)\big|\,ds, \quad A_{k,j} = \int\limits_a^b \ell_{k,j}(1)(s)\,ds \ \ \text{for } j = 0,1. \tag{3.13}$$

Then the problem (1.3), (1.4) *has a unique solution.*

Remark 3.2. The strict inequality (3.11) in Theorem 3.3 cannot be replaced by the nonstrict one (see [9, Example 4.2]). Moreover, the strict inequality (3.12) cannot be replaced by the nonstrict one provided $A_{k,m} = 0$ (see [9, Example 4.3]).

Theorem 3.4 below is proved using the so-called weak theorem on differential inequalities stated in [26]. We first give a definition.

Definition 3.1 ([26, Def. 3.2])**.** A pair $(p,g) \in \mathcal{L}_{ab} \times \mathcal{L}_{ab}$ is said to belong to the set $\widehat{\mathcal{S}}_{ab}^2(a)$ if for any u, $v \in \widetilde{C}([a,b];\mathbb{R})$ such that

$$u'(t) \geq p(v)(t), \quad v'(t) \geq g(u)(t) \ \text{ for a.e. } t \in [a,b]$$

and

$$u(a) \geq 0, \quad v(a) \geq 0$$

the condition

$$u(t) \geq 0 \ \text{ for } t \in [a,b]$$

is satisfied.

If $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}_{ab}^2(a)$, then we say that the *weak theorem on differential inequalities* holds for the system (1.3).

Remark 3.3. Let $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}_{ab}^2(a)$. Then it is easy to see that the homogeneous problem

$$x_1'(t) = \ell_1(x_2)(t), \quad x_2'(t) = \ell_2(x_1)(t), \tag{3.14}$$

$$x_1(a) = 0, \quad x_2(a) = 0 \tag{3.15}$$

corresponding to (1.3), (1.4) has only the trivial solution. Therefore, according to the Fredholm property of linear problems (see, e.g, [23], [16], [14], [8]), the problem (1.3), (1.4) has a unique solution for every q_1, $q_2 \in L([a,b];\mathbb{R})$

and c_1, $c_2 \in \mathbb{R}$. However, the inclusion $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$ guarantees, in addition, that the unique solution (x_1, x_2) to this problem satisfies $x_1(t) \geq 0$ for $t \in [a, b]$ whenever

$$q_k(t) \geq 0 \ \text{ for a.e. } \ t \in [a, b], \quad c_k \geq 0 \ \ (k = 1, 2).$$

Theorem 3.4. *Let $k \in \{1, 2\}$, $m \in \{0, 1\}$, $(-1)^m \ell_k \in \mathcal{P}_{ab}$, and let there exist operators $g_0 \in \mathcal{L}_{ab}$ and $g_1 \in \mathcal{P}_{ab}$ such that*

$$\big((-1)^m \ell_k, g_0\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \big((-1)^m \ell_k, g_0 + g_1\big) \in \widehat{\mathcal{S}}^2_{ab}(a) \tag{3.16}$$

and the inequality

$$\big|\ell_{3-k}(z)(t) + (-1)^{1-m} g_0(z)(t)\big| \leq g_1(|z|)(t) \ \text{ for a.e. } \ t \in [a, b] \tag{3.17}$$

holds on the set $\{z \in C([a, b]; \mathbb{R}) : \ z(a) = 0\}$. Then the problem (1.3), (1.4) *has a unique solution.*

Remark 3.4. The assumption (3.16) in the previous theorem can be replaced neither by the assumption

$$\big((-1)^m \ell_k, g_0\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \big((-1)^m (1-\varepsilon_1)\ell_k, (1-\varepsilon_2)(g_0+g_1)\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \tag{3.18}$$

nor by the assumption

$$\big((-1)^m (1-\varepsilon_1)\ell_k, (1-\varepsilon_2)g_0\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \big((-1)^m \ell_k, g_0+g_1\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \tag{3.19}$$

no matter how small $\varepsilon_1, \varepsilon_2 \in [0, 1[$ with $\varepsilon_1 + \varepsilon_2 > 0$ are (see Examples 6.1 and 6.2).

Theorem 3.4 yields

Corollary 3.1. *Let $k \in \{1, 2\}$, $m \in \{0, 1\}$, $(-1)^m \ell_k \in \mathcal{P}_{ab}$, and let $\ell_{3-k} = \ell_{3-k,0} - \ell_{3-k,1}$ with $\ell_{3-k,0}, \ell_{3-k,1} \in \mathcal{P}_{ab}$. If*

$$\big((-1)^m \ell_k, \ell_{3-k,m}\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \Big((-1)^m \ell_k, -\frac{1}{2}\, \ell_{3-k,1-m}\Big) \in \widehat{\mathcal{S}}^2_{ab}(a), \tag{3.20}$$

then the problem (1.3), (1.4) *has a unique solution.*

Remark 3.5. In [26] the following assertion is proved: *If $\ell_1 \in \mathcal{P}_{ab}$ and $\ell_2 = \ell_{2,0} - \ell_{2,1}$ with $\ell_{2,0}, \ell_{2,1} \in \mathcal{P}_{ab}$ are such that*

$$(\ell_1, \ell_{2,0}) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad (\ell_1, -\ell_{2,1}) \in \widehat{\mathcal{S}}^2_{ab}(a),$$

then $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$ as well. It is easy to find operators $\ell_1, \ell_2 \in \mathcal{L}_{ab}$ such that under the assumption

$$(\ell_1, \ell_{2,0}) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \Big(\ell_1, -\frac{1}{2}\, \ell_{2,1}\Big) \in \widehat{\mathcal{S}}^2_{ab}(a)$$

the weak theorem on differential inequalities does not hold for the system (1.3). However, Corollary 3.1 guarantees that the problem (1.3), (1.4) remains to be uniquely solvable.

As it was said above, the Cauchy problem for second order functional differential equations can be regarded as a particular case of (1.3), (1.4). As an example, we consider the problem

$$x''(t) = \frac{1}{(1-t)^\nu} \int\limits_0^t \frac{d_1 x(\tau(s)) - d_2 x(\lambda s)}{(1-s)^\nu}\, ds + q(t), \quad t \in [0,1], \tag{3.21}$$

$$x(0) = c_1, \quad x'(0) = c_2, \tag{3.22}$$

where d_1, $d_2 \in \mathbb{R}_+$, $\nu < 1$, $\lambda \in [0,1]$, $\tau : [0,1] \to [0,1]$ is a measurable function, $q \in L([0,1];\mathbb{R})$, and c_1, $c_2 \in \mathbb{R}$.

Corollary 3.2. *Let at least one of the following conditions be fulfilled:*

(a) *The deviation τ is a delay, i.e.,*

$$\tau(t) \le t \ \text{ for a.e. } \ t \in [0,1];$$

(b) *The numbers d_1 and d_2 satisfy*

$$d_1 < (3-2\nu)(2-\nu), \quad d_2 \le 2(3-2\nu)(2-\nu). \tag{3.23}$$

Then the problem (3.21), (3.22) *has a unique solution.*

3.2. **Systems with Argument Deviations.** In this section, we give some corollaries of Theorems 3.1–3.4 for systems with deviating arguments. All statements formulated below are proved in Subsection 3.3.

Consider the differential system

$$x_1'(t) = h_1(t) x_2(\tau_1(t)) + q_1(t), \quad x_2'(t) = h_2(t) x_1(\tau_2(t)) + q_2(t), \tag{3.24}$$

where h_1, h_2, q_1, $q_2 \in L([a,b];\mathbb{R})$ and τ_1, $\tau_2 : [a,b] \to [a,b]$ are measurable functions.

In order to simplify the formulation of the following statement, we put

$$h_{i,0} \stackrel{\text{def}}{\equiv} [h_i]_+, \quad h_{i,1} \stackrel{\text{def}}{\equiv} [h_i]_- \ \text{ for } \ i = 1,2. \tag{3.25}$$

Theorem 3.1 implies

Corollary 3.3. *Let $k \in \{1,2\}$, $m \in \{0,1\}$, and let the functions $h_{i,j}$ ($i = 1,2$, $j = 0,1$) be defined by* (3.25). *Assume that there exist numbers $\alpha_i \in \mathbb{R}_+$ ($i = 1,\dots,4$), at least one of which is positive, and $\lambda \in [0,1[$ such that*

$$\int\limits_{\omega_1}^{\omega_2} \frac{ds}{\alpha_1 + (\alpha_2+\alpha_3)s + \alpha_4 s^2} > \frac{(b-a)^{1-\lambda}}{1-\lambda}\,, \tag{3.26}$$

$$\alpha_1 (b-t)^\lambda \Bigg(\int\limits_t^{\tau_{3-k}(t)} \frac{ds}{(b-s)^\lambda} \Bigg) |h_{3-k}(t)| \le$$

$$\le \alpha_2 \Bigg[1 + \int\limits_{\tau_{3-k}(t)}^t \frac{\alpha_3}{(b-s)^\lambda}\, ds \Bigg] \ \text{ for a.e. } \ t \in [a,b], \tag{3.27}$$

$$(b-t)^{\lambda}|h_{3-k}(t)| \leq \alpha_4\left[1+\int\limits_{\tau_{3-k}(t)}^{t}\frac{\alpha_3}{(b-s)^{\lambda}}\,ds\right] \text{ for a.e. } t\in[a,b], \tag{3.28}$$

$$(b-t)^{\lambda}|h_k(t)| \leq \alpha_1\left[1+\int\limits_{t}^{\tau_k(t)}\frac{\alpha_2}{(b-s)^{\lambda}}\,ds\right] \text{ for a.e. } t\in[a,b], \tag{3.29}$$

and

$$\alpha_4(b-t)^{\lambda}\left(\int\limits_{\tau_k(t)}^{t}\frac{ds}{(b-s)^{\lambda}}\right)|h_k(t)| \leq$$
$$\leq \alpha_3\left[1+\int\limits_{t}^{\tau_k(t)}\frac{\alpha_2}{(b-s)^{\lambda}}\,ds\right] \text{ for a.e. } t\in[a,b], \tag{3.30}$$

where $\omega_1 = \|h_{k,1-m}\|_L$ *and* ω_2 *has the following properties:*

(i) *If* $h_{k,1-m}\equiv 0$ *and* $h_{3-k,m}\equiv 0$, *then* $\omega_2=+\infty$;

(ii) *If* $h_{k,1-m}\equiv 0$ *and* $h_{3-k,m}\not\equiv 0$, *then* $\omega_2=\|h_{3-k,m}\|_L^{-1}$;

(iii) *If* $h_{k,1-m}\not\equiv 0$ *and* $h_{3-k,m}\not\equiv 0$, *then* $\|h_{k,1-m}\|_L<\omega_2\leq\|h_{3-k,m}\|_L^{-1}$ *and*

$$\int\limits_a^b h_{3-k,1-m}(s)\left(\int\limits_a^{\tau_{3-k}(s)} h_{k,1-m}(\xi)d\xi\right)ds \leq$$
$$\leq \big(1-\omega_2\|h_{3-k,m}\|_L\big)\exp\left(-\int\limits_a^b\frac{\alpha_2+\alpha_4\omega_2}{(b-s)^{\lambda}}\,ds\right); \tag{3.31}$$

(iv) *If* $h_{k,1-m}\not\equiv 0$ *and* $h_{3-k,m}\equiv 0$, *then* $\|h_{k,1-m}\|_L<\omega_2<+\infty$ *and*

$$\int\limits_a^b h_{3-k,1-m}(s)\left(\int\limits_a^{\tau_{3-k}(s)} h_{k,1-m}(\xi)\,d\xi\right)ds<\exp\left(-\int\limits_a^b\frac{\alpha_2+\alpha_4\omega_2}{(b-s)^{\lambda}}\,ds\right). \tag{3.32}$$

Then the problem (3.24), (1.4) *has a unique solution.*

If neither of the functions h_1 and h_2 changes its sign and at least one of the deviations τ_1 and τ_2 is a delay, then we can derive the following statement from Theorem 3.2.

Corollary 3.4. *Let* $k\in\{1,2\}$, $m\in\{0,1\}$,

$$(-1)^m h_k(t)\geq 0,\quad (-1)^{1-m}h_{3-k}(t)\geq 0 \text{ for a.e. } t\in[a,b], \tag{3.33}$$

and

$$|h_{3-k}(t)|(\tau_{3-k}(t)-t)\leq 0 \text{ for a.e. } t\in[a,b]. \tag{3.34}$$

Assume that there exist numbers α_1, α_2, $\alpha_3 \in \mathbb{R}_+$ *at least one of which is positive,* $\lambda \in [0, 1[$ *, and* $\nu \in [0, \lambda]$ *such that*

$$\int_0^{+\infty} \frac{ds}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} > \frac{(b-a)^{1-\lambda}}{1-\lambda}\,, \tag{3.35}$$

$$(b-t)^{\lambda+\nu}|h_k(t)| \le \alpha_1 \ \text{ for a.e. } t \in [a,b], \tag{3.36}$$

$$\alpha_3(b-t)^{\nu}|h_k(t)|(t-\tau_k(t)) \le \alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}} \ \text{ for a.e. } t \in [a,b], \tag{3.37}$$

and

$$(b-t)^{\lambda-\nu}|h_{3-k}(t)| \le$$

$$\le \alpha_3\Bigg[1 + \sigma_{3-k}(t) \int_{\tau_{3-k}(t)}^{t} \Big(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^{\lambda}}\Big)\,ds\Bigg] \ \text{ for a.e. } t \in [a,b], \tag{3.38}$$

where

$$\sigma_{3-k}(t) \overset{\text{def}}{=} \frac{1}{2}\big(1 + \operatorname{sgn}(t - \tau_{3-k}(t))\big) \ \text{ for a.e. } t \in [a,b].$$

Then the problem (3.24), (1.4) *has a unique solution.*

Corollary 3.4 implies

Corollary 3.5. *Let*

$$h_1(t) \ge 0, \quad h_2(t) \le 0 \ \text{ for a.e. } t \in [a,b].$$

Assume that there exist numbers α, $\beta \in \mathbb{R}_+$, $\lambda \in [0, 1[$ *, and* $\nu \in [0, \lambda]$ *such that*

$$\int_0^{+\infty} \frac{ds}{\alpha + \beta s^2} > \frac{(b-a)^{1-\lambda}}{1-\lambda}\,,$$

and let either

$$h_1(t)(\tau_1(t) - t) \le 0, \quad |h_2(t)|(\tau_2(t) - t) \ge 0 \ \text{ for a.e. } t \in [a,b],$$
$$(b-t)^{\lambda-\nu}h_1(t) \le \beta, \quad (b-t)^{\lambda+\nu}|h_2(t)| \le \alpha \ \text{ for a.e. } t \in [a,b]$$

or

$$h_1(t)(\tau_1(t) - t) \ge 0, \quad |h_2(t)|(\tau_2(t) - t) \le 0 \ \text{ for a.e. } t \in [a,b],$$
$$(b-t)^{\lambda+\nu}h_1(t) \le \alpha, \quad (b-t)^{\lambda-\nu}|h_2(t)| \le \beta \ \text{ for a.e. } t \in [a,b]$$

be satisfied. Then the problem (3.24), (1.4) *has a unique solution.*

In order to illustrate Theorems 3.3 and 3.4, we consider the differential system

$$\begin{aligned} x_1'(t) &= f(t)x_2(\mu(t)) + q_1(t), \\ x_2'(t) &= h_0(t)x_1(\tau_0(t)) - h_1(t)x_1(\tau_1(t)) + q_2(t), \end{aligned} \tag{3.39}$$

where f, h_0, $h_1 \in L([a,b];\mathbb{R}_+)$, μ, τ_0, $\tau_1 : [a,b] \to [a,b]$ are measurable functions and q_1, $q_2 \in L([a,b];\mathbb{R})$.

In the next corollary of Theorem 3.3, the solvability conditions are given in terms of norms of the functions f, h_0, and h_1.

Corollary 3.6. *Let*

$$PG_0 < 1 \tag{3.40}$$

and

$$PG_1 < 4 + 4\sqrt{1 - PG_0}\,, \tag{3.41}$$

where

$$P = \int_a^b f(s)\,ds, \quad G_i = \int_a^b h_i(s)\,ds \;\; \text{for } \; i = 0, 1. \tag{3.42}$$

Then the problem (3.39), (1.4) *has a unique solution.*

The following statements can be derived from Theorem 3.4 and the results given in [26] (see also Section 5).

Corollary 3.7. *Let*

$$\mu(t) \le t, \quad \tau_1(t) \le t \;\; \text{for a.e.} \;\; t \in [a, b], \tag{3.43}$$

and let the functions f, μ, h_0, τ_0 satisfy at least one of the following conditions:

(a)

$$\int_t^{\tau_0(t)} \omega(s)\,ds \le \frac{1}{e} \;\; \text{for a.e.} \;\; t \in [a, b],$$

where

$$\omega(t) \overset{\text{def}}{=} \max\{f(t), h_0(t)\} \;\; \text{for a.e.} \;\; t \in [a, b]; \tag{3.44}$$

(b)

$$\int_a^b \cosh\Big(\int_s^b \omega(\xi)\,d\xi\Big) h_0(s)\sigma_1(s)\Big(\int_s^{\tau_0(s)} f(\xi)\,d\xi\Big) ds < 1,$$

where the function ω is defined by (3.44) *and*

$$\sigma_1 = \frac{1}{2}\big(1 + \operatorname{sgn}(\tau_0(t) - t)\big) \;\; \text{for a.e.} \;\; t \in [a, b];$$

(c) *either*

$$\int_a^{\tau_0^*} f(s)\Big(\int_a^{\mu(s)} h_0(\xi)\,d\xi\Big) ds < 1$$

or

$$\int_a^{\mu^*} h_0(s)\Big(\int_a^{\tau_0(s)} f(\xi)\,d\xi\Big) ds < 1,$$

where

$$\tau_0^* = \operatorname{ess\,sup}\big\{\tau_0(t) :\ t \in [a,b]\big\}, \quad \mu^* = \operatorname{ess\,sup}\big\{\mu(t) :\ t \in [a,b]\big\}.$$

Furthermore, assume that the functions f, μ, h_1, τ_1 satisfy at least one of the following conditions with $\gamma^* = 2$:

(A)

$$\int_a^b f(s)\bigg(\int_a^{\mu(s)} h_1(\xi)\,d\xi\bigg)\,ds \le \gamma^*;$$

(B) *there exist numbers α_1, $\alpha_2 \in \mathbb{R}_+$, $\alpha_3 > 0$, $\lambda \in [0,1[$, and $\nu \in [0,\lambda]$ such that* (3.35) *holds and*

$$(b-t)^{\lambda-\nu} f(t) \le \alpha_3\bigg[1+\sigma_2(t)\int_{\mu(t)}^{t}\Big(\frac{\nu}{b-s}+\frac{\alpha_2}{(b-s)^{\lambda}}\Big)\,ds\bigg] \ \text{ for a.e. } t\in[a,b],$$

$$(b-t)^{\lambda+\nu} h_1(t) \le \gamma^*\alpha_1 \ \text{ for a.e. } t \in [a,b],$$

$$\alpha_3(b-t)^{\nu} h_1(t)(t-\tau_1(t)) \le \gamma^*\Big(\alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}}\Big) \ \text{ for a.e. } t \in [a,b],$$

where

$$\sigma_2(t) \overset{\text{def}}{=} \frac{1}{2}\big(1+\operatorname{sgn}(t-\mu(t))\big) \ \text{ for a.e. } t \in [a,b].$$

Then the problem (3.39), (1.4) *has a unique solution.*

3.3. **Proofs.** Now we prove the statements formulated above. We first note that the linear problem (1.3), (1.4) has the so-called Fredholm property, i.e., the following lemma holds (see, e.g, [23], [16], [14], [8]).

Lemma 3.1. *The problem* (1.3), (1.4) *has a unique solution for every q_1, $q_2 \in L([a,b];\mathbb{R})$ and c_1, $c_2 \in \mathbb{R}$ if and only if the corresponding homogeneous problem* (3.14), (3.15) *has only the trivial solution.*

Remark 3.6. It is clear that (x_1, x_2) is a solution to the problem (3.14), (3.15) if and only if $(-x_1, x_2)$ is a solution to the problem

$$u_1'(t) = -\ell_1(u_2)(t), \quad u_2'(t) = -\ell_2(u_1)(t),$$
$$u_1(a) = 0, \quad u_2(a) = 0.$$

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Let $\ell_i = \ell_{i,0} - \ell_{i,1}$ with $\ell_{i,0}$, $\ell_{i,1} \in \mathcal{P}_{ab}$ $(i = 1,2)$. Assume that there exist functions α_1, α_2, β_1, $\beta_2 \in \widetilde{C}([a,b];\mathbb{R})$ such that*

$$\alpha_i(t) \le \beta_i(t) \ \text{ for } t \in [a,b], \ \ i = 1,2, \tag{3.45}$$

$$\alpha_1'(t) \le \ell_{1,0}(\alpha_2)(t) - \ell_{1,1}(\beta_2)(t) \ \text{ for a.e. } t \in [a,b], \tag{3.46}$$

$$\alpha_2'(t) \ge \ell_{2,0}(\beta_1)(t) - \ell_{2,1}(\alpha_1)(t) \ \text{ for a.e. } t \in [a,b], \tag{3.47}$$

$$\beta_1'(t) \ge \ell_{1,0}(\beta_2)(t) - \ell_{1,1}(\alpha_2)(t) \ \text{ for a.e. } t \in [a,b], \tag{3.48}$$

and

$$\beta_2'(t) \le \ell_{2,0}(\alpha_1)(t) - \ell_{2,1}(\beta_1)(t) \quad \textit{for a.e.} \;\; t \in [a,b]. \tag{3.49}$$

Then for arbitrary $c_1 \in [\alpha_1(a), \beta_1(a)]$ *and* $c_2 \in [\alpha_2(b), \beta_2(b)]$ *the system* (3.14) *has at least one solution* (x_1, x_2) *satisfying* $x_1(a) = c_1$, $x_2(b) = c_2$, *and*

$$\alpha_i(t) \le x_i(t) \le \beta_i(t) \quad \textit{for} \;\; t \in [a,b], \quad i = 1,2. \tag{3.50}$$

Proof. For $k = 1,2$ and $z \in C([a,b];\mathbb{R})$, we put

$$\chi_k(z)(t) \stackrel{\text{def}}{=} \frac{1}{2}\big(|z(t) - \alpha_k(t)| - |z(t) - \beta_k(t)| + \alpha_k(t) + \beta_k(t)\big) \quad \text{for} \;\; t \in [a,b].$$

It is clear that $\chi_1, \chi_2 : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ are continuous operators and

$$\alpha_k(t) \le \chi_k(z)(t) \le \beta_k(t) \quad \text{for} \;\; t \in [a,b], \quad z \in C([a,b];\mathbb{R}), \quad k = 1,2. \tag{3.51}$$

Put

$$T_1(z)(t) \stackrel{\text{def}}{=} c_1 + \int_a^t \ell_1(\chi_2(z))(s)\,ds \quad \text{for} \;\; t \in [a,b], \quad z \in C([a,b];\mathbb{R}),$$

$$T_2(z)(t) \stackrel{\text{def}}{=} c_2 - \int_t^b \ell_2(\chi_1(z))(s)\,ds \quad \text{for} \;\; t \in [a,b], \quad z \in C([a,b];\mathbb{R}).$$

By virtue of (3.51) and the assumptions $\ell_{i,0}, \ell_{i,1} \in \mathcal{P}_{ab}$ $(i = 1,2)$, for any $z \in C([a,b];\mathbb{R})$ the functions $T_1(z)$ and $T_2(z)$ belong to the set $\widetilde{C}([a,b];\mathbb{R})$,

$$|T_k(z)(t)| \le M_k \quad \text{for} \;\; t \in [a,b], \quad k = 1,2, \tag{3.52}$$

and

$$\begin{aligned} \ell_{k,0}(\alpha_{3-k})(t) - \ell_{k,1}(\beta_{3-k})(t) \le \frac{d}{dt}\,T_k(z)(t) \le \\ \le \ell_{k,0}(\beta_{3-k})(t) - \ell_{k,1}(\alpha_{3-k})(t) \quad \text{for a.e.} \;\; t \in [a,b], \quad k = 1,2, \end{aligned} \tag{3.53}$$

where

$$M_k = |c_k| + \int_a^b (\ell_{k,0} + \ell_{k,1})\big(|\alpha_{3-k}| + |\beta_{3-k}|\big)(s)\,ds \quad \text{for} \;\; k = 1,2.$$

Now define $T : C([a,b];\mathbb{R}) \times C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R}) \times C([a,b];\mathbb{R})$ by

$$T(z_1, z_2)(t) \stackrel{\text{def}}{=} \big(T_1(z_2)(t), T_2(z_1)(t)\big) \quad \text{for} \;\; t \in [a,b], \quad z_1, \, z_2 \in C([a,b];\mathbb{R}).$$

In view of (3.52) and (3.53), it is clear that T maps continuously the Banach space $C([a,b];\mathbb{R}) \times C([a,b];\mathbb{R})$ into its relatively compact subset. Therefore, by Schauder's fixed point theorem, the operator T has a fixed point, i.e., there exist $x_1, \, x_2 \in C([a,b];\mathbb{R})$ such that

$$x_1(t) = T_1(x_2)(t), \quad x_2(t) = T_2(x_1)(t) \quad \text{for} \;\; t \in [a,b]. \tag{3.54}$$

Obviously, $x_1,\, x_2 \in \widetilde{C}([a,b];\mathbb{R})$, $x_1(a)=c_1$, $x_2(b)=c_2$, and thus

$$\alpha_1(a) \le x_1(a) \le \beta_1(a), \quad \alpha_2(b) \le x_2(b) \le \beta_2(b). \tag{3.55}$$

On the other hand, by virtue of (3.48), (3.53) and (3.54), we get

$$\begin{aligned} x_1'(t) - \beta_1'(t) &= \frac{d}{dt}\, T_1(x_2)(t) - \beta_1'(t) \le \\ &\le \ell_{1,0}(\beta_2)(t) - \ell_{1,1}(\alpha_2)(t) - \beta_1'(t) \le 0 \ \text{ for a.e. } t \in [a,b], \end{aligned}$$

which, together with (3.55), implies $x_1(t) \le \beta_1(t)$ for $t \in [a,b]$. One can prove the other inequalities in (3.50) analogously using (3.46), (3.47) and (3.49). However, this means that

$$x_1(t) = c_1 + \int_a^t \ell_1(x_2)(s)\,ds, \quad x_2(t) = c_2 - \int_t^b \ell_2(x_1)(s)\,ds \ \text{ for } t \in [a,b],$$

that is, (x_1,x_2) is a solution to the system (3.14) satisfying $x_1(a)=c_1$, $x_2(b)=c_2$ and (3.50). $\square$

The next lemma follows from [25, Theorem 3.1].

Lemma 3.3. *Let there exist $g_1, g_2 \in \mathcal{P}_{ab}$ such that $(g_1,g_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$ and, for any $z \in C([a,b];\mathbb{R})$, the inequality*

$$\ell_k(z)(t)\operatorname{sgn} z(t) \le g_k(|z|)(t) \ \text{ for a.e. } t \in [a,b], \quad k=1,2,$$

holds. Then the problem (3.14), (3.15) *has only the trivial solution.*

Lemma 3.4. *Let $\ell_i = \ell_{i,0} - \ell_{i,1}$ with $\ell_{i,0},\, \ell_{i,1} \in \mathcal{P}_{ab}$ $(i=1,2)$. Assume that there exist functions $\beta_1,\, \beta_2 \in \widetilde{C}([a,b];\mathbb{R})$ satisfying* (3.1) *and*

$$\beta_1'(t) \ge \ell_{1,0}(\beta_2) + \ell_{1,1}(\beta_2) \ \text{ for a.e. } t \in [a,b], \tag{3.56}$$

$$\beta_2'(t) \le -\ell_{2,0}(\beta_1) - \ell_{2,1}(\beta_1) \ \text{ for a.e. } t \in [a,b]. \tag{3.57}$$

Then the problem

$$x_1(a)=0, \quad x_2(b)=0 \tag{3.58}$$

for the system (3.14) *has only the trivial solution.*

Proof. Let $\psi : L([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ be defined by

$$\psi(h)(t) \overset{\text{def}}{=} h(a+b-t) \ \text{ for a.e. } t \in [a,b], \ \text{ and all } h \in L([a,b];\mathbb{R}),$$

and let ω be the restriction of the operator ψ to the space $C([a,b];\mathbb{R})$. For any $z \in C([a,b];\mathbb{R})$ and $m=0,1$, we put

$$p_{1,m}(z)(t) \overset{\text{def}}{=} \ell_{1,m}(\omega(z))(t), \ \ p_{2,m}(z)(t) \overset{\text{def}}{=} \psi(\ell_{2,m}(z))(t) \ \text{ for a.e. } t\in[a,b].$$

It is clear that if (x_1,x_2) is a solution to the problem (3.14), (3.58), then the pair $(x_1,\omega(x_2))$ is a solution to the problem

$$v_1'(t) = p_{1,0}(v_2)(t) - p_{1,1}(v_2)(t), \quad v_2'(t) = p_{2,1}(v_1)(t) - p_{2,0}(v_1)(t), \tag{3.59}$$

$$v_1(a)=0, \quad v_2(a)=0, \tag{3.60}$$

and vice versa, if (v_1, v_2) is a solution to the problem (3.59), (3.60), then the pair $(v_1, \omega(v_2))$ is a solution to the problem (3.14), (3.58).

On the other hand, it follows from (3.56) and (3.57) that the functions $\gamma_1 \equiv \beta_1$ and $\gamma_2 \equiv \omega(\beta_2)$ satisfy

$$\gamma_1'(t) \geq p_{1,0}(\gamma_2)(t) + p_{1,1}(\gamma_2)(t), \quad \gamma_2'(t) \geq p_{2,0}(\gamma_1)(t) + p_{2,1}(\gamma_1)(t) \quad \text{for a.e. } t \in [a,b],$$

and since $p_{k,m} \in \mathcal{P}_{ab}$ $(k = 1,2;\ m = 0,1)$, Proposition 5.1 (see Section 5 below) implies

$$\big(p_{1,0} + p_{1,1}, p_{2,0} + p_{2,1}\big) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

It is also easy to verify that the inequalities

$$\big[p_{1,0}(z)(t) - p_{1,1}(z)(t)\big] \operatorname{sgn} z(t) \leq p_{1,0}(|z|)(t) + p_{1,1}(|z|)(t) \quad \text{for a.e. } t \in [a,b],$$
$$\big[p_{2,1}(z)(t) - p_{2,0}(z)(t)\big] \operatorname{sgn} z(t) \leq p_{2,0}(|z|)(t) + p_{2,1}(|z|)(t) \quad \text{for a.e. } t \in [a,b]$$

hold on the set $C([a,b];\mathbb{R})$. Therefore, by virtue of Lemma 3.3 and the above mentioned equivalence, we get the assertion of the lemma. □

Proof of Theorem 3.1. According to Lemma 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (3.14), (3.15) has only the trivial solution. In view of Remark 3.6, we can assume without loss of generality that $k = 1$ and $m = 0$. Let (x_1, x_2) be a solution to the problem (3.14), (3.15).

We first note that it follows from (3.1)–(3.3) that

$$\beta_1(t) \geq \beta_1(a) + \chi(\ell_{1,0}(\beta_2))(t), \quad \beta_2(t) \geq \beta_2(b) \quad \text{for } t \in [a,b],$$

and thus (3.5) yields

$$\beta_1(a) \int_a^b \ell_{2,0}(1)(s)\, ds + \beta_2(b) \int_a^b \ell_{2,0}\big(\chi(\ell_{1,0}(1))\big)(s)\, ds + \\ + \beta_2(b) \int_a^b \ell_{2,1}\big(\chi(\ell_{1,1}(1))\big)(s)\, ds \leq \beta_2(b).$$

Consequently, using (3.1) we get

$$\int_a^b \ell_{2,0}\big(\chi(\ell_{1,0}(1))\big)(s) ds + \int_a^b \ell_{2,1}\big(\chi(\ell_{1,1}(1))\big)(s) ds < 1, \tag{3.61}$$

because we suppose that the inequality (3.5) is strict if $\ell_{2,0} = 0$.

Put

$$\alpha_1(t) = -\int_a^t \ell_{1,1}(\beta_2)(s)\, ds \quad \text{for } t \in [a,b] \tag{3.62}$$

and

$$\alpha_2(t) = \int\limits_a^t \ell_{2,0}(\beta_1)(s)\,ds - \int\limits_a^t \ell_{2,1}(\alpha_1)(s)ds \text{ for } t \in [a,b]. \tag{3.63}$$

It is clear that

$$\alpha_2(t) \geq 0 \text{ for } t \in [a,b], \tag{3.64}$$

and using (3.4) one can easily verify that

$$-\alpha_1(t) = \int\limits_a^t \ell_{1,1}(\beta_2)(s)\,ds \leq \beta_1(a) \leq \beta_1(t) \text{ for } t \in [a,b]. \tag{3.65}$$

By virtue of (3.64) and (3.65), from (3.2), (3.3), (3.62) and (3.63) we get

$$\begin{aligned} &\alpha_1'(t) = -\ell_{1,1}(\beta_2)(t) \leq \ell_{1,0}(\alpha_2)(t) - \ell_{1,1}(\beta_2)(t) \text{ for a.e. } t \in [a,b], \\ &\alpha_2'(t) = \ell_{2,0}(\beta_1)(t) - \ell_{2,1}(\alpha_1)(t) \text{ for a.e. } t \in [a,b], \\ &\beta_1'(t) \geq \ell_{1,0}(\beta_2)(t) \geq \ell_{1,0}(\beta_2)(t) - \ell_{1,1}(\alpha_2)(t) \text{ for a.e. } t \in [a,b], \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} \beta_2'(t) \leq -\ell_{2,0}(\beta_1)(t) - \ell_{2,1}(\beta_1)(t) \leq \\ \leq \ell_{2,0}(\alpha_1)(t) - \ell_{2,1}(\beta_1)(t) \text{ for a.e. } t \in [a,b], \end{aligned} \tag{3.67}$$

i.e., the inequalities (3.46)–(3.49) are satisfied. Moreover, it is clear that

$$\alpha_1(t) \leq \beta_1(t) \text{ for } t \in [a,b]. \tag{3.68}$$

On the other hand, (3.5), (3.62) and (3.63) result in

$$\alpha_2(b) = \int\limits_a^b \ell_{2,0}(\beta_1)(s)\,ds + \int\limits_a^b \ell_{2,1}\big(\chi(\ell_{1,1}(\beta_2))\big)(s)\,ds \leq \beta_2(b).$$

Furthermore, (3.66)–(3.68) yield

$$\begin{aligned} \alpha_2'(t) = \ell_{2,0}(\beta_1)(t) - \ell_{2,1}(\alpha_1)(t) \geq \\ \geq \ell_{2,0}(\alpha_1)(t) - \ell_{2,1}(\beta_1)(t) \geq \beta_2'(t) \text{ for a.e. } t \in [a,b]. \end{aligned}$$

Hence, the last two relations result in $\alpha_2(t) \leq \beta_2(t)$ for $t \in [a,b]$, and thus the condition (3.45) is satisfied.

Therefore, by virtue of Lemma 3.2, the system (3.14) has a solution (u_1, u_2) satisfying

$$u_1(a) = 0, \quad u_2(b) = \beta_2(b), \tag{3.69}$$

and

$$\alpha_k(t) \leq u_k(t) \leq \beta_k(t) \text{ for } t \in [a,b], \quad k = 1,2. \tag{3.70}$$

We will show that

$$u_2(a) > 0. \tag{3.71}$$

Indeed, (3.64) and (3.70) imply $u_2(t) \geq 0$ for $t \in [a,b]$, and since (u_1, u_2) is a solution to the system (3.14), the first equation in (3.14) yields

$$u_1(t) \leq \int_a^t \ell_{1,0}(u_2)(s)ds, \quad -u_1(t) \leq \int_a^t \ell_{1,1}(u_2)(s)ds \ \text{ for } \ t \in [a,b].$$

Using these relations in the second equation of (3.14), we get

$$u_2'(t) \leq \ell_{2,0}\big(\chi(\ell_{1,0}(u_2))\big)(t) + \ell_{2,1}\big(\chi(\ell_{1,1}(u_2))\big)(t) \ \text{ for a.e. } \ t \in [a,b]. \tag{3.72}$$

Put $M = \max\{u_2(t) : \ t \in [a,b]\}$ and choose $t_M \in [a,b]$ such that $u_2(t_M) = M$. Integration of (3.72) from a to t_M yields

$$\begin{aligned} M &\leq u_2(a) + \int_a^{t_M} \ell_{2,0}\big(\chi(\ell_{1,0}(u_2))\big)(s)\, ds + \int_a^{t_M} \ell_{2,1}\big(\chi(\ell_{1,1}(u_2))\big)(s)\, ds \leq \\ &\leq u_2(a) + M\left[\int_a^b \ell_{2,0}\big(\chi(\ell_{1,0}(1))\big)(s)\, ds + \int_a^b \ell_{2,1}\big(\chi(\ell_{1,1}(1))\big)(s)\, ds\right]. \end{aligned} \tag{3.73}$$

In view of (3.1) and (3.69), we have $M > 0$. Therefore, (3.61) and (3.73) result in $M < u_2(a) + M$, i.e., the inequality (3.71) is true.

Finally, we put

$$v_k(t) = u_2(b)x_k(t) - u_k(t)x_2(b) \ \text{ for } \ t \in [a,b], \quad k = 1, 2.$$

Obviously, (v_1, v_2) is a solution to the problem (3.14), (3.58). Therefore, Lemma 3.4 yields $v_1 \equiv 0$ and $v_2 \equiv 0$. Consequently, we have

$$0 = v_2(a) = -u_2(a)x_2(b),$$

which, together with (3.71), implies $x_2(b) = 0$. However, this means that (x_1, x_2) is a solution to the problem (3.14), (3.58), and thus Lemma 3.4 yields $x_1 \equiv 0$ and $x_2 \equiv 0$.

Consequently, the homogeneous problem (3.14), (3.15) has only the trivial solution. □

Proof of Theorem 3.2. According to Lemma 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (3.14), (3.15) has only the trivial solution. In view of Remark 3.6, we can assume without loss of generality that $k = 1$ and $m = 0$. Assume that, on the contrary, (x_1, x_2) is a nontrivial solution to the problem (3.14), (3.15). Then it is clear that $x_1 \not\equiv 0$ and $x_2 \not\equiv 0$.

First suppose that x_2 does not change its sign. Then we can assume without loss of generality that $x_2(t) \geq 0$ for $t \in [a,b]$. Since the operator ℓ_1 is nondecreasing, the first equation in (3.14) implies $x_1'(t) \geq 0$ for a.e. $t \in [a,b]$. Therefore, by virtue of (3.15), we have $x_1(t) \geq 0$ for $t \in [a,b]$. On the other hand, the operator ℓ_2 is supposed to be nonincreasing, and thus the second equation in (3.14) yields $x_2'(t) \leq 0$ for a.e. $t \in [a,b]$. Consequently, using the condition $x_2(a) = 0$, we get the contradiction $x_2 \equiv 0$.

Now suppose that x_2 changes its sign. Put

$$\lambda_1 = \inf \mathcal{A}, \quad \lambda_2 = \max\left\{\frac{x_2(t)}{\gamma_2(t)} : \ t \in [a,b]\right\}, \tag{3.74}$$

where

$$\mathcal{A} = \Big\{\lambda > 0 : \ \lambda\gamma_1(t) - x_1(t) \geq 0 \text{ for } t \in [a,b[\ \Big\}. \tag{3.75}$$

It is clear that

$$0 \leq \lambda_1 < +\infty, \quad 0 < \lambda_2 < +\infty, \tag{3.76}$$

and there exists $t_0 \in\,]a,b]$ such that

$$\frac{x_2(t_0)}{\gamma_2(t_0)} = \lambda_2. \tag{3.77}$$

Without loss of generality, we can assume that $t_0 < b$ and there exists $b_0 \in\,]t_0, b[$ such that

$$x_2(b_0) = 0. \tag{3.78}$$

Indeed, if either $t_0 = b$ or $x_2(t) > 0$ for $t \in [t_0, b[\,$, then there exists $t^* \in\,]a, t_0[$ with the properties

$$x_2(t) > 0 \ \text{ for } \ t \in\,]t^*, b[\quad x_2(t^*) = 0.$$

Then we can redefine the numbers λ_1, λ_2, t_0 for the solution $(-x_1, -x_2)$ of the problem (3.14), (3.15), and we can take $b_0 = t^*$.

Now we put

$$w_1(t) = \lambda_1\gamma_1(t) - x_1(t) \ \text{ for } \ t \in [a,b[\,,$$
$$w_2(t) = \lambda_2\gamma_2(t) - x_2(t) \ \text{ for } \ t \in [a,b].$$

By virtue of (3.7), (3.74) and (3.77), it is clear that

$$w_1(t) \geq 0 \ \text{ for } \ t \in [a,b[\,, \quad w_2(t) \geq 0 \ \text{ for } \ t \in [a,b], \tag{3.79}$$

and

$$w_2(t_0) = 0. \tag{3.80}$$

Obviously, either $\lambda_1 < \lambda_2$ or $\lambda_1 \geq \lambda_2$.

First suppose that $\lambda_1 < \lambda_2$. Then, in view of (3.7), (3.10), (3.14), (3.76), (3.79) and the fact that ℓ_2 is a nonincreasing a-Volterra operator, we get

$$w_2'(t) \leq \ell_2^{ab_0}(\lambda_2\gamma_1 - x_1)(t) \leq \ell_2^{ab_0}(w_1)(t) \leq 0 \ \text{ for a.e. } \ t \in [a, b_0].$$

Therefore, by virtue of (3.7), (3.76) and (3.78), the last relation yields

$$w_2(t_0) \geq w_2(b_0) = \lambda_2\gamma_2(b_0) > 0,$$

which contradicts (3.80).

Now suppose that $\lambda_1 \geq \lambda_2$. Then (3.76) implies

$$\lambda_1 > 0. \tag{3.81}$$

Using (3.7), (3.8), (3.14), (3.15), (3.79), (3.81) and the assumption $\ell_1 \in \mathcal{P}_{ab}$, we get

$$w_1'(t) \geq \ell_1(\lambda_1\gamma_2 - x_2)(t) \geq \ell_1(w_2)(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b]$$

and

$$w_1(a) = \lambda_1 \gamma_1(a) > 0.$$

Consequently, we have $w_1(t) > 0$ for $t \in [a, b[$. Therefore, there exists $\varepsilon > 0$ such that

$$w_1(t) \geq \varepsilon x_1(t) \quad \text{for} \quad t \in [a, b[\,,$$

i.e.,

$$\frac{\lambda_1}{1+\varepsilon}\, \gamma_1(t) - x_1(t) \geq 0 \quad \text{for} \quad t \in [a, b[\,.$$

However, in view of (3.75), the last relation implies $\lambda_1/(1+\varepsilon) \in \mathcal{A}$, which contradicts the first equality in (3.74).

The contradictions obtained prove that the homogeneous problem (3.14), (3.15) has only the trivial solution. □

Proof of Theorem 3.3. According to Lemma 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (3.14), (3.15) has only the trivial solution. In view of Remark 3.6, we can assume without loss of generality that $k = 2$ and $m = 0$. Assume that, on the contrary, (x_1, x_2) is a nontrivial solution to the problem (3.14), (3.15). Then it is clear that $x_1 \not\equiv 0$ and $x_2 \not\equiv 0$.

Put

$$M_i = \max\big\{x_i(t) : \ t \in [a, b]\big\}, \quad m_i = -\min\big\{x_i(t) : \ t \in [a, b]\big\} \quad \text{for} \quad i = 1, 2 \tag{3.82}$$

and choose α_i, $\beta_i \in [a, b]$ $(i = 1, 2)$ such that

$$x_i(\alpha_i) = M_i, \quad x_i(\beta_i) = -m_i \quad \text{for} \quad i = 1, 2. \tag{3.83}$$

For the sake of clarity we will divide the discussion into the following cases.

(a) The function x_1 does not change its sign. Then we can assume without loss of generality that

$$x_1(t) \geq 0 \quad \text{for} \quad t \in [a, b]; \tag{3.84}$$

(b) The function x_1 changes its sign. Then, in view of the assumption $\ell_1 \in \mathcal{P}_{ab}$, the function x_2 changes its sign as well. Moreover, we can assume without loss of generality that $\alpha_2 < \beta_2$. Further, one of the following conditions is satisfied:

(b1) $\alpha_1 < \beta_1$;

(b2) $\alpha_1 > \beta_1$.

Case (a): $x_1(t) \geq 0$ *for* $t \in [a, b]$. Obviously,

$$M_1 > 0, \quad m_1 = 0, \quad M_2 \geq 0, \quad m_2 \geq 0. \tag{3.85}$$

Integration of the first equation in (3.14) from a to α_1, in view of (3.15), (3.3), (3.83), (3.85), and the assumption $\ell_1 \in \mathcal{P}_{ab}$, implies

$$M_1 = \int_a^{\alpha_1} \ell_1(x_2)(s)ds \le M_2 \int_a^{\alpha_1} \ell_1(1)(s)ds \le M_2 A_1. \tag{3.86}$$

On the other hand, the integration of the second equation in (3.14) from a to α_2 on account of (3.15), (3.3)–(3.85), and the assumptions $\ell_{2,0}$, $\ell_{2,1} \in \mathcal{P}_{ab}$ yields

$$\begin{aligned} M_2 = \int_a^{\alpha_2} \ell_{2,0}(x_1)(s)\,ds - \int_a^{\alpha_2} \ell_{2,1}(x_1)(s)\,ds \le \\ \le M_1 \int_a^{\alpha_2} \ell_{2,0}(1)(s)\,ds \le M_1 A_{2,0}. \end{aligned} \tag{3.87}$$

Now, using (3.85), the relations (3.86) and (3.87) result in $M_2 > 0$ and $1 \le A_1 A_{2,0}$, which contradicts (3.11).

Case (b): *Both functions x_1 and x_2 change their signs and $\alpha_2 < \beta_2$.* It is clear that

$$M_i > 0, \quad m_i > 0 \ \text{ for } \ i = 1, 2. \tag{3.88}$$

Put

$$A_{2,i}^1 = \int_a^{\alpha_2} \ell_{2,i}(1)(s)\,ds, \quad A_{2,i}^2 = \int_{\alpha_2}^{\beta_2} \ell_{2,i}(1)(s)\,ds \ \text{ for } \ i = 0, 1. \tag{3.89}$$

Integration of the second equation in (3.14) from a to α_2 and from α_2 to β_2 in view of (3.15), (3.3), (3.83), and the assumptions $\ell_{2,0}, \ell_{2,1} \in \mathcal{P}_{ab}$ implies

$$\begin{aligned} M_2 = \int_a^{\alpha_2} \ell_{2,0}(x_1)(s)\,ds - \int_a^{\alpha_2} \ell_{2,1}(x_1)(s)ds \le \\ \le M_1 \int_a^{\alpha_2} \ell_{2,0}(1)(s)\,ds + m_1 \int_a^{\alpha_2} \ell_{2,1}(1)(s)\,ds = M_1 A_{2,0}^1 + m_1 A_{2,1}^1 \end{aligned}$$

and

$$\begin{aligned} M_2 + m_2 = -\int_{\alpha_2}^{\beta_2} \ell_{2,0}(x_1)(s)\,ds + \int_{\alpha_2}^{\beta_2} \ell_{2,1}(x_1)(s)\,ds \le \\ \le m_1 \int_{\alpha_2}^{\beta_2} \ell_{2,0}(1)(s)\,ds + M_1 \int_{\alpha_2}^{\beta_2} \ell_{2,1}(1)(s)\,ds = m_1 A_{2,0}^2 + M_1 A_{2,1}^2\,. \end{aligned}$$

Using (3.13), (3.88) and (3.89), from the last two relations we get

$$\frac{M_2}{m_1} + \frac{M_2}{M_1} + \frac{m_2}{M_1} \leq \frac{M_1}{m_1} A_{2,0}^1 + \frac{m_1}{M_1} A_{2,0}^2 + A_{2,1}. \tag{3.90}$$

Now we are in a position to discuss the cases (b1) and (b2).

Case (b1): $\alpha_1 < \beta_1$. Integration of the first equation in (3.14) from a to α_1 and from α_1 to β_1 by virtue of (3.15), (3.3), (3.83) and the assumption $\ell_1 \in \mathcal{P}_{ab}$ yields

$$M_1 = \int_a^{\alpha_1} \ell_1(x_2)(s)\,ds \leq M_2 \int_a^{\alpha_1} \ell_1(1)(s)\,ds$$

and

$$M_1 + m_1 = -\int_{\alpha_1}^{\beta_1} \ell_1(x_2)(s)\,ds \leq m_2 \int_{\alpha_1}^{\beta_1} \ell_1(1)(s)\,ds.$$

In view of (3.88), it follows from the last two relations that

$$\frac{M_1}{M_2} + \frac{M_1}{m_2} + \frac{m_1}{m_2} \leq \int_a^{\alpha_1} \ell_1(1)(s)\,ds + \int_{\alpha_1}^{\beta_1} \ell_1(1)(s)\,ds \leq A_1. \tag{3.91}$$

Now, (3.90) and (3.91) imply

$$\begin{aligned} A_1 A_{2,1} + \frac{M_1}{m_1} A_1 A_{2,0}^1 + \frac{m_1}{M_1} A_1 A_{2,0}^2 \geq \\ \geq \frac{M_1}{m_1} + \frac{M_2 M_1}{m_2 m_1} + \frac{M_2}{m_2} + 1 + \frac{M_2}{m_2} + \frac{M_2 m_1}{M_1 m_2} + \frac{m_2}{M_2} + 1 + \frac{m_1}{M_1}\,. \end{aligned} \tag{3.92}$$

If we take (3.11), (3.13), (3.89) and the relation

$$d_1 + d_2 \geq 2\sqrt{d_1 d_2} \quad \text{for} \quad d_1,\, d_2 \geq 0 \tag{3.93}$$

into account, it is easy to verify that

$$\begin{aligned} \frac{M_1}{m_1}(1 - A_1 A_{2,0}^1) + \frac{m_1}{M_1}(1 - A_1 A_{2,0}^2) \geq 2\sqrt{(1 - A_1 A_{2,0}^1)(1 - A_1 A_{2,0}^2)} \geq \\ \geq 2\sqrt{1 - A_1(A_{2,0}^1 + A_{2,0}^2)} \geq 2\sqrt{1 - A_1 A_{2,0}} \end{aligned} \tag{3.94}$$

and

$$\frac{M_2 M_1}{m_2 m_1} + \frac{M_2 m_1}{M_1 m_2} \geq 2\,\frac{M_2}{m_2}\,, \quad 4\,\frac{M_2}{m_2} + \frac{m_2}{M_2} \geq 4. \tag{3.95}$$

Using (3.94) and (3.95) in (3.92), we get

$$A_1 A_{2,1} \geq 6 + 2\sqrt{1 - A_1 A_{2,0}} \geq 4 + 4\sqrt{1 - A_1 A_{2,0}}\,,$$

which contradicts (3.12).

Case (b2): $\alpha_1 > \beta_1$. Integration of the first equation in (3.14) from a to β_1 and from β_1 to α_1, by virtue of (3.15), (3.3), (3.83) and the assumption $\ell_1 \in \mathcal{P}_{ab}$ yields

$$m_1 = -\int\limits_a^{\beta_1} \ell_1(x_2)(s)\,ds \le m_2 \int\limits_a^{\beta_1} \ell_1(1)(s)\,ds$$

and

$$M_1 + m_1 = \int\limits_{\beta_1}^{\alpha_1} \ell_1(x_2)(s)\,ds \le M_2 \int\limits_{\beta_1}^{\alpha_1} \ell_1(1)(s)\,ds.$$

By virtue of (3.88), the last two relations result in

$$\frac{m_1}{m_2} + \frac{M_1}{M_2} + \frac{m_1}{M_2} \le \int\limits_a^{\beta_1} \ell_1(1)(s)ds + \int\limits_{\beta_1}^{\alpha_1} \ell_1(1)(s)ds \le A_1\,. \tag{3.96}$$

Now, it follows from (3.90) and (3.96) that

$$\begin{gathered} A_1 A_{2,1} + \frac{M_1}{m_1}\, A_1 A_{2,0}^1 + \frac{m_1}{M_1}\, A_1 A_{2,0}^2 \ge \\ \ge \frac{M_2}{m_2} + \frac{M_1}{m_1} + 1 + \frac{M_2 m_1}{m_2 M_1} + 1 + \frac{m_1}{M_1} + \frac{m_1}{M_1} + \frac{m_2}{M_2} + \frac{m_2 m_1}{M_1 M_2}\,. \end{gathered} \tag{3.97}$$

In view of (3.93), it is clear that

$$\frac{M_2 m_1}{m_2 M_1} + \frac{m_2 m_1}{M_1 M_2} \ge 2\,\frac{m_1}{M_1}\,, \quad \frac{M_2}{m_2} + \frac{m_2}{M_2} \ge 2. \tag{3.98}$$

Using (3.11), (3.13), (3.89), (3.93) and (3.98) in (3.97), we get

$$\begin{aligned} A_1 A_{2,1} &\ge 4 + \frac{M_1}{m_1}\,(1 - A_1 A_{2,0}^1) + \frac{m_1}{M_1}\,(4 - A_1 A_{2,0}^2) \ge \\ &\ge 4 + 2\sqrt{(1 - A_1 A_{2,0}^1)(4 - A_1 A_{2,0}^2)} \ge \\ &\ge 4 + 2\sqrt{4 - A_1(4A_{2,0}^1 + A_{2,0}^2)} \ge 4 + 4\sqrt{1 - A_1(A_{2,0}^1 + A_{2,0}^2)} \ge \\ &\ge 4 + 4\sqrt{1 - A_1 A_{2,0}}\,, \end{aligned}$$

which contradicts (3.12).

The contradictions obtained prove that the homogeneous problem (3.14), (3.15) has only the trivial solution. $\square$

Proof of Theorem 3.4. According to Lemma 3.1, to prove the theorem it is sufficient to show that the homogeneous problem (3.14), (3.15) has only the trivial solution. In view of Remark 3.6, we can assume without loss of generality that $k = 1$ and $m = 0$. Let (x_1, x_2) be a solution to the problem (3.14), (3.15).

By virtue of the assumption $(\ell_1, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a)$ and Remark 3.3, the problem

$$u_1'(t) = \ell_1(u_2)(t), \quad u_2'(t) = g_0(u_1)(t) + g_1(|x_1|)(t), \tag{3.99}$$

$$u_1(a) = 0, \quad u_2(a) = 0 \tag{3.100}$$

has a unique solution (u_1, u_2). Combining (3.14), (3.15), (3.17), (3.99) and (3.100), we get

$$\begin{aligned}
&u_1'(t) + x_1'(t) = \ell_1(u_2 + x_2)(t) \ \text{ for a.e. } \ t \in [a,b], \\
&u_2'(t) + x_2'(t) = g_0(u_1 + x_1)(t) + \ell_2(x_1)(t) - g_0(x_1) + g_1(|x_1|)(t) \geq \\
&\qquad\qquad \geq g_0(u_1 + x_1)(t) \ \text{ for a.e. } \ t \in [a,b], \\
&u_1(a) + x_1(a) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&u_1'(t) - x_1'(t) = \ell_1(u_2 - x_2)(t) \ \text{ for a.e. } \ t \in [a,b], \\
&u_2'(t) - x_2'(t) = g_0(u_1 - x_1)(t) - \ell_2(x_1)(t) + g_0(x_1) + g_1(|x_1|)(t) \geq \\
&\qquad\qquad \geq g_0(u_1 - x_1)(t) \ \text{ for a.e. } \ t \in [a,b], \\
&u_1(a) - x_1(a) = 0.
\end{aligned}$$

Consequently, the inclusion $(\ell_1, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a)$ implies

$$u_1(t) + x_1(t) \geq 0, \quad u_1(t) - x_1(t) \geq 0 \ \text{ for } \ t \in [a,b],$$

that is,

$$|x_1(t)| \leq u_1(t) \ \text{ for } \ t \in [a,b]. \tag{3.101}$$

Taking now the assumption $g_1 \in \mathcal{P}_{ab}$ into account, we get from (3.99) that

$$u_1'(t) = \ell_1(u_2)(t), \quad u_2'(t) \leq (g_0 + g_1)(u_1)(t) \ \text{ for a.e. } \ t \in [a,b]. \tag{3.102}$$

However, we also suppose that $(\ell_1, g_0 + g_1) \in \widehat{\mathcal{S}}^2_{ab}(a)$, and thus the relations (3.100) and (3.102) result in $u_1(t) \leq 0$ for $t \in [a,b]$. Therefore, (3.101) yields $x_1 \equiv 0$. Consequently, (3.14) and (3.15) imply $x_2 \equiv 0$, i.e., the homogeneous problem (3.14), (3.15) has only the trivial solution. □

Proof of Corollary 3.1. It is not difficult to verify that the assumptions of Theorem 3.4 are satisfied with $g_0 = -\frac{1}{2}\,\ell_{3-k,1-m}$ and $g_1 = \ell_{3-k,m} + \frac{1}{2}\,\ell_{3-k,1-m}$. □

Proof of Corollary 3.2. It is clear that (3.21), (3.22) is a particular case of (1.3), (1.4) with $a = 0$, $b = 1$, $q_1 \equiv 0$, $q_2 \equiv q$, $\ell_2 = \ell_{2,0} - \ell_{2,1}$, and ℓ_1, $\ell_{2,0}$, $\ell_{2,1}$ are defined by the formulae $\ell_1(z)(t) \stackrel{\text{def}}{=} z(t)$ and

$$\ell_{2,0}(z)(t) \stackrel{\text{def}}{=} \frac{d_1}{(1-t)^\nu} \int_0^t \frac{z(\tau(s))}{(1-s)^\nu}\, ds, \quad \ell_{2,1}(z)(t) \stackrel{\text{def}}{=} \frac{d_2}{(1-t)^\nu} \int_0^t \frac{x(\lambda s)}{(1-s)^\nu}\, ds$$

for a.e. $t \in [0,1]$ and all $z \in C([0,1];\mathbb{R})$. Obviously, ℓ_1, $\ell_{2,0}$, $\ell_{2,1} \in \mathcal{P}_{01}$ and the operators ℓ_1, $\ell_{2,1}$ are 0-Volterra ones.

Case (a): Since τ is a delay, the operator $\ell_{2,0}$ is a 0-Volterra one. Therefore, [24, Proposition 3.4] yields $(\ell_1, \ell_{2,0}+\ell_{2,1}) \in \widehat{\mathcal{S}}^2_{01}(0)$. On the other hand, for any $z \in C([0,1];\mathbb{R})$, we have

$$\ell_1(z)(t)\operatorname{sgn} z(t) = \ell_1(|z|)(t) \ \text{ for a.e. } \ t \in [0,1],$$
$$\ell_2(z)(t)\operatorname{sgn} z(t) \le \ell_{2,0}(|z|)(t) + \ell_{2,1}(|z|)(t) \ \text{ for a.e. } \ t \in [0,1],$$

and thus, the validity of the corollary follows from Lemmas 3.1 and 3.3.

Case (b): Using (3.23), we get

$$\int_a^b \ell_1\big(\chi(\ell_{2,0}(1))\big)(s)\,ds < 1, \quad \int_a^b \ell_1\big(\chi(\ell_{2,1}(1))\big)(s)\,ds \le 2,$$

where χ is defined by (3.6). Therefore, [26, Corollaries 3.2 and 3.3] guarantee

$$(\ell_1, \ell_{2,0}) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \Big(\ell_1, -\frac{1}{2}\,\ell_{2,1}\Big) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

Consequently, the assumptions of Corollary 3.1 with $k = 1$ and $m = 0$ are satisfied. □

In order to prove Corollary 3.3, we need the following lemma.

Lemma 3.5. *Let the numbers $\alpha_i \in \mathbb{R}_+$ $(i = 1,\dots,4)$, at least one of which is positive, $\varrho_b > \varrho_a > 0$, and $\lambda \in [0,1[$ be such that*

$$\int_{\varrho_a}^{\varrho_b} \frac{ds}{\alpha_1 + (\alpha_2+\alpha_3)s + \alpha_4 s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}\,. \tag{3.103}$$

Then there exist functions β_1, $\beta_2 \in \widetilde{C}([a,b];\mathbb{R})$ satisfying (3.1),

$$\beta_1'(t) = \frac{\alpha_3}{(b-t)^\lambda}\,\beta_1(t) + \frac{\alpha_1}{(b-t)^\lambda}\,\beta_2(t) \ \ \textit{for a.e. } \ t \in [a,b], \tag{3.104}$$

$$\beta_2'(t) = -\frac{\alpha_4}{(b-t)^\lambda}\,\beta_1(t) - \frac{\alpha_2}{(b-t)^\lambda}\,\beta_2(t) \ \ \textit{for a.e. } \ t \in [a,b], \tag{3.105}$$

$$\beta_1(a) = \varrho_a, \quad \beta_1(b) = \varrho_b\beta_2(b), \quad \beta_2(a) = 1, \tag{3.106}$$

and

$$\beta_2(b) \ge \exp\bigg(-\int_a^b \frac{\alpha_2 + \alpha_4\varrho_b}{(b-s)^\lambda}\,ds\bigg). \tag{3.107}$$

Proof. Define the function $\varrho : [a,b] \to \mathbb{R}_+$ by setting

$$\int_{\varrho(t)}^{\varrho_b} \frac{ds}{\alpha_1 + (\alpha_2+\alpha_3)s + \alpha_4 s^2} = \frac{(b-t)^{1-\lambda}}{1-\lambda} \ \ \text{for } \ t \in [a,b].$$

In view of (3.103), we get

$$\varrho(t) > 0 \ \text{ for } \ t \in [a,b], \quad \varrho(a) = \varrho_a, \quad \varrho(b) = \varrho_b, \tag{3.108}$$

and

$$\varrho'(t) = \frac{\alpha_1 + (\alpha_2 + \alpha_3)\varrho(t) + \alpha_4 \varrho^2(t)}{(b-t)^\lambda} \quad \text{for} \quad t \in [a, b[\,. \tag{3.109}$$

Put

$$\beta_2(t) = \exp\bigg(- \int_a^t \frac{\alpha_2 + \alpha_4 \varrho(s)}{(b-s)^\lambda}\, ds \bigg), \quad \beta_1(t) = \varrho(t)\beta_2(t) \quad \text{for} \quad t \in [a, b].$$

It is not difficult to verify that $\beta_1,\, \beta_2 \in \widetilde{C}([a, b]; \mathbb{R})$ and the conditions (3.104) and (3.105) are satisfied. Moreover, by virtue of (3.108) and (3.109), it is clear that (3.1), (3.106) and (3.107) hold as well. □

Proof of Corollary 3.3. Let $\ell_i, \ell_{i,j} \in \mathcal{L}_{ab}$ ($i = 1, 2;\ j = 0, 1$) be defined by the formulae

$$\ell_i(z)(t) \stackrel{\text{def}}{=} h_i(t) z(\tau_i(t)) \quad \text{for a.e.} \quad t \in [a, b] \quad \text{and all} \quad z \in C([a, b]; \mathbb{R}) \tag{3.110}$$

and

$$\ell_{i,j}(z)(t) \stackrel{\text{def}}{=} h_{i,j}(t) z(\tau_i(t)) \quad \text{for a.e.} \quad t \in [a, b] \quad \text{and all} \quad z \in C([a, b]; \mathbb{R}).$$

It is clear that $\ell_{i,j} \in \mathcal{P}_{ab}$ ($i = 1, 2;\ j = 0, 1$) and $\ell_i = \ell_{i,0} - \ell_{i,1}$ for $i = 1, 2$. By virtue of (3.26), there exist $\varrho_a,\, \varrho_b \in \mathbb{R}_+$ such that $\omega_1 < \varrho_a < \varrho_b < \omega_2$ and the equality (3.103) is fulfilled. According to Lemma 3.5, we can find $\beta_1,\, \beta_2 \in \widetilde{C}([a, b]; \mathbb{R})$ satisfying (3.1) and (3.104)–(3.107). Using these conditions, we get

$$\beta_1'(t) \ge 0, \quad \beta_2'(t) \le 0 \quad \text{for a.e.} \quad t \in [a, b]. \tag{3.111}$$

Put

$$A_i = \big\{ t \in [a, b] :\ h_i(t) \neq 0 \big\} \quad \text{for} \quad i = 1, 2. \tag{3.112}$$

If we take (3.1), (3.104), (3.105) and (3.111) into account, by direct calculation we obtain

$$\begin{aligned}
\beta_2(\tau_k(t)) = \beta_2(t) - \int_{\tau_k(t)}^t \beta_2'(s)\, ds = \\
= \beta_2(t) + \int_{\tau_k(t)}^t \frac{\alpha_4}{(b-s)^\lambda}\, \beta_1(s)\, ds + \int_{\tau_k(t)}^t \frac{\alpha_2}{(b-s)^\lambda}\, \beta_2(s)\, ds \le \\
\le \beta_2(t) + \beta_1(t) \int_{\tau_k(t)}^t \frac{\alpha_4}{(b-s)^\lambda}\, ds + \beta_2(\tau_k(t)) \int_{\tau_k(t)}^t \frac{\alpha_2}{(b-s)^\lambda}\, ds
\end{aligned}$$

and

$$-\beta_1(\tau_{3-k}(t)) = -\beta_1(t) + \int_{\tau_{3-k}(t)}^{t} \beta_1'(s)\,ds =$$

$$= -\beta_1(t) + \int_{\tau_{3-k}(t)}^{t} \frac{\alpha_3}{(b-s)^\lambda}\,\beta_1(s)\,ds + \int_{\tau_{3-k}(t)}^{t} \frac{\alpha_1}{(b-s)^\lambda}\,\beta_2(s)\,ds \geq$$

$$\geq -\beta_1(t) + \beta_1(\tau_{3-k}(t)) \int_{\tau_{3-k}(t)}^{t} \frac{\alpha_3}{(b-s)^\lambda}\,ds + \beta_2(t) \int_{\tau_{3-k}(t)}^{t} \frac{\alpha_1}{(b-s)^\lambda}\,ds$$

for a.e. $t \in [a,b]$. Therefore, by virtue of (3.1), (3.27)–(3.30), (3.104) and (3.105), we get from the last two relations

$$|h_k(t)|\beta_2(\tau_k(t)) \leq \frac{|h_k(t)| \int\limits_{\tau_k(t)}^{t} \frac{\alpha_4}{(b-s)^\lambda}\,ds}{1 + \int\limits_{t}^{\tau_k(t)} \frac{\alpha_2}{(b-s)^\lambda}\,ds}\,\beta_1(t) + \frac{|h_k(t)|}{1 + \int\limits_{t}^{\tau_k(t)} \frac{\alpha_2}{(b-s)^\lambda}\,ds}\,\beta_2(t) \leq$$

$$\leq \frac{\alpha_3}{(b-t)^\lambda}\,\beta_1(t) + \frac{\alpha_1}{(b-t)^\lambda}\,\beta_2(t) = \beta_1'(t) \quad \text{for a.e. } t \in A_k$$

and

$$-|h_{3-k}(t)|\beta_1(\tau_{3-k}(t)) \geq$$

$$\geq -\frac{|h_{3-k}(t)|}{1 + \int\limits_{\tau_{3-k}(t)}^{t} \frac{\alpha_3}{(b-s)^\lambda}\,ds}\,\beta_1(t) - \frac{|h_{3-k}(t)| \int\limits_{t}^{\tau_{3-k}(t)} \frac{\alpha_1}{(b-s)^\lambda}\,ds}{1 + \int\limits_{\tau_{3-k}(t)}^{t} \frac{\alpha_3}{(b-s)^\lambda}\,ds}\,\beta_2(t) \geq$$

$$\geq -\frac{\alpha_4}{(b-t)^\lambda}\,\beta_1(t) - \frac{\alpha_2}{(b-t)^\lambda}\,\beta_2(t) = \beta_2'(t) \quad \text{for a.e. } t \in A_{3-k},$$

which, together with (3.111), guarantees

$$\beta_1'(t) \geq |h_k(t)|\beta_2(\tau_k(t)), \quad \beta_2'(t) \leq -|h_{3-k}(t)|\beta_1(\tau_{3-k}(t)) \quad \text{for a.e. } t \in [a,b].$$

Consequently, the functions β_1, β_2 satisfy (3.2) and (3.3).

On the other hand, in view of (3.106) and (3.111) we get

$$\int_a^b h_{k,1-m}(s)\beta_2(\tau_k(s))\,ds \leq \beta_2(a)\|h_{k,1-m}\|_L = \omega_1 < \varrho_a = \beta_1(a)$$

and thus the inequality (3.4) holds. At last we show that the inequality (3.5) is satisfied in all cases (i)–(iv). Note that, in view of (3.106) and (3.111), we have

$$\Phi := \int_a^b h_{3-k,m}(s)\beta_1(\tau_{3-k}(s))\,ds+$$

$$+\int_a^b h_{3-k,1-m}(s)\Bigg(\int_a^{\tau_{3-k}(s)} h_{k,1-m}(\xi)\beta_2(\tau_k(\xi))\,d\xi\Bigg)\,ds \le$$

$$\le \varrho_b\beta_2(b)\|h_{3-k,m}\|_L + \int_a^b h_{3-k,1-m}(s)\Bigg(\int_a^{\tau_{3-k}(s)} h_{k,1-m}(\xi)\,d\xi\Bigg)\,ds. \quad (3.113)$$

Case (i): $h_{k,1-m} \equiv 0$ *and* $h_{3-k,m} \equiv 0$. In this case, we have $\Phi = 0$ and thus the inequality (3.5) trivially holds as a strict one.

Case (ii): $h_{k,1-m} \equiv 0$ *and* $h_{3-k,m} \not\equiv 0$. The relation (3.113) yields

$$\Phi \le \varrho_b\beta_2(b)\|h_{3-k,m}\|_L < \omega_2\|h_{3-k,m}\|_L\beta_2(b) = \beta_2(b),$$

i.e., (3.5) is true.

Case (iii): $h_{k,1-m} \not\equiv 0$ *and* $h_{3-k,m} \not\equiv 0$. In view of (3.31) and (3.107), the relation (3.113) implies

$$\Phi \le \varrho_b\beta_2(b)\|h_{3-k,m}\|_L + \big(1-\omega_2\|h_{3-k,m}\|_L\big)\exp\Bigg(-\int_a^b \frac{\alpha_2+\alpha_4\omega_2}{(b-s)^\lambda}\,ds\Bigg) \le$$

$$\le \varrho_b\beta_2(b)\|h_{3-k,m}\|_L + \big(1-\varrho_b\|h_{3-k,m}\|_L\big)\exp\Bigg(-\int_a^b \frac{\alpha_2+\alpha_4\varrho_b}{(b-s)^\lambda}\,ds\Bigg) \le$$

$$\le \varrho_b\beta_2(b)\|h_{3-k,m}\|_L + \big(1-\varrho_b\|h_{3-k,m}\|_L\big)\beta_2(b) = \beta_2(b),$$

i.e., (3.5) is satisfied.

Case (iv): $h_{k,1-m} \not\equiv 0$ *and* $h_{3-k,m} \equiv 0$. Using (3.32) and (3.107), we get from (3.113) the relation

$$\Phi < \exp\Bigg(-\int_a^b \frac{\alpha_2+\alpha_4\omega_2}{(b-s)^\lambda}\,ds\Bigg) \le \exp\Bigg(-\int_a^b \frac{\alpha_2+\alpha_4\varrho_b}{(b-s)^\lambda}\,ds\Bigg) \le \beta_2(b),$$

and thus the inequality (3.5) holds as a strict one.

Consequently, the assumptions of Theorem 3.1 are satisfied. □

Proof of Corollary 3.4. Let $\ell_i \in \mathcal{L}_{ab}$ $(i = 1, 2)$ be defined by (3.110). It is clear that $(-1)^m\ell_k$, $(-1)^{1-m}\ell_{3-k} \in \mathcal{P}_{ab}$, and the operator ℓ_{3-k} is an a-Volterra one. By virtue of (3.35), there exist ϱ_a, $\varrho_b \in]0,+\infty[$ such that

$\varrho_a < \varrho_b$ and

$$\int\limits_{\varrho_a}^{\varrho_b} \frac{ds}{\alpha_1 + \alpha_2 s + \alpha_3 s^2} = \frac{(b-a)^{1-\lambda}}{1-\lambda}\,.$$

Therefore, according to Lemma 3.5, we can find functions $\omega_1, \omega_2 \in \widetilde{C}([a,b];\mathbb{R})$ satisfying

$$\omega_1'(t) = \frac{\alpha_2}{(b-t)^\lambda}\,\omega_1(t) + \frac{\alpha_1}{(b-t)^\lambda}\,\omega_2(t) \ \text{ for a.e. } \ t \in [a,b], \tag{3.114}$$

$$\omega_2'(t) = -\frac{\alpha_3}{(b-t)^\lambda}\,\omega_1(t) \ \text{ for a.e. } \ t \in [a,b], \tag{3.115}$$

and

$$\omega_i(t) > 0 \ \text{ for } \ t \in [a,b], \ \ i = 1,2.$$

Put

$$\gamma_1(t) = \frac{\omega_1(t)}{(b-t)^\nu} \ \text{ for } \ t \in [a,b[\,, \quad \gamma_2(t) = \omega_2(t) \ \text{ for } \ t \in [a,b].$$

It is easy to see that $\gamma_1 \in \widetilde{C}_{loc}([a,b[\,;\mathbb{R})$, $\gamma_2 \in \widetilde{C}([a,b];\mathbb{R})$, and the condition (3.7) holds. Using (3.114) and (3.115), we get

$$\gamma_1'(t) = \Big(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^\lambda}\Big)\gamma_1(t) + \\ + \frac{\alpha_1}{(b-t)^{\lambda+\nu}}\,\gamma_2(t) \ \text{ for a.e. } \ t \in [a,b], \tag{3.116}$$

$$\gamma_2'(t) = -\frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,\gamma_1(t) \ \text{ for a.e. } \ t \in [a,b]. \tag{3.117}$$

Consequently, it is clear that γ_2' is continuous and nonincreasing on $[a,b[$ and

$$\gamma_1'(t) \geq 0, \quad \gamma_2'(t) \leq 0 \ \text{ for a.e. } \ t \in [a,b]. \tag{3.118}$$

Define the set A_{3-k} by (3.112). If we take (3.7), (3.34) and (3.116)–(3.118) into account, by direct calculation we obtain

$$\gamma_2(\tau_k(t)) = \gamma_2(t) + \int\limits_t^{\tau_k(t)} \gamma_2'(s)\,ds \leq \gamma_2(t) + \gamma_2'(t)(\tau_k(t) - t) = \\ = \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,(t - \tau_k(t))\gamma_1(t) + \gamma_2(t) \ \text{ for a.e. } \ t \in [a,b]$$

and

$$-\gamma_1(\tau_{3-k}(t)) = -\gamma_1(t) + \int\limits_{\tau_{3-k}(t)}^t \gamma_1'(s)\,ds =$$

$$= -\gamma_1(t) + \int\limits_{\tau_{3-k}(t)}^t \Big[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\Big]\gamma_1(s)\,ds + \int\limits_{\tau_{3-k}(t)}^t \frac{\alpha_1}{(b-s)^{\lambda+\nu}}\,\gamma_2(s)\,ds \geq$$

$$\geq -\gamma_1(t) + \gamma_1(\tau_{3-k}(t)) \int\limits_{\tau_{3-k}(t)}^{t} \left[\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\right] ds \quad \text{for a.e. } t \in A_{3-k}\,.$$

Therefore, by virtue of (3.7), (3.33), (3.34), (3.36)–(3.38), (3.116) and (3.117), we get from the last relations

$$(-1)^m h_k(t)\gamma_2(\tau_k(t)) \leq \frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,|h_k(t)|(t-\tau_k(t))\gamma_1(t) + |h_k(t)|\gamma_2(t) \leq$$

$$\leq \left(\frac{\nu}{b-t} + \frac{\alpha_2}{(b-t)^\lambda}\right)\gamma_1(t) + \frac{\alpha_1}{(b-t)^{\lambda+\nu}}\,\gamma_2(t) = \gamma_1'(t) \quad \text{for a.e. } t \in [a,b]$$

and

$$(-1)^m h_{3-k}(t)\gamma_1(\tau_{3-k}(t)) \geq -\frac{|h_{3-k}(t)|}{1 + \int\limits_{\tau_{3-k}(t)}^{t} \left(\frac{\nu}{b-s} + \frac{\alpha_2}{(b-s)^\lambda}\right) ds}\,\gamma_1(t) \geq$$

$$\geq -\frac{\alpha_3}{(b-t)^{\lambda-\nu}}\,\gamma_1(t) = \gamma_2'(t) \quad \text{for a.e. } t \in A_{3-k},$$

which, together with (3.118), guarantees

$$\gamma_1'(t) \geq (-1)^m h_k(t)\gamma_2(\tau_k(t)), \quad \gamma_2'(t) \leq (-1)^m h_{3-k}(t)\gamma_1(\tau_{3-k}(t))$$

for a.e. $t \in [a,b]$, and thus γ_1, γ_2 satisfy (3.8) and (3.9).

Consequently, the assumptions of Theorem 3.2 are fulfilled. □

Proof of Corollary 3.5. The validity of the corollary follows from Corollary 3.4 with $\alpha_1 = \alpha$, $\alpha_2 = 0$, $\alpha_3 = \beta$, and $k = 2$, $m = 1$ (resp. $k = 1$, $m = 0$). □

Proof of Corollary 3.6. Let the operator $\ell_1 \in \mathcal{L}_{ab}$ be defined by the formula

$$\ell_1(z)(t) \overset{\text{def}}{=} f(t)z(\mu(t)) \quad \text{for a.e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}), \tag{3.119}$$

and let $\ell_2 = \ell_{2,0} - \ell_{2,1}$, where

$$\ell_{2,i}(z)(t) \overset{\text{def}}{=} h_i(t)z(\tau_i(t)) \quad \text{for a.e. } t \in [a,b], \text{ all } z \in C([a,b];\mathbb{R}), \; i = 0,1. \tag{3.120}$$

Obviously, ℓ_1, $\ell_{2,0}$, $\ell_{2,1} \in \mathcal{P}_{ab}$, $\ell_1(1) \equiv f$, $\ell_{2,0}(1) \equiv h_0$, and $\ell_{2,1}(1) \equiv h_1$. Therefore, in view of (3.40) and (3.41), it is clear that the validity of the corollary follows immediately from Theorem 3.3 with $k = 2$ and $m = 0$. □

Proof of Corollary 3.7. Let the operator $\ell_1 \in \mathcal{L}_{ab}$ be defined by the formula (3.119) and let $\ell_2 = \ell_{2,0} - \ell_{2,1}$, where $\ell_{2,0}$, $\ell_{2,1}$ are given by (3.120). Obviously, ℓ_1, $\ell_{2,0}$, $\ell_{2,1} \in \mathcal{P}_{ab}$. By virtue of the condition (a) (resp. (b), resp. (c)) of the corollary, it follows from Proposition 5.3 (resp. Proposition 5.4, resp. Proposition 5.5) that

$$(\ell_1, \ell_{2,0}) \in \widehat{\mathcal{S}}_{ab}^2(a).$$

On the other hand, in view of the condition (A) (resp. (B)) with $\gamma^* = 2$, Proposition 5.6 (resp. Proposition 5.7) yields

$$\Big(\ell_1, -\frac{1}{2}\,\ell_{2,1}\Big) \in \widehat{\mathcal{S}}_{ab}^{2}(a).$$

Consequently, the assumptions of Corollary 3.1 with $k = 1$ and $m = 0$ are satisfied. $\square$

4. Nonlinear Problem

In this section, we establish new efficient conditions sufficient for the solvability as well as unique solvability of the nonlinear problem (1.1), (1.2) under one-sided restrictions imposed on the right-hand side of the system considered. The main results are finally applied to the case where (1.1) is a differential system with argument deviations.

Throughout this section, the following assumptions are used:

(H_1) $F_1,\, F_2 : C([a,b];\mathbb{R}) \times C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are continuous operators such that the relation

$$\sup\Big\{\big|F_i(u_1,u_2)(\cdot)\big| :\ u_1,\, u_2 \in C([a,b];\mathbb{R}),\ \|u_1\|_C + \|u_2\|_C \le r\Big\} \in \\ \in L([a,b];\mathbb{R}_+)$$

is satisfied for every $r > 0$ and $i = 1, 2$.

(H_2) $\varphi_1,\, \varphi_2 : C([a,b];\mathbb{R}) \times C([a,b];\mathbb{R}) \to \mathbb{R}$ are continuous functionals such that the condition

$$\sup\Big\{\big|\varphi_i(u_1,u_2)\big| :\ u_1, u_2 \in C([a,b];\mathbb{R}),\ \|u_1\|_C + \|u_2\|_C \le r\Big\} < +\infty$$

holds for every $r > 0$ and $i = 1, 2$.

4.1. **Main Results.**

We first formulate the main results. Their proofs are given later in Subsection 4.3.

Theorem 4.1. *Let $k \in \{1,2\}$, the assumptions (H_1) and (H_2) be satisfied, and let there exist $p, g_0, g_1 \in \mathcal{P}_{ab}$ such that, for any $u_1, u_2 \in C([a,b];\mathbb{R})$, the inequalities*

$$\varphi_i(u_1,u_2)\operatorname{sgn} u_i(a) \le \eta_i\big(\|u_1\|_C + \|u_2\|_C\big) \ \ \textit{for}\ \ i = 1,2, \tag{4.1}$$

$$\big[F_k(u_1,u_2)(t) - p(u_{3-k})(t)\big]\operatorname{sgn} u_k(t) \le \\ \le \omega_k\big(t, \|u_1\|_C + \|u_2\|_C\big) \ \ \textit{for a.e.}\ \ t \in [a,b], \tag{4.2}$$

and

$$\big[F_{3-k}(u_1,u_2)(t) - g_0(u_k)(t) + g_1(u_k)(t)\big]\operatorname{sgn} u_{3-k}(t) \le \\ \le \omega_{3-k}\big(t, \|u_1\|_C + \|u_2\|_C\big) \ \ \textit{for a.e.}\ \ t \in [a,b] \tag{4.3}$$

are fulfilled, where $\omega_1,\, \omega_2 \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ and $\eta_1,\, \eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy

$$\lim_{r\to+\infty} \frac{1}{r}\left(\eta_i(r) + \int_a^b \omega_i(s,r)\,ds\right) = 0 \ \ \textit{for}\ \ i = 1,2. \tag{4.4}$$

If, moreover,

$$PG_0 < 1, \quad PG_1 < 4\sqrt{1 - PG_0}\,, \tag{4.5}$$

where

$$P = \int_a^b p(1)(s)\,ds, \quad G_i = \int_a^b g_i(1)(s)\,ds \ \text{ for } \ i = 0, 1, \tag{4.6}$$

then the problem (1.1), (1.2) *has at least one solution.*

Remark 4.1. The first strict inequality in (4.5) cannot be replaced by the nonstrict one (see Example 6.3). Furthermore, the second strict inequality in (4.5) cannot be replaced by the nonstrict one provided $G_0 = 0$ (see Example 6.4).

Using the weak theorem on differential inequalities, we can prove

Theorem 4.2. *Let $k \in \{1, 2\}$, the assumptions (H_1) and (H_2) be satisfied, and let there exist $p, g_0, g_1 \in \mathcal{P}_{ab}$ such that for any u_1, $u_2 \in C([a, b]; \mathbb{R})$ the inequalities* (4.1), (4.2) *and*

$$\begin{gathered}\big[F_{3-k}(u_1, u_2)(t) + g_1(u_k)(t)\big] \operatorname{sgn} u_{3-k}(t) \leq \\ \leq g_0(|u_k|)(t) + \omega_{3-k}\big(t, \|u_1\|_C + \|u_2\|_C\big) \ \text{ for a.e. } \ t \in [a, b]\end{gathered} \tag{4.7}$$

are fulfilled, where ω_1, $\omega_2 \in K([a, b] \times \mathbb{R}_+; \mathbb{R}_+)$ and η_1, $\eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (4.4). *If, moreover,*

$$(p, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad (p, -g_1) \in \widehat{\mathcal{S}}^2_{ab}(a), \tag{4.8}$$

then the problem (1.1), (1.2) *has at least one solution.*

Remark 4.2. The assumption (4.8) in the previous theorem can be replaced neither by the assumption

$$\big((1 - \varepsilon_1)p, (1 - \varepsilon_2)g_0\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad (p, -g_1) \in \widehat{\mathcal{S}}^2_{ab}(a) \tag{4.9}$$

nor by the assumption

$$(p, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a), \quad \big((1 - \varepsilon_1)p, -(1 - \varepsilon_2)g_1\big) \in \widehat{\mathcal{S}}^2_{ab}(a), \tag{4.10}$$

no matter how small ε_1, $\varepsilon_2 \in [0, 1[$ with $\varepsilon_1 + \varepsilon_2 > 0$ are (see Examples 6.5 and 6.6).

Example 4.1. On the interval $[0, \pi/4]$, we consider the problem

$$\begin{aligned} x_1'(t) &= d_1 \sin t \int_0^{t/2} s\, x_2(s/2)\,ds - e^{x_1(t/2)x_2(\lambda t)} x_1(t) + q_1(t), \\ x_2'(t) &= d_2 \cos(2t) \int_0^t \cos(2s)\big(x_1(\tau(s)) - x_1(\lambda s)\big)\,ds + \\ &\quad + q_2(t) \operatorname{arctg}(x_2(t)), \end{aligned} \tag{4.11}$$

$$x_1(0) = c_1 \operatorname{arctg}(x_2(t_0)), \quad x_2(0) = -e^{\int_0^{\pi/4} x_1(s/2)x_2(\lambda s)\,ds} x_2(0) + c_2, \tag{4.12}$$

where $d_1,\ d_2 \in \mathbb{R}_+$, $\lambda \in [0,1]$, $q_1,\ q_2 \in L([0,\pi/4];\mathbb{R})$, $\tau : [0,\pi/4] \to [0,\pi/4]$ is a measurable function, $t_0 \in [0,\pi/4]$, and $c_1,\ c_2 \in \mathbb{R}$.

It is clear that (4.11), (4.12) is a particular case of (1.1), (1.2) in which $a = 0$, $b = \pi/4$, F_1, F_2 and φ_1, φ_2 are given by the formulae

$$F_1(z_1,z_2)(t) \stackrel{\text{def}}{=} d_1 \sin t \int_0^{t/2} s\, z_2(s/2)\, ds - e^{z_1(t/2)z_2(\lambda t)} z_1(t) + q_1(t),$$

$$F_2(z_1,z_2)(t) \stackrel{\text{def}}{=} d_2 \cos(2t)\int_0^t \cos(2s)\big(z_1(\tau(s)) - z_1(\lambda s)\big)\, ds + q_2(t)\operatorname{arctg}(z_2(t))$$

for a.e. $t \in [0,\pi/4]$ and all $z_1,\ z_2 \in C([0,\pi/4];\mathbb{R})$, and

$$\varphi_1(z_1,z_2) \stackrel{\text{def}}{=} c_1 \operatorname{arctg}(z_2(t_0)), \quad \varphi_2(z_1,z_2) \stackrel{\text{def}}{=} -e^{\int_0^{\pi/4} z_1(s/2)z_2(\lambda s)\, ds} z_2(0) + c_2$$

for $z_1,\ z_2 \in C([0,\pi/4];\mathbb{R})$, respectively.

Let p, g_0, and g_1 be defined by the formulae

$$p(z)(t) \stackrel{\text{def}}{=} d_1 \sin t \int_0^{t/2} s\, z(s/2)\, ds,$$

$$g_0(z)(t) \stackrel{\text{def}}{=} d_2 \cos(2t) \int_0^t \cos(2s) z(\tau(s))\, ds,$$

$$g_1(z)(t) \stackrel{\text{def}}{=} d_2 \cos(2t) \int_0^t \cos(2s) z(\lambda s)\, ds$$

for a.e. $t \in [0,\pi/4]$ and all $z \in C([0,\pi/4];\mathbb{R})$. It is clear that $p,\ g_0,\ g_1 \in \mathcal{P}_{0\frac{\pi}{4}}$ and the operators p, g_1 are 0-Volterra ones.

(a) Suppose that $\tau(t) \le t$ for a.e. $t \in [0,\pi/4]$. Then the operator g_0 is a 0-Volterra one and thus, according to [24, Prop. 3.4] and [25, Theorem 4.2], the problem (4.11), (4.12) has at least one solution.

(b) Assume that d_1, d_2 satisfy

$$d_1 d_2 < \frac{2^{12}}{4\pi(1+2\sqrt{2}) - \pi^2(1+\sqrt{2}) - 24}\,.$$

Then [26, Corollaries 3.2 and 3.3] imply the validity of the condition (4.8). Moreover, for any $u_1,\ u_2 \in C([0,\pi/4];\mathbb{R})$ the inequalities (4.1), (4.2) and (4.7) are fulfilled, where $\eta_1 \equiv |c_1|\pi/2$, $\eta_2 \equiv |c_2|$, $\omega_1 \equiv |q_1|$, and $\omega_2 \equiv |q_2|\pi/2$. Consequently, according to Theorem 4.2, the problem (4.11), (4.12) has at least one solution.

Now we establish statements concerning the unique solvability of the problem (1.1), (1.2).

Theorem 4.3. *Let $k \in \{1,2\}$, the assumptions (H_1) and (H_2) be satisfied, and let there exist $p, g_0, g_1 \in \mathcal{P}_{ab}$ such that for any u_1, u_2, v_1, $v_2 \in C([a,b];\mathbb{R})$ the inequalities*

$$\big[\varphi_i(u_1,u_2)-\varphi_i(v_1,v_2)\big]\operatorname{sgn}(u_i(a)-v_i(a)) \le 0 \quad \textit{for } i=1,2, \tag{4.13}$$

$$\begin{aligned}\Big[F_k(u_1,u_2)(t)-F_k(v_1,v_2)(t)-p(u_{3-k}-v_{3-k})(t)\Big]\times\\ \times\operatorname{sgn}(u_k(t)-v_k(t)) \le 0 \quad \textit{for a.e. } t\in[a,b],\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\Big[F_{3-k}(u_1,u_2)(t)-F_{3-k}(v_1,v_2)(t)-g_0(u_k-v_k)(t)+g_1(u_k-v_k)(t)\Big]\times\\ \operatorname{sgn}\big(u_{3-k}(t)-v_{3-k}(t)\big) \le 0 \quad \textit{for a.e. } t\in[a,b]\end{aligned}$$

are fulfilled. If, moreover, the condition (4.5) holds, where P, G_0, G_1 are defined by (4.6), then the problem (1.1), (1.2) has a unique solution.

Theorem 4.4. *Let $k \in \{1,2\}$, the assumptions (H_1) and (H_2) be satisfied, and let there exist $p, g_0, g_1 \in \mathcal{P}_{ab}$ such that for any $u_1, u_2, v_1, v_2 \in C([a,b];\mathbb{R})$ the inequalities (4.13), (4.14) and*

$$\begin{aligned}\Big[F_{3-k}(u_1,u_2)(t)-F_{3-k}(v_1,v_2)(t)+g_1(u_k-v_k)(t)\Big]\times\\ \times\operatorname{sgn}\big(u_{3-k}(t)-v_{3-k}(t)\big) \le g_0(|u_k-v_k|)(t) \quad \textit{for a.e. } t\in[a,b]\end{aligned}$$

are fulfilled. If, moreover, the condition (4.8) holds, then the problem (1.1), (1.2) has a unique solution.

As an example, we consider the differential system

$$\begin{aligned}x_1'(t)&=f(t)x_2(\mu(t))+k_1\big(t,x_1(t),x_2(t),x_1(\zeta_{1,1}(t)),x_2(\zeta_{1,2}(t))\big),\\ x_2'(t)&=h_0(t)x_1(\tau_0(t))-h_1(t)x_1(\tau_1(t))+\\ &\quad+k_2\big(t,x_1(t),x_2(t),x_1(\zeta_{2,1}(t)),x_2(\zeta_{2,2}(t))\big),\end{aligned} \tag{4.15}$$

where f, $h_m \in L([a,b];\mathbb{R}_+)$, μ, τ_m, $\zeta_{i,j} : [a,b]\to[a,b]$ are measurable functions, and $k_i \in K([a,b]\times\mathbb{R}^4;\mathbb{R})$, $m=0,1$, $i,j=1,2$.

The next statement follows from Theorem 4.1.

Corollary 4.1. *Let the assumption (H_2) be satisfied, the condition (4.1) hold for arbitrary u_1, $u_2 \in C([a,b];\mathbb{R})$, and*

$$\begin{aligned}k_i(t,y_1,y_2,z_1,z_2)\operatorname{sgn} y_i \le \omega_i\big(t,|y_1|+|y_2|\big)\\ \textit{for a.e. } t\in[a,b] \textit{ and every } y_1,\ y_2,\ z_1,\ z_2\in\mathbb{R},\quad i=1,2,\end{aligned} \tag{4.16}$$

where η_1, $\eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$, and the nondecreasing in the second argument functions ω_1, $\omega_2 \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ satisfy (4.4). If, moreover, the condition (4.5) holds with P, G_0, G_1 defined by (3.42), then the problem (4.15), (1.2) has at least one solution.

In view of the results stated in [26], Theorem 4.2 yields

Corollary 4.2. *Let the assumption* (H_2) *be satisfied, the condition* (4.1) *hold for arbitrary* u_1, $u_2 \in C([a,b];\mathbb{R})$, *and let the condition* (4.16) *be fulfilled, where* η_1, $\eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$, *and the nondecreasing in the second argument functions* ω_1, $\omega_2 \in K([a,b] \times \mathbb{R}_+;\mathbb{R}_+)$ *fulfil* (4.4)*. Assume that* (3.43) *holds, the functions* f, μ, h_0, τ_0 *satisfy at least one of the conditions* (a)–(c) *of Corollary* 3.7*, whereas the functions* f, μ, h_1, τ_1 *fulfil the condition* (A) *or/and* (B) *of Corollary* 3.7 *with* $\gamma^* = 1$*. Then the problem* (4.15), (1.2) *has at least one solution.*

Analogously to Corollaries 4.1 and 4.2, one can derive from Theorems 4.3 and 4.4 conditions sufficient for the unique solvability of the problem (4.15), (1.2).

4.2. **Auxiliary Statements.** In order to prove Theorems 4.1–4.4, we need several auxiliary statements. We first formulate a result from [15] in a form suitable for us.

Lemma 4.1 ([15, Corollary 3])**.** *Let there exist operators* p, $g \in \widetilde{\mathcal{L}}_{ab}$ *and a number* $\varrho > 0$ *such that the homogeneous problem*

$$x_1'(t) = p(x_2)(t), \quad x_2'(t) = g(x_1)(t), \tag{4.17}$$

$$x_1(a) = 0, \quad x_2(a) = 0 \tag{4.18}$$

has only the trivial solution and for every $\delta \in]0,1[$ *arbitrary functions* $x_1, x_2 \in \widetilde{C}([a,b];\mathbb{R})$ *satisfying the relations*

$$x_1'(t) = p(x_2)(t) + \delta\big[F_1(x_1,x_2)(t) - p(x_2)(t)\big] \ \text{ for a.e. } \ t \in [a,b], \tag{4.19}$$

$$x_2'(t) = g(x_1)(t) + \delta\big[F_2(x_1,x_2)(t) - g(x_1)(t)\big] \ \text{ for a.e. } \ t \in [a,b] \tag{4.20}$$

and

$$x_1(a) = \delta\varphi_1(x_1,x_2), \quad x_2(a) = \delta\varphi_2(x_1,x_2) \tag{4.21}$$

admit the estimate

$$\|x_1\|_C + \|x_2\|_C \le \varrho. \tag{4.22}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Definition 4.1. We say that a triplet $(p,g,\ell) \in \widetilde{\mathcal{L}}_{ab} \times \widetilde{\mathcal{L}}_{ab} \times \mathcal{P}_{ab}$ belongs to the set $\mathcal{A}_{ab}$ if there exist ϱ_1, $\varrho_2 \in]0,+\infty[$ such that for arbitrary c_1^*, $c_2^* \in \mathbb{R}_+$ and q_1^*, $q_2^* \in L([a,b];\mathbb{R}_+)$ every pair of functions x_1, $x_2 \in \widetilde{C}([a,b];\mathbb{R})$ satisfying the conditions

$$|x_i(a)| \le c_i^* \ \text{ for } \ i = 1,2, \tag{4.23}$$

$$\big[x_1'(t) - p(x_2)(t)\big] \operatorname{sgn} x_1(t) \le q_1^*(t) \ \text{ for a.e. } \ t \in [a,b], \tag{4.24}$$

and

$$\big[x_2'(t) - g(x_1)(t)\big] \operatorname{sgn} x_2(t) \le \ell(|x_1|)(t) + q_2^*(t) \ \text{ for a.e. } \ t \in [a,b] \tag{4.25}$$

admits the estimate

$$\|x_1\|_C + \|x_2\|_C \le \varrho_1\big(c_1^* + \|q_1^*\|_L\big) + \varrho_2\big(c_2^* + \|q_2^*\|_L\big). \tag{4.26}$$

Lemma 4.2. *Let the assumptions* (H_1) *and* (H_2) *be satisfied and there exist a triplet* $(p, g, \ell) \in \mathcal{A}_{ab}$ *such that for any* u_1, $u_2 \in C([a,b];\mathbb{R})$ *the inequalities* (4.1),

$$\begin{aligned}\big[F_1(u_1,u_2)(t) - p(u_2)(t)\big] \operatorname{sgn} u_1(t) \le \\ \le \omega_1\big(t, \|u_1\|_C + \|u_2\|_C\big) \ \text{ for a.e. } \ t \in [a,b]\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}\big[F_2(u_1,u_2)(t) - g(u_1)(t)\big] \operatorname{sgn} u_2(t) \le \\ \le \ell(|u_1|)(t) + \omega_2\big(t, \|u_1\|_C + \|u_2\|_C\big) \ \text{ for a.e. } \ t \in [a,b]\end{aligned} \tag{4.28}$$

are fulfilled, where ω_1, $\omega_2 \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ *and* η_1, $\eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy* (4.4). *Then the problem* (1.1), (1.2) *has at least one solution.*

Proof. By virtue of the inclusions $(p, g, \ell) \in \mathcal{A}_{ab}$ and $\ell \in \mathcal{P}_{ab}$, the homogeneous problem (4.17), (4.18) has only the trivial solution.

Let ϱ_1, ϱ_2 be the numbers appearing in Definition 4.1. According to (4.4), there exists $\varrho > 0$ such that

$$\frac{\eta_i(r)}{r} + \frac{1}{r}\int_a^b \omega_i(s,r)\,ds < \frac{1}{2\varrho_i} \quad \text{for } \ r > \varrho, \ \ i = 1, 2. \tag{4.29}$$

Suppose that $x_1, x_2 \in \widetilde{C}([a,b];\mathbb{R})$ satisfy (4.19)–(4.21) with some $\delta \in]0,1[$. Then, using (4.1), (4.27) and (4.28), we obtain

$$\begin{aligned}|x_i(a)| = x_i(a)\operatorname{sgn} x_i(a) &= \delta\varphi_i(x_1,x_2)\operatorname{sgn} x_i(a) \le \\ &\le \eta_i\big(\|x_1\|_C + \|x_2\|_C\big) \ \text{ for } \ i = 1, 2, \\ \big[x_1'(t) - p(x_2)(t)\big]\operatorname{sgn} x_1(t) &= \delta\big[F_1(x_1,x_2)(t) - p(x_2)(t)\big]\operatorname{sgn} x_1(t) \le \\ &\le \omega_1\big(t, \|x_1\|_C + \|x_2\|_C\big) \ \text{ for a.e. } \ t \in [a,b],\end{aligned}$$

and

$$\begin{aligned}\big[x_2'(t) - g(x_1)(t)\big]\operatorname{sgn} x_2(t) = \delta\big[F_2(x_1,x_2)(t) - g(x_1)(t)\big]\operatorname{sgn} x_2(t) \le \\ \le \ell(|x_1|)(t) + \omega_2\big(t, \|x_1\|_C + \|x_2\|_C\big) \ \text{ for a.e. } \ t \in [a,b],\end{aligned}$$

i.e., the inequalities (4.23)–(4.25) are fulfilled, where $c_i^* = \eta_i(\|x_1\|_C + \|x_2\|_C)$ and $q_i^* \equiv \omega_i(\cdot, \|x_1\|_C + \|x_2\|_C)$ for $i = 1, 2$. Hence, by virtue of the assumption $(p, g, \ell) \in \mathcal{A}_{ab}$, we get

$$\|x_1\|_C + \|x_2\|_C \le \sum_{i=1}^{2} \varrho_i\left(\eta_i\big(\|x_1\|_C + \|x_2\|_C\big) + \int_a^b \omega_i\big(s, \|x_1\|_C + \|x_2\|_C\big)\,ds\right).$$

Consequently, in view of (4.29), the estimate (4.22) is satisfied.

Since ϱ depends neither on x_1, x_2 nor on δ, it follows from Lemma 4.1 that the problem (1.1), (1.2) has at least one solution. □

Lemma 4.3. *Let the assumptions* (H_1) *and* (H_2) *be satisfied and let there exist a triplet* $(p, g, \ell) \in \mathcal{A}_{ab}$ *such that for any* u_1, u_2, v_1, $v_2 \in C([a,b];\mathbb{R})$ *the inequalities* (4.13),

$$\big[F_1(u_1,u_2)(t) - F_1(v_1,v_2)(t) - p(u_2 - v_2)(t)\big]\times \\ \times \operatorname{sgn}\big(u_1(t) - v_1(t)\big) \le 0 \ \ \textit{for a.e.} \ \ t \in [a,b] \tag{4.30}$$

and

$$\big[F_2(u_1,u_2)(t) - F_2(v_1,v_2)(t) - g(u_1 - v_1)(t)\big] \operatorname{sgn}\big(u_2(t) - v_2(t)\big) \le \\ \le \ell(|u_1 - v_1|)(t) \ \ \textit{for a.e.} \ \ t \in [a,b] \tag{4.31}$$

are fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

Proof. It follows from (4.13), (4.30) and (4.31) that for any u_1, $u_2 \in C([a,b];\mathbb{R})$ the inequalities (4.1), (4.27) and (4.28) are satisfied, where $\eta_i \equiv |\varphi_i(0,0)|$ and $\omega_i \equiv |F_i(0,0)|$ for $i = 1,2$. Consequently, the assumptions of Lemma 4.2 are fulfilled, and thus the problem (1.1), (1.2) has at least one solution. It remains to show that this problem has at most one solution.

Indeed, let (x_1, x_2) and (y_1, y_2) be solutions of the problem (1.1), (1.2). Put

$$z_i(t) = x_i(t) - y_i(t) \ \ \text{for} \ \ t \in [a,b], \ \ i = 1,2.$$

Using (4.13), (4.30) and (4.31), we get

$$|z_i(a)| = \big[\varphi_i(x_1,x_2) - \varphi_i(y_1,y_2)\big] \operatorname{sgn}\big(x_1(a) - y_1(a)\big) \le 0 \ \ \text{for} \ \ i = 1,2,$$

$$\big[z_1'(t) - p(z_2)(t)\big] \operatorname{sgn} z_1(t) = \\ = \big[F_1(x_1,x_2)(t) - F_1(y_1,y_2)(t) - p(x_2 - y_2)(t)\big] \operatorname{sgn}\big(x_1(t) - y_1(t)\big) \le 0 \\ \text{for a.e. } \ t \in [a,b],$$

and

$$\big[z_2'(t) - g(z_1)(t)\big] \operatorname{sgn} z_2(t) = \\ = \big[F_2(x_1,x_2)(t) - F_2(y_1,y_2)(t) - g(x_1 - y_1)(t)\big] \operatorname{sgn}\big(x_2(t) - y_2(t)\big) \le \\ \le \ell(|z_1|)(t) \ \ \text{for a.e. } \ t \in [a,b].$$

Therefore, the assumption $(p, g, \ell) \in \mathcal{A}_{ab}$ yields $\|z_1\|_C + \|z_2\|_C = 0$, i.e., $x_1 \equiv y_1$ and $x_2 \equiv y_2$. □

Lemma 4.4 ([25, Lemma 4.4])**.** *Let* $p, g_0 \in \mathcal{L}_{ab}$ *and let the homogeneous problem*

$$z_1'(t) = p(z_2)(t), \quad z_2'(t) = g_0(z_1)(t), \\ z_1(a) = 0, \quad z_2(a) = 0$$

have only the trivial solution. Then there exists a number $\varrho_0 > 0$ *such that for arbitrary* c_1^*, $c_2^* \in \mathbb{R}$ *and* q_1^*, $q_2^* \in L([a,b];\mathbb{R})$ *the solution* (z_1, z_2) *of the* *problem*

$$z_1'(t) = p(z_2)(t) + q_1^*(t), \quad z_2'(t) = g_0(z_1)(t) + q_2^*(t), \tag{4.32}$$

$$z_1(a) = c_1^*, \quad z_2(a) = c_2^* \tag{4.33}$$

admits the estimate

$$\|z_1\|_C + \|z_2\|_C \le \varrho_0\big(|c_1^*| + \|q_1^*\|_L\big) + \varrho_0\big(|c_2^*| + \|q_2^*\|_L\big).$$

4.3. **Proofs.** We give the following two lemmas on a priori estimates before we prove Theorems 4.1–4.4.

Lemma 4.5. *Let p, g_0, $g_1 \in \mathcal{P}_{ab}$ satisfy* (4.5)*, where P, G_0, G_1 are defined by* (4.6)*. Then $(p, g_0 - g_1, 0) \in \mathcal{A}_{ab}$.*

Proof. Let c_1^*, $c_2^* \in \mathbb{R}_+$, q_1^*, $q_2^* \in L([a,b];\mathbb{R}_+)$, and x_1, $x_2 \in \widetilde{C}([a,b];\mathbb{R})$ satisfy (4.23)–(4.25) with $g = g_0 - g_1$ and $\ell = 0$. We will show that the estimate (4.26) is true, where

$$\varrho_1 = \frac{16(PG_1 + 1)(G_0 + G_1 + 1)}{16(1 - PG_0) - P^2G_1^2} \tag{4.34}$$

and

$$\varrho_2 = \frac{4P(PG_1 + 4)(G_0 + G_1 + 1)}{16(1 - PG_0) - P^2G_1^2} + 1. \tag{4.35}$$

It is clear that x_1, x_2 satisfy

$$x_1'(t) = p(x_2)(t) + \widetilde{q}_1(t) \quad \text{for a.e. } t \in [a,b], \tag{4.36}$$

$$x_2'(t) = g_0(x_1)(t) - g_1(x_1)(t) + \widetilde{q}_2(t) \quad \text{for a.e. } t \in [a,b], \tag{4.37}$$

where

$$\widetilde{q}_1(t) = x_1'(t) - p(x_2)(t), \quad \widetilde{q}_2(t) = x_2'(t) - g_0(x_1)(t) + g_1(x_1)(t) \quad \text{for a.e. } t \in [a,b].$$

Using (4.24) and (4.25), we get

$$\widetilde{q}_1(t)\operatorname{sgn} x_1(t) \le q_1^*(t), \quad \widetilde{q}_2(t)\operatorname{sgn} x_2(t) \le q_2^*(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.38}$$

For the sake of clarity we will divide the discussion into the following cases.

(a) Neither of the functions x_1 and x_2 changes its sign and

$$x_1(t)x_2(t) \ge 0 \quad \text{for } t \in [a,b]; \tag{4.39}$$

(b) Neither of the functions x_1 and x_2 changes its sign and

$$x_1(t)x_2(t) \le 0 \quad \text{for } t \in [a,b]; \tag{4.40}$$

(c) The function x_1 changes its sign. It is clear that one of the following conditions is satisfied.
   (c1) $x_2(t) \ge 0$ for $t \in [a,b]$;
   (c2) $x_2(t) \le 0$ for $t \in [a,b]$;
   (c3) The function x_2 changes its sign.

Case (a): *Neither of the functions x_1 and x_2 changes its sign and* (4.39) *holds.* By virtue of (4.38) and the assumptions p, g_0, $g_1 \in \mathcal{P}_{ab}$, from (4.36) and (4.37) we get

$$|x_1(t)|' \le p(|x_2|)(t) + q_1^*(t) \quad \text{for a.e. } t \in [a,b], \tag{4.41}$$

$$|x_2(t)|' \le g_0(|x_1|)(t) + q_2^*(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.42}$$

It is clear that there exist $t_1, t_2 \in [a,b]$ such that

$$|x_1(t_1)| = \|x_1\|_C \quad \text{and} \quad |x_2(t_2)| = \|x_2\|_C\,. \tag{4.43}$$

Integration of (4.41) and (4.42) from a to t_1 and from a to t_2, respectively, in view of (4.23), (4.43), and the assumptions p, $g_0 \in \mathcal{P}_{ab}$ implies

$$\|x_1\|_C \le c_1^* + \int_a^{t_1} p(|x_2|)(s)\,ds + \int_a^{t_1} q_1^*(s)\,ds \le \|x_2\|_C P + f_1$$

and

$$\|x_2\|_C \le c_2^* + \int_a^{t_2} g_0(|x_1|)(s)\,ds + \int_a^{t_2} q_2^*(s)\,ds \le \|x_1\|_C G_0 + f_2\,,$$

where

$$f_i = c_i^* + \|q_i^*\|_L \quad \text{for } i = 1, 2. \tag{4.44}$$

The last two inequalities yield

$$\|x_1\|_C \le \|x_1\|_C P G_0 + P f_2 + f_1, \quad \|x_2\|_C \le \|x_2\|_C P G_0 + G_0 f_1 + f_2,$$

and thus, using the first inequality in (4.5), we get

$$\|x_1\|_C \le \frac{1}{1-PG_0}\, f_1 + \frac{P}{1-PG_0}\, f_2, \quad \|x_2\|_C \le \frac{G_0}{1-PG_0}\, f_1 + \frac{1}{1-PG_0}\, f_2.$$

Consequently, the estimate (4.26) holds with ϱ_1 and ϱ_2 given by (4.34) and (4.35).

Case (b): *Neither of the functions x_1 and x_2 changes its sign and* (4.40) *holds.* By virtue of (4.38) and the assumptions $p, g_0, g_1 \in \mathcal{P}_{ab}$, from (4.36) and (4.37) we obtain

$$|x_1(t)|' \le q_1^*(t) \quad \text{for a.e. } t \in [a,b], \tag{4.45}$$

$$|x_2(t)|' \le g_1(|x_1|)(t) + q_2^*(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.46}$$

It is clear that there exist t_1, $t_2 \in [a,b]$ such that (4.43) is satisfied. It follows from (4.23), (4.43) and (4.45) that

$$\|x_1\|_C \le c_1^* + \int_a^{t_1} q_1^*(s)\,ds \le f_1,$$

where f_1 is defined by (4.44). Therefore integration of (4.46) from a to t_2, on account of (4.23), (4.43), (4.44) and the assumption $g_1 \in \mathcal{P}_{ab}$, implies

$$\|x_2\|_C \le c_2^* + \int_a^{t_2} g_1(|x_1|)(s)\,ds + \int_a^{t_2} q_2^*(s)\,ds \le \|x_1\|_C G_1 + f_2 \le G_1 f_1 + f_2.$$

Consequently, the estimate (4.26) holds with ϱ_1 and ϱ_2 given by (4.34) and (4.35).

Case (c): *The function x_1 changes its sign.* For $i = 1, 2$, we put

$$M_i = \max\bigl\{x_i(t) : \ t \in [a,b]\bigr\}, \quad m_i = -\min\bigl\{x_i(t) : \ t \in [a,b]\bigr\} \tag{4.47}$$

and we choose α_i, $\beta_i \in [a,b]$ $(i = 1,2)$ such that

$$x_i(\alpha_i) = M_i, \quad x_i(\beta_i) = -m_i \ \text{ for } \ i = 1,2. \tag{4.48}$$

Obviously, $M_1 > 0$ and $m_1 > 0$. Therefore, in view of (4.23), there exist $t_3 \in [a, \alpha_1]$ and $t_4 \in [a, \beta_1]$ such that

$$|x_1(t_3)| \le c_1^*, \quad |x_1(t_4)| \le c_1^*, \tag{4.49}$$

$$\text{if } \ t_3 < \alpha_1 \ \text{ then } \ x_1(t) > 0 \ \text{ for } \ t \in]t_3, \alpha_1], \tag{4.50}$$

and

$$\text{if } \ t_4 < \beta_1 \ \text{ then } \ x_1(t) < 0 \ \text{ for } \ t \in]t_4, \beta_1]. \tag{4.51}$$

It is clear that $[t_3, \alpha_1] \cap [t_4, \beta_1] = \varnothing$. Put

$$P_1 = \int_{t_3}^{\alpha_1} p(1)(s)\,ds, \quad P_2 = \int_{t_4}^{\beta_1} p(1)(s)\,ds.$$

Integration of (4.36) from t_3 to α_1 and from t_4 to β_1, in view of (4.38), (4.47)–(4.51) and the assumption $p \in \mathcal{P}_{ab}$ implies

$$\begin{aligned} M_1 &= x_1(t_3) + \int_{t_3}^{\alpha_1} p(x_2)(s)\,ds + \int_{t_3}^{\alpha_1} \widetilde{q}_1(s)\,ds \le \\ &\le c_1^* + M_2 \int_{t_3}^{\alpha_1} p(1)(s)\,ds + \int_{t_3}^{\alpha_1} q_1^*(s)\,ds \le M_2 P_1 + f_1 \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} m_1 &= -x_1(t_4) - \int_{t_4}^{\beta_1} p(x_2)(s)\,ds - \int_{t_4}^{\beta_1} \widetilde{q}_1(s)\,ds \le \\ &\le c_1^* + m_2 \int_{t_4}^{\beta_1} p(1)(s)\,ds + \int_{t_4}^{\beta_1} q_1^*(s)\,ds \le m_2 P_2 + f_1, \end{aligned} \tag{4.53}$$

where f_1 is defined by (4.44).

Now we are in a position to discuss the cases (c1)–(c3).

Case (c1): $x_2(t) \geq 0$ *for* $t \in [a,b]$. It is clear that $M_2 = \|x_2\|_C$ and $m_2 \leq 0$. Therefore, by virtue of the assumption $p \in \mathcal{P}_{ab}$, (4.52) and (4.53) yield

$$M_1 \leq M_2 P + f_1\,, \quad m_1 \leq f_1\,. \tag{4.54}$$

According to (4.38), (4.47) and the assumptions g_0, $g_1 \in \mathcal{P}_{ab}$, from (4.37) we get

$$x_2'(t) \leq M_1 g_0(1)(t) + m_1 g_1(1)(t) + q_2^*(t) \ \text{ for a.e. } \ t \in [a,b].$$

Integration of the last inequality from a to α_2 in view of (4.23) and (4.48) yields

$$\begin{aligned} M_2 &\leq c_2^* + M_1 \int_a^{\alpha_2} g_0(1)(s)\,ds + m_1 \int_a^{\alpha_2} g_1(1)(s)\,ds + \int_a^{\alpha_2} q_2^*(s)\,ds \leq \\ &\leq M_1 G_0 + m_1 G_1 + f_2. \end{aligned} \tag{4.55}$$

Combining (4.54) and (4.55), we get

$$M_2 \leq M_2 P G_0 + G_0 f_1 + G_1 f_1 + f_2$$

and thus, using the first inequality in (4.5) , we obtain

$$\|x_2\|_C = M_2 \leq \frac{G_0 + G_1}{1 - PG_0}\, f_1 + \frac{1}{1 - PG_0}\, f_2.$$

Now (4.54) yields

$$\|x_1\|_C = \max\{M_1, m_1\} \leq \frac{1 + PG_1}{1 - PG_0}\, f_1 + \frac{P}{1 - PG_0}\, f_2. \tag{4.56}$$

Consequently, the estimate (4.26) holds with ϱ_1 and ϱ_2 given by (4.34) and (4.35).

Case (c2): $x_2(t) \leq 0$ *for* $t \in [a,b]$. Obviously, $M_2 \leq 0$ and $m_2 = \|x_2\|_C$. Therefore, (4.52) and (4.53) imply

$$M_1 \leq f_1, \quad m_1 \leq m_2 P + f_1. \tag{4.57}$$

By virtue of (4.38), (4.47) and the assumptions g_0, $g_1 \in \mathcal{P}_{ab}$, from (4.37) we get

$$-x_2'(t) \leq m_1 g_0(1)(t) + M_1 g_1(1)(t) + q_2^*(t) \ \text{ for a.e. } \ t \in [a,b].$$

Integration of the last inequality from a to β_2 in view of (4.23) and (4.48) yields

$$\begin{aligned} m_2 &\leq c_2^* + m_1 \int_a^{\beta_2} g_0(1)(s)\,ds + M_1 \int_a^{\beta_2} g_1(1)(s)\,ds + \int_a^{\beta_2} q_2^*(s)\,ds \leq \\ &\leq m_1 G_0 + M_1 G_1 + f_2. \end{aligned} \tag{4.58}$$

Now (4.57) and (4.58) result in

$$m_2 \leq m_2 P G_0 + G_0 f_1 + G_1 f_1 + f_2$$

and thus, using the first inequality in (4.5), we get

$$\|x_2\|_C = m_2 \leq \frac{G_0 + G_1}{1 - PG_0}\, f_1 + \frac{1}{1 - PG_0}\, f_2.$$

Therefore, (4.57) implies (4.56). Consequently, the estimate (4.26) holds with ϱ_1 and ϱ_2 given by (4.34) and (4.35).

Case (c3): *The function x_2 changes its sign.* It is clear that $M_2 > 0$ and $m_2 > 0$. Therefore, in view of (4.23), there exist $t_5 \in [a, \alpha_2]$ and $t_6 \in [a, \beta_2]$ such that

$$|x_2(t_5)| \leq c_2^*, \quad |x_2(t_6)| \leq c_2^*, \tag{4.59}$$

$$\text{if } \ t_5 < \alpha_2, \ \text{ then } \ x_2(t) > 0 \ \text{ for } \ t \in]t_5, \alpha_2], \tag{4.60}$$

and

$$\text{if } \ t_6 < \beta_2, \ \text{ then } \ x_2(t) < 0 \ \text{ for } \ t \in]t_6, \beta_2]. \tag{4.61}$$

It is clear that $[t_5, \alpha_2] \cap [t_6, \beta_2] = \varnothing$. Put

$$G_{i,1} = \int_{t_5}^{\alpha_2} g_i(1)(s)\, ds, \quad G_{i,2} = \int_{t_6}^{\beta_2} g_i(1)(s)\, ds \ \text{ for } \ i = 0, 1.$$

Integration of (4.37) from t_5 to α_2 and from t_6 to β_2 on account of (4.38), (4.47), (4.48), (4.59)–(4.61) and the assumptions g_0, $g_1 \in \mathcal{P}_{ab}$ implies

$$\begin{aligned} M_2 &= x_2(t_5) + \int_{t_5}^{\alpha_2} g_0(x_1)(s)\, ds - \int_{t_5}^{\alpha_2} g_1(x_1)(s)\, ds + \int_{t_5}^{\alpha_2} \widetilde{q}_2(s)\, ds \leq \\ &\leq c_2^* + M_1 \int_{t_5}^{\alpha_2} g_0(1)(s)\, ds + m_1 \int_{t_5}^{\alpha_2} g_1(1)(s)\, ds + \int_{t_5}^{\alpha_2} q_2^*(s)\, ds \leq \\ &\leq M_1 G_{0,1} + m_1 G_{1,1} + f_2 \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} m_2 &= -x_2(t_6) - \int_{t_6}^{\beta_2} g_0(x_1)(s)\, ds + \int_{t_6}^{\beta_2} g_1(x_1)(s)\, ds - \int_{t_6}^{\beta_2} \widetilde{q}_2(s)\, ds \leq \\ &\leq c_2^* + m_1 \int_{t_6}^{\beta_2} g_0(1)(s)\, ds + M_1 \int_{t_6}^{\beta_2} g_1(1)(s)\, ds + \int_{t_6}^{\beta_2} q_2^*(s)\, ds \leq \\ &\leq m_1 G_{0,2} + M_1 G_{1,2} + f_2, \end{aligned} \tag{4.63}$$

where f_1 is defined by (4.44). Using (4.62) and (4.63) in (4.52) and (4.53), respectively, we get

$$\begin{aligned} M_1 &\leq M_1 P_1 G_{0,1} + m_1 P_1 G_{1,1} + P_1 f_2 + f_1, \\ m_1 &\leq m_1 P_2 G_{0,2} + M_1 P_2 G_{1,2} + P_2 f_2 + f_1. \end{aligned}$$

Therefore, in view of the first inequality in (4.5), the last two relations yield

$$0 < M_1(1 - P_1 G_{0,1}) \le m_1 P_1 G_{1,1} + P_1 f_2 + f_1, \tag{4.64}$$

$$0 < m_1(1 - P_2 G_{0,2}) \le M_1 P_2 G_{1,2} + P_2 f_2 + f_1. \tag{4.65}$$

Combining (4.64) and (4.65), we get

$$M_1(1 - P_1 G_{0,1})(1 - P_2 G_{0,2}) \le M_1 P_1 P_2 G_{1,1} G_{1,2} + \\ + P_1 P_2 G_{1,1} f_2 + P_1 G_{1,1} f_1 + (1 - P_2 G_{0,2})(P_1 f_2 + f_1). \tag{4.66}$$

It is easy to verify that

$$P_1 P_2 \le \frac{1}{4}\,(P_1 + P_2)^2 \le \frac{1}{4}\,P^2, \quad G_{1,1} G_{1,2} \le \frac{1}{4}\,(G_{1,1} + G_{1,2})^2 \le \frac{1}{4}\,G_1^2$$

and

$$(1 - P_1 G_{0,1})(1 - P_2 G_{0,2}) \ge 1 - P_1 G_{0,1} - P_2 G_{0,2} \ge 1 - P G_0.$$

Hence, (4.3) implies

$$M_1(1 - P G_0) \le \frac{M_1}{16}\,P^2 G_1^2 + \frac{1}{4}\,P^2 G_1 f_2 + P G_1 f_1 + P f_2 + f_1$$

and thus, by virtue of the second inequality in (4.5), we get

$$M_1 \le \frac{16(P G_1 + 1)}{16(1 - P G_0) - P^2 G_1^2}\,f_1 + \frac{4P(P G_1 + 4)}{16(1 - P G_0) - P^2 G_1^2}\,f_2.$$

One can show analogously that the number m_1 has the same upper bound as M_1. Consequently,

$$\|x_1\|_C = \max\{M_1, m_1\} \le \\ \le \frac{16(P G_1 + 1)}{16(1 - P G_0) - P^2 G_1^2}\,f_1 + \frac{4P(P G_1 + 4)}{16(1 - P G_0) - P^2 G_1^2}\,f_2. \tag{4.67}$$

On the other hand, it follows from (4.62) and (4.63) that

$$\|x_2\|_C = \max\{M_2, m_2\} \le (G_0 + G_1)\|x_1\|_C + f_2. \tag{4.68}$$

Therefore, the inequalities (4.67) and (4.68) guarantee the estimate (4.26) with ϱ_1 and ϱ_2 given by (4.34) and (4.35). □

Lemma 4.6. *Let p, g_0, $g_1 \in \mathcal{P}_{ab}$ satisfy* (4.8)*. Then $(p, -g_1, g_0) \in \mathcal{A}_{ab}$.*

Proof. By virtue of the inclusion $(p, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a)$, the assumptions of Lemma 4.4 are fulfilled. Let ϱ_0 be the number appearing in the lemma indicated. Assume that c_1^*, $c_2^* \in \mathbb{R}_+$, q_1^*, $q_2^* \in L([a,b]; \mathbb{R}_+)$ and x_1, $x_2 \in \widetilde{C}([a,b]; \mathbb{R})$ satisfy (4.23)–(4.25) with $g = -g_1$ and $\ell = g_0$. We will show that the estimate (4.26) holds, where

$$\varrho_1 = \varrho_0(1 + G_0 + G_1), \quad \varrho_2 = \varrho_0(1 + G_0 + G_1) + 1, \tag{4.69}$$

and G_0, G_1 are defined by (4.6).

It is clear that x_1 and x_2 satisfy (4.36) and

$$x_2'(t) = -g_1(x_1)(t) + \widetilde{q}_2(t) \ \text{ for a.e. } \ t \in [a,b], \tag{4.70}$$

where

$$\widetilde{q}_1(t) = x_1'(t) - p(x_2)(t), \quad \widetilde{q}_2(t) = x_2'(t) + g_1(x_1)(t) \text{ for a.e. } t \in [a,b].$$

Using (4.24) and (4.25), we get

$$\widetilde{q}_1(t) \operatorname{sgn} x_1(t) \le q_1^*(t) \text{ for a.e. } t \in [a,b], \tag{4.71}$$

$$\widetilde{q}_2(t) \operatorname{sgn} x_2(t) \le g_0(|x_1|)(t) + q_2^*(t) \text{ for a.e. } t \in [a,b]. \tag{4.72}$$

In view of (4.71) and the assumption $p \in \mathcal{P}_{ab}$, the relation (4.36) yields

$$\begin{aligned}[x_1(t)]_+' &= \frac{1}{2}\big(p(x_2)(t) \operatorname{sgn} x_1(t) + p(x_2)(t)\big) + \frac{\operatorname{sgn} x_1(t) + 1}{2}\,\widetilde{q}_1(t) \le \\ &\le p([x_2]_+)(t) + q_1^*(t) \text{ for a.e. } t \in [a,b]\end{aligned} \tag{4.73}$$

and

$$\begin{aligned}[x_1(t)]_-' &= \frac{1}{2}\big(p(x_2)(t) \operatorname{sgn} x_1(t) - p(x_2)(t)\big) + \frac{\operatorname{sgn} x_1(t) - 1}{2}\,\widetilde{q}_1(t) \le \\ &\le p([x_2]_-)(t) + q_1^*(t) \text{ for a.e. } t \in [a,b].\end{aligned} \tag{4.74}$$

On the other hand, by virtue of (4.72) and the assumptions g_0, $g_1 \in \mathcal{P}_{ab}$, from (4.70) we get

$$\begin{aligned}[x_2(t)]_+' &= \frac{1}{2}\big(-g_1(x_1)(t) \operatorname{sgn} x_2(t) - g_1(x_1)(t)\big) + \frac{\operatorname{sgn} x_2(t) + 1}{2}\,\widetilde{q}_2(t) \le \\ &\le g_1([x_1]_-)(t) + g_0(|x_1|)(t) + q_2^*(t) = \\ &= -g_1([x_1]_+)(t) + g_1(|x_1|)(t) + g_0(|x_1|)(t) + q_2^*(t) \\ &\qquad \text{for a.e. } t \in [a,b]\end{aligned} \tag{4.75}$$

and

$$\begin{aligned}[x_2(t)]_-' &= \frac{1}{2}\big(-g_1(x_1)(t) \operatorname{sgn} x_2(t) + g_1(x_1)(t)\big) + \frac{\operatorname{sgn} x_2(t) - 1}{2}\,\widetilde{q}_2(t) \le \\ &\le g_1([x_1]_+)(t) + g_0(|x_1|)(t) + q_2^*(t) = \\ &= -g_1([x_1]_-)(t) + g_1(|x_1|)(t) + g_0(|x_1|)(t) + q_2^*(t) \\ &\qquad \text{for a.e. } t \in [a,b].\end{aligned} \tag{4.76}$$

Furthermore, (4.23) implies

$$[x_i(a)]_+ \le c_i^*, \quad [x_i(a)]_- \le c_i^* \text{ for } i = 1,2. \tag{4.77}$$

According to the assumption $(p, -g_1) \in \widehat{\mathcal{S}}_{ab}^2(a)$ and Remark 3.3, the problem

$$u_1'(t) = p(u_2)(t) + q_1^*(t), \tag{4.78}$$

$$u_2'(t) = -g_1(u_1)(t) + g_1(|x_1|)(t) + g_0(|x_1|)(t) + q_2^*(t), \tag{4.79}$$

$$u_1(a) = c_1^*, \quad u_2(a) = c_2^* \tag{4.80}$$

has a unique solution (u_1, u_2).

On the other hand, using the inclusion $(p, -g_1) \in \widehat{\mathcal{S}}_{ab}^2(a)$, from (4.73)–(4.80) we get

$$[x_1(t)]_+ \le u_1(t), \quad [x_1(t)]_- \le u_1(t) \text{ for } t \in [a,b],$$

i.e.,

$$|x_1(t)| \leq u_1(t) \text{ for } t \in [a,b]. \tag{4.81}$$

Taking now the assumptions $g_0, g_1 \in \mathcal{P}_{ab}$ into account, it follows from (4.78) and (4.79) that

$$u_1'(t) = p(u_2)(t) + q_1^*(t), \quad u_2'(t) \leq g_0(u_1)(t) + q_2^*(t) \text{ for a.e. } t \in [a,b].$$

However, we also suppose that $(p, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a)$ and thus the function u_1 satisfies

$$u_1(t) \leq z_1(t) \text{ for } t \in [a,b],$$

where (z_1, z_2) is a solution to the problem (4.32), (4.33). From (4.81) and Lemma 4.4 it is clear that

$$\|x_1\|_C \leq \varrho_0\big(c_1^* + \|q_1^*\|_L\big) + \varrho_0\big(c_2^* + \|q_2^*\|_L\big). \tag{4.82}$$

Now observe that by virtue of (4.72) and the assumptions g_0, $g_1 \in \mathcal{P}_{ab}$ the relation (4.70) yields

$$\begin{aligned}|x_2(t)|' &= -g_1(x_1)(t)\operatorname{sgn} x_2(t) + \widetilde{q}_2(t)\operatorname{sgn} x_2(t) \leq \\ &\leq g_1(|x_1|)(t) + g_0(|x_1|)(t) + q_2^*(t) \leq \\ &\leq \big(g_0(1)(t) + g_1(1)(t)\big)\|x_1\|_C + q_2^*(t) \text{ for a.e. } t \in [a,b].\end{aligned}$$

Therefore in view of (4.23) it follows from the last relation that

$$\begin{aligned}|x_2(t)| \leq c_2^* + \bigg(\int_a^t g_0(1)(s)\,ds + \int_a^t g_1(1)(s)\,ds\bigg)\|x_1\|_C + \\ + \int_a^t q_2^*(s)\,ds \text{ for } t \in [a,b],\end{aligned}$$

and thus

$$\|x_2\|_C \leq (G_0 + G_1)\|x_1\|_C + c_2^* + \|q_2^*\|_L, \tag{4.83}$$

where G_0, G_1 are defined by (4.6).

Therefore, (4.82) and (4.83) guarantee the estimate (4.26) with ϱ_1 and ϱ_2 given by (4.69). □

Proof of Theorem 4.1. Without loss of generality we can assume that $k = 1$. According to Lemma 4.5, the condition (4.5) yields the inclusion $(p, g_0 - g_1, 0) \in \mathcal{A}_{ab}$. Consequently, the validity of the theorem follows from Lemma 4.2. □

Proof of Theorem 4.2. Without loss of generality we can assume that $k = 1$. By virtue of Lemma 4.6, the condition (4.8) implies the inclusion $(p, -g_1, g_0) \in \mathcal{A}_{ab}$. Consequently, the validity of the theorem follows from Lemma 4.2. □

Proof of Theorem 4.3. Without loss of generality we can assume that $k = 1$. According to Lemma 4.5, the condition (4.5) yields the inclusion $(p, g_0 - g_1, 0) \in \mathcal{A}_{ab}$. Consequently, the validity of the theorem follows from Lemma 4.3. □

Proof of Theorem 4.4. Without loss of generality we can assume that $k = 1$. By virtue of Lemma 4.6, the condition (4.8) implies the inclusion $(p, -g_1, g_0) \in \mathcal{A}_{ab}$. Consequently, the validity of the theorem follows from Lemma 4.3. □

Proof of Corollary 4.1. Put

$$F_1(z_1, z_2)(t) \stackrel{\text{def}}{=} f(t)z_2(\mu(t)) + k_1\big(t, z_1(t), z_2(t), z_1(\zeta_{1,1}(t), z_2(\zeta_{1,2}(t))\big)$$
$$\text{for a.e. } t \in [a, b] \text{ and all } z_1,\, z_2 \in C([a, b]; \mathbb{R}) \tag{4.84}$$

and

$$F_2(z_1, z_2)(t) \stackrel{\text{def}}{=} h_0(t)z_1(\tau_0(t)) - h_1(t)z_1(\tau_1(t)) +$$
$$+ k_2\big(t, z_1(t), z_2(t), z_1(\zeta_{2,1}(t)), z_2(\zeta_{2,2}(t))\big)$$
$$\text{for a.e. } t \in [a, b] \text{ and all } z_1,\, z_2 \in C([a, b]; \mathbb{R}). \tag{4.85}$$

It is clear that F_1 and F_2 satisfy the condition (H_1). Moreover, it follows from (4.16) that for any u_1, $u_2 \in C([a, b]; \mathbb{R})$ the inequalities (4.2) and (4.3) with $k = 1$ are fulfilled, where

$$p(z)(t) \stackrel{\text{def}}{=} f(t)z(\mu(t)) \text{ for a.e. } t \in [a, b] \text{ and all } z \in C([a, b]; \mathbb{R}), \tag{4.86}$$

$$g_i(z)(t) \stackrel{\text{def}}{=} h_i(t)z(\tau_i(t)) \text{ for a.e. } t \in [a, b], \text{ all } z \in C([a, b]; \mathbb{R}),\ i = 0, 1. \tag{4.87}$$

Furthermore, $p(1) \equiv f$, $g_0(1) \equiv h_0$ and $g_1(1) \equiv h_1$, and thus the assumptions of Theorem 4.1 with $k = 1$ are satisfied. □

Proof of Corollary 4.2. Let F_1 and F_2 be defined by (4.84) and (4.85), respectively. It is clear that F_1 and F_2 satisfy the condition (H_1). Moreover, it follows from (4.16) that for any u_1, $u_2 \in C([a, b]; \mathbb{R})$ the inequalities (4.2) and (4.7) with $k = 1$ are fulfilled, where p and g_0, g_1 are defined by (4.86) and (4.87), respectively.

By virtue of the condition (a) (resp. (b), resp. (c)) of the corollary, it follows from Proposition 5.3 (resp. Proposition 5.4, resp. Proposition 5.5) that

$$(p, g_0) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

On the other hand, in view of the condition (A) (resp. (B)) with $\gamma^* = 1$, Proposition 5.6 (resp. Proposition 5.7) yields

$$(p, -g_1) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

Consequently, the assumptions of Theorem 4.2 with $k = 1$ are satisfied. □

5. On the Set $\widehat{\mathcal{S}}_{ab}^2(a)$

In this section, we give some sufficient conditions stated in [26] guaranteeing the validity of the inclusion $(p,g) \in \widehat{\mathcal{S}}_{ab}^2(a)$. We first formulate rather general results.

Proposition 5.1 ([26, Theorem 3.2])**.** *Let $p,\, g \in \mathcal{P}_{ab}$. Then $(p,g) \in \widehat{\mathcal{S}}_{ab}^2(a)$ if and only if there exist functions $\gamma_1,\, \gamma_2 \in \widetilde{C}([a,b];\mathbb{R})$ such that*

$$\gamma_1(t) > 0, \quad \gamma_2(t) > 0 \;\text{ for }\; t \in [a,b],$$
$$\gamma_1'(t) \ge p(\gamma_2)(t), \quad \gamma_2'(t) \ge g(\gamma_1)(t) \;\text{ for a.e. }\; t \in [a,b].$$

Proposition 5.2 ([26, Theorem 3.3])**.** *Let $p \in \mathcal{P}_{ab}$, $-g \in \mathcal{P}_{ab}$, and let p, g be a-Volterra operators. Then $(p,g) \in \widehat{\mathcal{S}}_{ab}^2(a)$ if and only if there exist functions $\gamma_1,\, \gamma_2 \in \widetilde{C}_{loc}([a,b[\,;\mathbb{R})$ such that $\gamma_1 \in C([a,b];\mathbb{R})$,*

$$\gamma_1'(t) \le p(\gamma_2)(t) \;\text{ for a.e. }\; t \in [a,b],^{2} \tag{5.1}$$
$$\gamma_2'(t) \le g(\gamma_1)(t) \;\text{ for a.e. }\; t \in [a,b],$$
$$\gamma_1(t) \ge 0 \;\text{ for }\; t \in [a,b], \quad \gamma_1(a) > 0, \quad \gamma_2(a) \le 0,$$

and

$$|\gamma_1(t)| + |\gamma_2(t)| \ne 0 \;\text{ for }\; t \in \,]a,b[\,.$$

Remark 5.1. Since possibly $\gamma_2(t) \to -\infty$ as $t \to b-$, the condition (5.1) of the previous proposition is understood in the sense that for any $b_0 \in \,]a,b[$ the relation

$$\gamma_1'(t) \le p^{ab_0}(\gamma_2)(t) \;\text{ for a.e. }\; t \in [a,b_0]$$

holds, where p^{ab_0} is a restriction of the operator p to the space $C([a,b_0];\mathbb{R})$.

Choosing suitable functions γ_1 and γ_2 in the propositions stated above, one can derive several efficient conditions sufficient for the validity of the inclusion $(p,g) \in \widehat{\mathcal{S}}_{ab}^2(a)$. These conditions are not formulated here in the general form; we present however some corollaries for "operators with argument deviations".

Proposition 5.3 ([26, Theorem 5.1])**.** *Let $h_k \in L([a,b];\mathbb{R}_+)$ and $\tau_k : [a,b] \to [a,b]$ be measurable functions $(k=1,2)$ such that*

$$\int\limits_t^{\tau_k(t)} \omega(s)\,ds \le \frac{1}{e} \;\text{ for a.e. }\; t \in [a,b], \;\; k=1,2,$$

where

$$\omega(t) \overset{\text{def}}{=} \max\{h_1(t), h_2(t)\} \;\text{ for a.e. }\; t \in [a,b]. \tag{5.2}$$

Then $(\ell_1,\ell_2) \in \widehat{\mathcal{S}}_{ab}^2(a)$, where

$$\ell_k(z)(t) \overset{\text{def}}{=} h_k(t) z(\tau_k(t)) \;\text{ for a.e. }\; t \in [a,b], \;\text{ all }\; z \in C([a,b];\mathbb{R}), \;\; k=1,2. \tag{5.3}$$

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[2] See Remark 5.1.

Proposition 5.4 ([26, Theorem 5.2]). *Let $h_k \in L([a,b];\mathbb{R}_+)$, $\tau_k : [a,b] \to [a,b]$ be measurable functions ($k = 1,2$), and let* $\max\{\lambda_1, \lambda_2\} < 1$, *where*

$$\lambda_k = \int_a^b \cosh\bigg(\int_s^b \omega(\xi)\,d\xi\bigg) h_k(s)\sigma_k(s)\bigg(\int_s^{\tau_k(s)} h_{3-k}(\xi)\,d\xi\bigg)\,ds + \\ + \int_a^b \sinh\bigg(\int_s^b \omega(\xi)\,d\xi\bigg) h_{3-k}(s)\sigma_{3-k}(s)\bigg(\int_s^{\tau_{3-k}(s)} h_k(\xi)\,d\xi\bigg)\,ds \quad \text{for } k=1,2,$$

the function ω is defined by (5.2), *and*

$$\sigma_k(t) \overset{\text{def}}{=} \frac{1}{2}\big(1 + \operatorname{sgn}(\tau_k(t) - t)\big) \quad \text{for a.e. } t \in [a,b].$$

Then $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$, where ℓ_1, ℓ_2 are defined by (5.3).

Proposition 5.5 ([26, Theorems 5.3 and 5.3′]). *Let $h_k \in L([a,b];\mathbb{R}_+)$, $\tau_k : [a,b] \to [a,b]$ be measurable functions ($k = 1,2$), and let there exist $m \in \{1,2\}$ such that*

$$\int_a^{\tau_m^*} h_{3-m}(s)\bigg(\int_a^{\tau_{3-m}(s)} h_m(\xi)\,d\xi\bigg)\,ds < 1,$$

where $\tau_k^* = \operatorname{ess\,sup}\{\tau_k(t) : t \in [a,b]\}$ for $k = 1,2$. Then $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$, where ℓ_1, ℓ_2 are defined by (5.3).

Proposition 5.6 ([26, Theorem 5.5]). *Let $h_k \in L([a,b];\mathbb{R}_+)$ and $\tau_k : [a,b] \to [a,b]$ be measurable functions ($k = 1,2$) such that*

$$h_k(t)(\tau_k(t) - t) \le 0 \quad \text{for a.e. } t \in [a,b], \quad k = 1,2. \tag{5.4}$$

If

$$\int_a^b h_1(s)\bigg(\int_a^{\tau_1(s)} h_2(\xi)\,d\xi\bigg)\,ds \le 1,$$

then $(\ell_1, \ell_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$, where

$$\ell_k(z)(t) \overset{\text{def}}{=} (-1)^{k+1} h_k(t) z(\tau_k(t)) \\ \text{for a.e. } t \in [a,b] \text{ and all } z \in C([a,b];\mathbb{R}), \quad k = 1,2. \tag{5.5}$$

Proposition 5.7 ([26, Theorem 5.6]). *Let $h_k \in L([a,b];\mathbb{R}_+)$ and $\tau_k : [a,b] \to [a,b]$ be measurable functions ($k = 1,2$) fulfilling* (5.4). *Assume that there exist numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_3 > 0$, $\lambda \in [0,1[$ and $\nu \in [0,\lambda]$ such* *that* (3.35) *holds,*

$$(b-t)^{\lambda-\nu}h_1(t) \leq \alpha_3\left[1+\sigma_3(t)\int\limits_{\tau_1(t)}^{t}\Big(\frac{\nu}{b-s}+\frac{\alpha_2}{(b-s)^{\lambda}}\Big)\,ds\right] \ \textit{for a.e.} \ t\in[a,b],$$

$$(b-t)^{\lambda+\nu}h_2(t) \leq \alpha_1 \ \textit{for a.e.} \ t \in [a,b],$$

and

$$\alpha_3(b-t)^{\nu}h_2(t)(t-\tau_2(t)) \leq \alpha_2 + \frac{\nu}{(b-t)^{1-\lambda}} \ \textit{for a.e.} \ t \in [a,b],$$

where

$$\sigma_3(t) \stackrel{\text{def}}{=} \frac{1}{2}\big(1+\operatorname{sgn}(t-\tau_1(t))\big) \ \textit{for a.e.} \ t \in [a,b].$$

Then $(\ell_1,\ell_2) \in \widehat{\mathcal{S}}^2_{ab}(a)$, *where* ℓ_1, ℓ_2 *are defined by* (5.5).

6. Counter-Examples

In this section, we give examples verifying that the results obtained are unimprovable in a certain sense.

Example 6.1. Let ε_1, $\varepsilon_2 \in [0,1[$, $\varepsilon_1+\varepsilon_2>0$, and let ℓ_1, $\ell_2 \in \mathcal{L}_{ab}$ be defined by

$$\begin{aligned} &\ell_1(z)(t) \stackrel{\text{def}}{=} f(t)z(\mu(t)) \ \text{ for a.e. } t\in[a,b] \ \text{ and all } \ z\in C([a,b];\mathbb{R}),\\ &\ell_2(z)(t) \stackrel{\text{def}}{=} h(t)z(b) \ \text{ for a.e. } t\in[a,b] \ \text{ and all } \ z\in C([a,b];\mathbb{R}), \end{aligned} \tag{6.1}$$

where f, $h \in L([a,b];\mathbb{R}_+)$ and $\mu : [a,b] \to [a,b]$ is a measurable function such that

$$\int\limits_a^b f(s)\bigg(\int\limits_a^{\mu(s)} h(\xi)\,d\xi\bigg)\,ds = 1. \tag{6.2}$$

It is clear that for any $z \in C([a,b];\mathbb{R})$ the inequality (3.17) with $k=1$ and $m=0$ is satisfied, where $g_0=0$ and $g_1=\ell_2$. Moreover,

$$(1-\varepsilon_1)(1-\varepsilon_2)\int\limits_a^b f(s)\bigg(\int\limits_a^{\mu(s)} h(\xi)\,d\xi\bigg)\,ds < 1 \tag{6.3}$$

and thus, using Proposition 5.5, we get

$$\big((1-\varepsilon_1)\,\ell_1,(1-\varepsilon_2)\,\ell_2\big) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

It is clear that $(\ell_1,0) \in \widehat{\mathcal{S}}^2_{ab}(a)$ (see, e.g., Proposition 5.5). Consequently, the assumptions of Theorem 3.4 with $k=1$ and $m=0$ are satisfied, except the condition (3.16), instead of which the condition (3.18) is fulfilled.

On the other hand, the homogeneous problem (3.14), (3.15) has a nontrivial solution (x_1, x_2), where

$$x_1(t) = \int_a^t f(s)\Big(\int_a^{\mu(s)} h(\xi)\,d\xi\Big)\,ds, \quad x_2(t) = \int_a^t h(s)\,ds \ \text{ for } \ t \in [a,b].$$

This example shows that the assumption (3.16) of Theorem 3.4 cannot be replaced by the assumption (3.18), no matter how small ε_1, $\varepsilon_2 \in [0,1[$ with $\varepsilon_1 + \varepsilon_2 > 0$ are.

Example 6.2. Let $\alpha \in]0,1[$, ε_1, $\varepsilon_2 \in [0,1[$, $\varepsilon_1 + \varepsilon_2 > 0$, and $a < t_1 < t_2 < b$. Put $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and choose f, $h \in L([a,b];\mathbb{R})$ such that

$$f(t) \geq 0, \quad (t - t_1)(t - t_2)h(t) \leq 0 \ \text{ for a.e. } \ t \in [a,b], \tag{6.4}$$

$$\int_a^{t_1} f(s)\Big(\int_a^s |h(\xi)|\,d\xi\Big)\,ds = \frac{\alpha}{1+\varepsilon}\,,$$

$$\int_{t_1}^{t_2} f(s)\Big((1+\varepsilon)\int_a^{t_1} |h(\xi)|\,d\xi + \int_{t_1}^s h(\xi)\,d\xi\Big)\,ds = 1 - \alpha,$$

$$\int_{t_2}^b f(s)\,ds = \varepsilon \min\left\{\frac{\int_{t_1}^{t_2} f(s)\,ds \int_a^{t_1} |h(s)|\,ds}{\int_a^{t_2} |h(s)|\,ds}\,,\ \frac{1}{(1+\varepsilon)\int_a^{t_1} |h(s)|\,ds + \int_{t_1}^{t_2} h(s)\,ds}\right\},$$

and

$$\int_{t_2}^b f(s)\Big(\int_{t_2}^s |h(\xi)|\,d\xi\Big)\,ds = 2(1+\varepsilon).$$

Furthermore, we put

$$x_2(t) = \begin{cases} (1+\varepsilon)\displaystyle\int_a^t |h(s)|\,ds & \text{for } \ t \in [a, t_1[\\ (1+\varepsilon)\displaystyle\int_a^{t_1} |h(s)|\,ds + \int_{t_1}^t h(s)\,ds & \text{for } \ t \in [t_1, b] \end{cases}$$

and

$$x_1(t) = \int_a^t f(s)x_2(s)\,ds \ \text{ for } \ t \in [a,b].$$

It is clear that $x_1(t_2) = 1$ and $x_1(b) \leq -(1+\varepsilon)$, and thus there exists $t_0 \in [t_2, b]$ such that $x_1(t_0) = -(1+\varepsilon)$. Let ℓ_1, $\ell_2 \in \mathcal{L}_{ab}$ be defined by

$$\ell_1(z)(t) \stackrel{\text{def}}{=} f(t)z(t) \ \text{ for a.e. } \ t \in [a,b] \ \text{ and all } \ z \in C([a,b];\mathbb{R}),$$

$$\ell_2(z)(t) \stackrel{\text{def}}{=} h(t)z(\tau(t)) \ \text{ for a.e. } \ t\in[a,b] \ \text{ and all } \ z\in C([a,b];\mathbb{R}),$$

where

$$\tau(t)=\begin{cases} t_0 & \text{for } \ t\in[a,t_1[\\ t_2 & \text{for } \ t\in[t_1,b]\end{cases}.$$

It is not difficult to verify that for any $z\in C([a,b];\mathbb{R})$ the inequality (3.17) with $k=1$ and $m=0$ is satisfied, where

$$g_0(z)(t) \stackrel{\text{def}}{=} -h_0(t)z(\tau_0(t)) \ \text{ for a.e. } \ t\in[a,b] \ \text{ and all } \ z\in C([a,b];\mathbb{R}),$$

$$g_1(z)(t) \stackrel{\text{def}}{=} h_1(t)z(\tau(t)) \ \text{ for a.e. } \ t\in[a,b] \ \text{ and all } \ z\in C([a,b];\mathbb{R}),$$

$$h_0(t)=\begin{cases} 0 & \text{for } \ t\in[a,t_2[\\ \dfrac{1}{2}\,|h(t)| & \text{for } \ t\in[t_2,b]\end{cases}, \quad h_1(t)=\begin{cases} |h(t)| & \text{for } \ t\in[a,t_2[\\ \dfrac{1}{2}|h(t)| & \text{for } \ t\in[t_2,b]\end{cases}$$

and

$$\tau_0(t)=\begin{cases} a & \text{for } \ t\in[a,t_2[\\ t_2 & \text{for } \ t\in[t_2,b]\end{cases}.$$

Obviously, $\ell_1\in\mathcal{P}_{ab}$ and

$$(g_0+g_1)(z)(t)=\widetilde{h}(t)z(\tau(t)) \ \text{ for a.e. } \ t\in[a,b] \ \text{ and all } \ z\in C([a,b];\mathbb{R}),$$

where

$$\widetilde{h}(t)=\begin{cases} |h(t)| & \text{for } \ t\in[a,t_2[\\ 0 & \text{for } \ t\in[t_2,b]\end{cases}.$$

Therefore, $g_0+g_1\in\mathcal{P}_{ab}$,

$$\int_a^b f(s)\bigg(\int_a^s \widetilde{h}(\xi)\,d\xi\bigg)\,ds=$$

$$=\int_a^{t_1} f(s)\bigg(\int_a^s |h(\xi)|\,d\xi\bigg)\,ds+\int_{t_1}^{t_2} f(s)\bigg((1+\varepsilon)\int_a^{t_1}|h(\xi)|\,d\xi+\int_{t_1}^s h(\xi)\,d\xi\bigg)\,ds-$$

$$-\bigg(\varepsilon\int_{t_1}^{t_2} f(s)\,ds\int_a^{t_1}|h(s)|\,ds-\int_{t_2}^b f(s)\,ds\int_a^{t_2}|h(s)|\,ds\bigg)\leq$$

$$\leq\frac{\alpha}{1+\varepsilon}+1-\alpha=\frac{1+\varepsilon(1-\alpha)}{1+\varepsilon}<1,$$

and thus Proposition 5.5 yields

$$(\ell_1,g_0+g_1)\in\widehat{\mathcal{S}}^2_{ab}(a).$$

Furthermore, $-g_0\in\mathcal{P}_{ab}$, the operators ℓ_1, g_0 are a-Volterra ones, and since

$$(1-\varepsilon_1)(1-\varepsilon_2)\int_a^b f(s)\bigg(\int_a^s h_0(\xi)\,d\xi\bigg)\,ds\leq$$

$$\leq \frac{1-\varepsilon}{2} \int\limits_{t_2}^{b} f(s)\bigg(\int\limits_{t_2}^{s} |h(\xi)|\, d\xi\bigg)\, ds = 1-\varepsilon^2 < 1,$$

using Proposition 5.6 we get

$$\big((1-\varepsilon_1)\,\ell_1, (1-\varepsilon_2)\,g_0\big) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

Consequently, the assumptions of Theorem 3.4 with $k = 1$ and $m = 0$ are satisfied, except the condition (3.16), instead of which the condition (3.19) is fulfilled.

On the other hand, (x_1, x_2) is a nontrivial solution to the homogeneous problem (3.14), (3.15).

This example shows that the assumption (3.16) of Theorem 3.4 cannot be replaced by the assumption (3.19), no matter how small ε_1, $\varepsilon_2 \in [0, 1[$ with $\varepsilon_1 + \varepsilon_2 > 0$ are.

The following lemma, which we need in examples concerning the nonlinear case, follows from the Riesz–Schauder theory and the Fredholm property of the problem (1.3), (1.4) (see, e.g., the proof of Theorem 1.1.1 in [14]).

Lemma 6.1. *If the homogeneous problem* (3.14), (3.15) *has a nontrivial solution, then there exist* q_1, $q_2 \in L([a, b]; \mathbb{R})$ *and* c_1, $c_2 \in \mathbb{R}$ *such that the problem* (1.3), (1.4) *has no solution.*

Example 6.3. Let $\varepsilon \in \mathbb{R}_+$. In [9, Example 4.2], operators ℓ_1, $\ell_2 \in \mathcal{P}_{ab}$ are constructed such that

$$\int\limits_a^b \ell_1(1)(s)\, ds \int\limits_a^b \ell_2(1)(s)\, ds = 1 + \varepsilon$$

and the homogeneous problem (3.14), (3.15) has a nontrivial solution. Then, according to Lemma 6.1, there exist q_1, $q_2 \in L([a, b]; \mathbb{R})$ and c_1, $c_2 \in \mathbb{R}$ such that the problem (1.3), (1.4) has no solution.

Having taken these operators ℓ_1, ℓ_2, we put

$$F_i(z_1, z_2)(t) \overset{\text{def}}{=} \ell_i(z_{3-i})(t) + q_i(t)$$
$$\text{for a.e. } t \in [a, b] \text{ and all } z_1,\, z_2 \in C([a, b]; \mathbb{R}), \quad i = 1, 2, \tag{6.5}$$

and

$$\varphi_i(z_1, z_2) \overset{\text{def}}{=} c_i \text{ for } z_1,\, z_2 \in C([a, b]; \mathbb{R}), \quad i = 1, 2. \tag{6.6}$$

It is clear that F_1, F_2 and φ_1, φ_2 satisfy the conditions (H_1) and (H_2), respectively. Moreover, for any u_1, $u_2 \in C([a, b]; \mathbb{R})$ the inequalities (4.1) and (4.2), (4.3) with $k = 1$ hold, where $p = \ell_1$, $g_0 = \ell_2$, $g_1 = 0$, and

$$\eta_i \equiv |c_i|, \quad \omega_i \equiv |q_i| \text{ for } i = 1, 2. \tag{6.7}$$

Consequently, the assumptions of Theorem 4.1 with $k = 1$ are satisfied, except the first inequality in (4.5), instead of which the equality $PG_0 = 1+\varepsilon$ holds. However, the problem (1.1), (1.2) has no solution.

This example shows that the first strict inequality of (4.5) in Theorem 4.1 cannot be weakened.

Example 6.4. Let $\varepsilon \in \mathbb{R}_+$, $a < t_1 < t_2 < t_3 \leq b$, and let the operators p and g_1 be defined by (4.86) and (4.87), respectively, where f, $h_1 \in L([a,b];\mathbb{R}_+)$ and μ, $\tau_1 : [a,b] \to [a,b]$ are measurable functions such that

$$\int_a^{t_1} f(s)\,ds = \int_a^{t_1} h_1(s)\,ds = 1, \quad \int_{t_2}^{t_3} f(s)\,ds = \int_{t_2}^{t_3} h_1(s)\,ds = 1,$$

$$f \equiv 0, \quad h_1 \equiv 0 \ \text{ on } \ [t_1,t_2],$$

$$\int_a^b f(s)\,ds \int_a^b h_1(s)\,ds = 4+\varepsilon,$$

and

$$\mu(t) = \begin{cases} t_3 & \text{for } \ t \in [a,t_2[\\ t_1 & \text{for } \ t \in [t_2,b] \end{cases}, \quad \tau_1(t) = \begin{cases} t_1 & \text{for } \ t \in [a,t_2[\\ t_3 & \text{for } \ t \in [t_2,b] \end{cases}.$$

For any z_1, $z_2 \in C([a,b];\mathbb{R})$ and $i = 1,2$, we put

$$T_i(z_1,z_2)(t) = \begin{cases} 0 & \text{for } \ t \in [a,t_1[\\ -z_i(t)|z_i(t)| & \text{for } \ t \in [t_1,t_2[\\ q_i & \text{for } \ t \in [t_2,b] \end{cases},$$

where q_1, $q_2 \in L([a,b];\mathbb{R})$ are such that

$$\int_{t_2}^{t_3} q_2(s)\,ds - \int_{t_2}^{t_3} q_1(s)\,ds \geq \frac{2}{t_2-t_1}\,. \tag{6.8}$$

Let

$$F_1(z_1,z_2)(t) \stackrel{\text{def}}{=} p(z_2)(t) + T_1(z_1,z_2)(t)$$
$$\text{for a.e. } \ t \in [a,b] \ \text{ and all } \ z_1,z_2 \in C([a,b];\mathbb{R}),$$
$$F_2(z_1,z_2)(t) \stackrel{\text{def}}{=} -g_1(z_1)(t) + T_2(z_1,z_2)(t)$$
$$\text{for a.e. } \ t \in [a,b] \ \text{ and all } \ z_1,z_2 \in C([a,b];\mathbb{R}),$$

and

$$\varphi_i(z_1,z_2) \stackrel{\text{def}}{=} 0 \ \text{ for } \ z_1,\ z_2 \in C([a,b];\mathbb{R}), \quad i = 1,2.$$

It is clear that the conditions (H_1) and (H_2) are satisfied and for any u_1, $u_2 \in C([a,b];\mathbb{R})$ the inequalities (4.1) and (4.2), (4.3) with $k = 1$ are fulfilled, where $g_0 = 0$ and

$$\eta_i \equiv 0, \quad \omega_i \equiv |q_i| \ \text{ for } \ i = 1,2.$$

Moreover, $p(1) \equiv f$ and $g_1(1) \equiv h_1$. Consequently, the assumptions of Theorem 4.1 with $k = 1$ are satisfied, except the second inequality in (4.5), instead of which the equality $PG_1 = 4+\varepsilon$ holds. However, the problem

$(1.1), (1.2)$ has no solution. Indeed, suppose that, on the contrary, (x_1, x_2) is a solution to this problem, i.e., $x_1(a) = 0$, $x_2(a) = 0$, and

$$x_1'(t) = f(t)x_2(\mu(t)) + T_1(x_1, x_2)(t) \quad \text{for a.e. } t \in [a, b],$$
$$x_2'(t) = -h_1(t)x_1(\tau_1(t)) + T_2(x_1, x_2)(t) \quad \text{for a.e. } t \in [a, b].$$

The last relations yield

$$x_1(t_1) = x_2(t_3), \quad x_2(t_1) = -x_1(t_1),$$
$$x_1(t_2) = \frac{x_1(t_1)}{1 + |x_1(t_1)|(t_2 - t_1)}, \quad x_2(t_2) = \frac{x_2(t_1)}{1 + |x_2(t_1)|(t_2 - t_1)},$$

and

$$x_1(t_3) = x_1(t_2) + x_2(t_1) + \int_{t_2}^{t_3} q_1(s)\, ds,$$
$$x_2(t_3) = x_2(t_2) - x_1(t_3) + \int_{t_2}^{t_3} q_2(s)\, ds,$$

whence we get

$$\int_{t_2}^{t_3} q_2(s) ds - \int_{t_2}^{t_3} q_1(s)\, ds < \frac{2}{t_2 - t_1},$$

which contradicts (6.8). The contradiction obtained proves that the problem $(1.1), (1.2)$ has no solution.

This example shows that the second strict inequality of (4.5) in Theorem 4.1 cannot be weakened.

Example 6.5. Let ε_1, $\varepsilon_2 \in [0, 1[$, $\varepsilon_1 + \varepsilon_2 > 0$, and let the operators ℓ_1, ℓ_2 be defined by (6.1), where f, $h \in L([a, b]; \mathbb{R}_+)$ and $\mu : [a, b] \to [a, b]$ is a measurable function such that (6.2) is satisfied. According to Example 6.1, the homogeneous problem $(3.14), (3.15)$ has a nontrivial solution. Therefore, by virtue of Lemma 6.1, there exist q_1, $q_2 \in L([a, b]; \mathbb{R})$ and c_1, $c_2 \in \mathbb{R}$ such that the problem $(1.3), (1.4)$ has no solution.

Let F_1, F_2 and φ_1, φ_2 be defined by (6.5) and (6.6), respectively. It is clear that the conditions (H_1) and (H_2) hold and for any $u_1, u_2 \in C([a, b]; \mathbb{R})$ the inequalities (4.1) and (4.2), (4.7) with $k = 1$ are fulfilled, where $p = \ell_1$, $g_0 = \ell_2$, $g_1 = 0$, and η_i, ω_i $(i = 1, 2)$ are defined by (6.7). Moreover, the inequality (6.3) holds and thus Proposition 5.5 implies

$$\big((1 - \varepsilon_1)\, \ell_1, (1 - \varepsilon_2)\, \ell_2\big) \in \widehat{\mathcal{S}}^2_{ab}(a).$$

It is clear that $(\ell_1, 0) \in \widehat{\mathcal{S}}^2_{ab}(a)$ (see, e.g., Proposition 5.5). Consequently, the assumptions of Theorem 4.2 with $k = 1$ are satisfied, except the condition (4.8), instead of which the condition (4.9) is fulfilled. However, the problem $(1.1), (1.2)$ has no solution.

This example shows that the assumption (4.8) of Theorem 4.2 cannot be replaced by the assumption (4.9), no matter how small $\varepsilon_1,\ \varepsilon_2 \in [0,1[$ with $\varepsilon_1+\varepsilon_2>0$ are.

Example 6.6. Let $\alpha\in]0,1[$, $\varepsilon_1,\ \varepsilon_2\in[0,1[$, $\varepsilon_1+\varepsilon_2>0$, and $a<t_1<t_2<t_3<b$. Put $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$ and choose $f,\ h\in L([a,b];\mathbb{R})$ such that (6.4) holds, $f\equiv 0$ and $h\equiv 0$ on $[t_2,t_3]$,

$$\int_a^{t_1} f(s)\bigg(\int_a^{s}|h(\xi)|\,d\xi\bigg)\,ds=\frac{3\alpha}{3+\varepsilon}\,,$$

$$\int_{t_1}^{t_2} f(s)\bigg(\Big(1+\frac{\varepsilon}{3}\Big)\int_a^{t_1}|h(\xi)|\,d\xi+\int_{t_1}^{s}h(\xi)\,d\xi\bigg)\,ds=1-\alpha,$$

$$\int_{t_3}^{b} f(s)\,ds=\frac{\varepsilon}{3}\,\min\left\{\frac{\int_{t_1}^{t_2}f(s)\,ds\int_a^{t_1}|h(s)|\,ds}{\int_a^{t_2}|h(s)|\,ds}\,,\,\frac{1}{\big(1+\frac{\varepsilon}{3}\big)\int_a^{t_1}|h(s)|\,ds+\int_{t_1}^{t_2}h(s)\,ds}\right\},$$

and

$$\int_{t_3}^{b} f(s)\bigg(\int_{t_3}^{s}|h(\xi)|\,d\xi\bigg)\,ds=1+\varepsilon.$$

Furthermore, we put

$$x_2(t)=\begin{cases}\Big(1+\dfrac{\varepsilon}{3}\Big)\displaystyle\int_a^{t}|h(s)|\,ds & \text{for } t\in[a,t_1[\\[2ex] \Big(1+\dfrac{\varepsilon}{3}\Big)\displaystyle\int_a^{t_1}|h(s)|\,ds+\int_{t_1}^{t}h(s)\,ds & \text{for } t\in[t_1,b]\end{cases}$$

and

$$x_1(t)=\begin{cases}\displaystyle\int_a^{t} f(s)x_2(s)\,ds & \text{for } t\in[a,t_2[\\[2ex] 1-\Big(1-\dfrac{\varepsilon}{3}\Big)(t_3-t_2)^{-1}(t-t_2) & \text{for } t\in[t_2,t_3[\,.\\[2ex] \dfrac{\varepsilon}{3}+\displaystyle\int_{t_3}^{t} f(s)x_2(s)\,ds & \text{for } t\in[t_3,b]\end{cases}$$

It is clear that $x_1(t_3)=\varepsilon/3$ and $x_1(b)\le-(1+\varepsilon/3)$, and thus there exists $t_0\in[t_3,b]$ such that $x_1(t_0)=-(1+\varepsilon/3)$. Let $g_0,\ g_1,\ p\in\mathcal{L}_{ab}$ be defined by (4.87) and

$$p(z)(t)\overset{\text{def}}{=}f(t)z(t)\ \text{ for a.e. } t\in[a,b]\ \text{ and all } z\in C([a,b];\mathbb{R}),$$

where

$$h_0(t)=\begin{cases}|h(t)| & \text{for } t\in[a,t_2[\\ 0 & \text{for } t\in[t_2,b]\end{cases},\quad h_1(t)=\begin{cases}0 & \text{for } t\in[a,t_3[\\ |h(t)| & \text{for } t\in[t_3,b]\end{cases},$$

$$\tau_0(t)=\begin{cases}t_0 & \text{for } t\in[a,t_1[\\ t_2 & \text{for } t\in[t_1,b]\end{cases},\quad \tau_1(t)=\begin{cases}a & \text{for } t\in[a,t_2[\\ t_2 & \text{for } t\in[t_2,b]\end{cases}.$$

Obviously, p, g_0, $g_1\in\mathcal{P}_{ab}$ and p, g_1 are a-Volterra operators. Moreover,

$$\int_a^b f(s)\bigg(\int_a^s h_0(\xi)\,d\xi\bigg)\,ds=$$

$$=\int_a^{t_1} f(s)\bigg(\int_a^s |h(\xi)|\,d\xi\bigg)\,ds+\int_{t_1}^{t_2} f(s)\bigg(\Big(1+\frac{\varepsilon}{3}\Big)\int_a^{t_1}|h(\xi)|\,d\xi+\int_{t_1}^{s}h(\xi)\,d\xi\bigg)\,ds-$$

$$-\bigg(\frac{\varepsilon}{3}\int_{t_1}^{t_2}f(s)\,ds\int_a^{t_1}|h(s)|\,ds-\int_{t_3}^{b}f(s)\,ds\int_a^{t_2}|h(s)|\,ds\bigg)\le$$

$$\le\frac{3\alpha}{3+\varepsilon}+1-\alpha=\frac{3+\varepsilon(1-\alpha)}{3+\varepsilon}<1,$$

and thus Proposition 5.5 yields

$$(p,g_0)\in\widehat{\mathcal{S}}^2_{ab}(a).$$

Furthermore, since

$$(1-\varepsilon_1)(1-\varepsilon_2)\int_a^b f(s)\bigg(\int_a^s h_1(\xi)\,d\xi\bigg)\,ds\le$$

$$\le(1-\varepsilon)\int_{t_3}^b f(s)\bigg(\int_{t_3}^s |h(\xi)|\,d\xi\bigg)\,ds=1-\varepsilon^2<1,$$

using Proposition 5.6 we get

$$\big((1-\varepsilon_1)\,p,-(1-\varepsilon_2)\,g_1\big)\in\widehat{\mathcal{S}}^2_{ab}(a).$$

On the other hand, it is easy to verify that (x_1,x_2) is a nontrivial solution to the homogeneous Cauchy problem (3.15) for the system

$$x_1'(t)=f(t)x_2(t)-f_0(t)x_1(t),\quad x_2'(t)=h(t)x_1(\tau_0(t)),$$

where

$$f_0(t)=\begin{cases}0 & \text{for } t\in[a,t_2[\,\cup[t_3,b]\\ \dfrac{3-\varepsilon}{3(t_3-t_2)-(3-\varepsilon)(t-t_2)} & \text{for } t\in[t_2,t_3[\end{cases}.$$

Therefore, according to Lemma 6.1, there exist $q_1,\, q_2 \in L([a,b];\mathbb{R})$ and $c_1,\, c_2 \in \mathbb{R}$ such that the Cauchy problem (1.4) for the system

$$x_1'(t) = f(t)x_2(t) - f_0(t)x_1(t) + q_1(t), \quad x_2'(t) = h(t)x_1(\tau_0(t)) + q_2(t)$$

has no solution.

Now let

$$F_1(z_1, z_2)(t) \overset{\text{def}}{=} f(t)z_2(t) - f_0(t)z_1(t) + q_1(t)$$
$$\text{for a.e. } t \in [a,b] \text{ and all } z_1,\, z_2 \in C([a,b];\mathbb{R}),$$
$$F_2(z_1, z_2)(t) \overset{\text{def}}{=} h(t)z_1(\tau_0(t)) + q_2(t)$$
$$\text{for a.e. } t \in [a,b] \text{ and all } z_1,\, z_2 \in C([a,b];\mathbb{R}),$$

and let φ_1, φ_2 be defined by (6.6). It is clear that the conditions (H_1) and (H_2) hold and for any $u_1,\, u_2 \in C([a,b];\mathbb{R})$ the inequalities (4.1) and (4.2), (4.7) with $k = 1$ are fulfilled, where η_i, ω_i $(i = 1, 2)$ are defined by (6.7). Consequently, the assumptions of Theorem 4.2 with $k = 1$ are satisfied, except the condition (4.8), instead of which the condition (4.10) is fulfilled. However, the problem (1.1), (1.2) has no solution.

This example shows that the assumption (4.8) of Theorem 4.2 cannot be replaced by the assumption (4.10), no matter how small $\varepsilon_1,\, \varepsilon_2 \in [0,1[$ with $\varepsilon_1 + \varepsilon_2 > 0$ are.

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