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MIXED TYPE BOUNDARY VALUE PROBLEMS IN THE LINEAR THEORY OF ELASTIC MIXTURES FOR BODIES WITH INTERIOR CUTS

Abstract. We consider two-dimensional mixed type boundary value problems for the equations of the linear theory of elastic mixtures. We assume that the elastic body under consideration contains interior cracks. On the exterior boundary of the body the mixed Dirichlet (displacement) and Neumann (traction) type conditions are given while on the crack sides the stress vector is prescribed. We apply generalized Kolosov–Muskhelishvili type representation formulas and reduce the mixed boundary value problem to the system of singular integral equations with discontinuous coefficients. Fredholm properties of the corresponding integral operator are studied and the index is found explicitly. With the help of the results obtained we prove unique solvability of the original mixed boundary value problem.

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რეზიუმე. ნაშრომში განხილულია დრეკად ნარევთა წრფივი თეორიის შერეული სასაზღვრო ამოცანები. იგულისხმება, რომ დრეკადი სხეული შეიცავს შიგა ბზარებს, რომლებზეც მოცემულია ძაბვის ვექტორის ზღვრული მნიშვნელობები. დრეკადი სხეულის გარე საზღვარზე მოცემულია დირიხლეს და ნეიმანის შერეული სასაზღვრო პირობები. კოლოსოვ-მუსხელიშვილის ტიპის განზოგადებულ წარმოდგენების გამოყენებით შერეული სასაზღვრო ამოცანა დაიყვანება წყვეტილკოეფიციენტებიან სინგულარულ ინტეგრალურ განტოლებათა სისტემაზე. შესწავლილია შესაბამისი სინგულარული ოპერატორის ფრედჰოლმურობა და დადგენილია მისი ინდექსი. ამ შედეგების გამოყენებით დამტკიცებულია შერეული სასაზღვრო ამოცანის ცალსახად ამოხსნადობა.

1. Introduction

Here we treat a two-dimensional mathematical model of the linear theory of elastic mixtures (for details concerning the mathematical and mechanical modelling see, e.g., [6], [12], [5], [11]). The corresponding system of differential equations of statics generates a second order 4×4 matrix strongly elliptic partial differential operator with constant coefficients.

Recently, in the references [2] and [3], M. Basheleishvili has constructed representation formulas for solutions to this system by means of analytic vector-functions. These formulas are nontrivial generalizations of the well-known formulas of Kolosov–Muskhelishvili which played a crucial role in the classical elasticity theory (see [9]).

With the help of the new representation formulas, the basic BVPs of the linear theory of elastic mixtures have been investigated in [3] and [4] for regular domains. The same problems by the potential method have been studied in [1]. The corresponding three-dimensional problems have been solved in [11] using the multi-dimensional boundary integral and pseudo-differential equations technique.

In this paper we consider a general two-dimensional mixed type boundary value problem for elastic bodies with interior cracks. The exterior boundary of the body is divided into several disjoint parts where the Dirichlet (displacement) and Neumann (traction) type conditions are given while on the crack sides the stress vector is prescribed. We apply the representation formulas obtained in [2] and reduce the mixed boundary value problem to the system of one-dimensional singular integral equations with discontinuous coefficients. We study the Fredholm properties of the corresponding matrix integral operator. The index of the operator is found explicitly. Further, we establish that the system of integral equations is solvable and study the smoothness of solutions, which are densities of the corresponding *complex potentials* (analytic vector-functions represented as Cauchy type integrals) involved in the general representation formulas. With the help of the results obtained we prove unique solvability of the original mixed boundary value problem.

2. Formulation of the Problem and General Representation Formulas of Solutions

Let Ω^+ be a domain of finite diameter in $\mathbb{R}^2$ and $S := \partial\Omega^+$ be its boundary of the class $C^{2,\alpha}$ with $1/2 < \alpha < 1$. Put $\overline{\Omega}^+ = \Omega^+ \cup S$. Further, let the contour S be divided into $2p$ disjoint open parts S_j, $j = 1, 2, \dots, 2p$. We denote the end points of the arc S_j by c_j and c_{j+1}. We set $c_{2p+1} = c_1$. For simplicity, in what follows we assume that S is a simple curve and the positive direction on it is selected so that if S is went around in this direction, the interior region Ω^+ is on the left. Thus we have the following decomposition

$$S = \overline{S}_1 \cup \overline{S}_2 \ \cdots \ \cup \overline{S}_{2p}, \quad \overline{S}_j = S_j \cup c_j \cup c_{j+1}.$$

Denote

$$S_D = S_1 \cup \cdots \cup S_{2p-1}, \quad S_T = S_2 \cup \cdots \cup S_{2p}, \quad \Omega^- = \mathbb{R}^2 \setminus (\Omega^+ \cup S).$$

Let the region $\overline{\Omega}^+$ be occupied by a material representing a two component elastic mixture. Moreover, we assume that the elastic body contains an interior crack along some simple arc $\Sigma \in C^{2,\alpha}$, $\frac{1}{2} < \alpha < 1$. We denote the end points of Σ by e_1 and e_2, and select the positive direction from e_1 to e_2. The totality of points $\{e_1, e_2, c_1, \dots, c_{2p}\}$ will be referred to as *singular points*. We assume that Σ is a part of some closed, simple, $C^{2,\alpha}$-smooth curve $S_0 \subset \Omega^+$ surrounding a region $\overline{\Omega_0} \subset \Omega^+$. Further, let $\Omega^+_\Sigma := \Omega^+ \setminus \overline{\Sigma}$.

The basic differential equations of statics in the linear theory of elastic mixtures have the form (for the mechanical description of the corresponding model see, e.g., [11], [1] and the references therein)

$$\begin{aligned} &a_1 \Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2 \Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{2.1}$$

where $u' = (u_1, u_2)^\top$ and $u'' = (u_3, u_4)^\top$ are the partial displacement vectors,

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2 \rho_2 \rho^{-1}, \quad a_2 = \mu_2 - \lambda_5,\\ c = \mu_3 + \lambda_5, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2 \rho_1 \rho^{-1}, \quad \rho = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4,\\ d = \mu_3 + \lambda_3 - \lambda_5 - \alpha_2 \rho_1 \rho^{-1} \equiv \mu_3 + \lambda_4 - \lambda_5 + \alpha_2 \rho_2 \rho^{-1}, \end{gathered} \tag{2.2}$$

$\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the two dimensional Laplace operator. Here λ_1, λ_2, λ_3, λ_4, λ_5, μ_1, μ_2, and μ_3 are material constants characterizing the mechanical properties of the elastic mixture and satisfying the inequalities

$$\begin{gathered} \lambda_5 < 0, \quad \mu_1 > 0, \quad \mu_1 \mu_2 > \mu_3^2, \quad \lambda_1 - \frac{\alpha_2 \rho_2}{\rho} + \frac{2}{3}\mu_1 > 0,\\ \Big(\lambda_1 - \frac{\alpha_2 \rho_2}{\rho} + \frac{2}{3}\mu_1\Big)\Big(\lambda_2 + \frac{\alpha_2 \rho_1}{\rho} + \frac{2}{3}\mu_2\Big) \geq \Big(\lambda_3 - \frac{\alpha_2 \rho_1}{\rho} + \frac{2}{3}\mu_3\Big)^2. \end{gathered} \tag{2.3}$$

These conditions imply that the density of the potential energy is positive definite with respect to the generalized deformations and the matrix differential operator generated by the left-hand side expressions in the equations (2.1) is self-adjoint and strongly elliptic (for details see [11]).

The partial stress tensors $[\tau'_{kj}(u)]_{2\times 2}$ and $[\tau''_{kj}(u)]_{2\times 2}$ are related to the partial displacement vectors by the formulas

$$\begin{aligned} \tau'_{11}(u) &= \Big(\lambda_1 - \frac{\alpha_2 \rho_2}{\rho}\Big) \operatorname{div} u' + \Big(\lambda_3 - \frac{\alpha_2 \rho_1}{\rho}\Big) \operatorname{div} u'' + 2\mu_1 \frac{\partial u_1}{\partial x_1} + 2\mu_3 \frac{\partial u_3}{\partial x_1},\\ \tau'_{21}(u) &= (\mu_1 - \lambda_5) \frac{\partial u_1}{\partial x_2} + (\mu_1 + \lambda_5) \frac{\partial u_2}{\partial x_1} + (\mu_3 + \lambda_5) \frac{\partial u_3}{\partial x_2} + (\mu_3 - \lambda_5) \frac{\partial u_4}{\partial x_1},\\ \tau'_{12}(u) &= (\mu_1 + \lambda_5) \frac{\partial u_1}{\partial x_2} + (\mu_1 - \lambda_5) \frac{\partial u_2}{\partial x_1} + (\mu_3 - \lambda_5) \frac{\partial u_3}{\partial x_2} + (\mu_3 + \lambda_5) \frac{\partial u_4}{\partial x_1},\\ \tau'_{22}(u) &= \Big(\lambda_1 - \frac{\alpha_2 \rho_2}{\rho}\Big) \operatorname{div} u' + \Big(\lambda_3 - \frac{\alpha_2 \rho_1}{\rho}\Big) \operatorname{div} u'' + 2\mu_1 \frac{\partial u_2}{\partial x_2} + 2\mu_3 \frac{\partial u_4}{\partial x_2}, \end{aligned} \tag{2.4}$$

$$\tau''_{11}(u)=\Big(\lambda_4+\frac{\alpha_2\rho_2}{\rho}\Big)\operatorname{div}u'+\Big(\lambda_2+\frac{\alpha_2\rho_1}{\rho}\Big)\operatorname{div}u''+2\mu_3\,\frac{\partial u_1}{\partial x_1}+2\mu_2\,\frac{\partial u_3}{\partial x_1},$$
$$\tau''_{21}(u)=(\mu_3+\lambda_5)\,\frac{\partial u_1}{\partial x_2}+(\mu_3-\lambda_5)\,\frac{\partial u_2}{\partial x_1}+(\mu_2-\lambda_5)\,\frac{\partial u_3}{\partial x_2}+(\mu_2+\lambda_5)\,\frac{\partial u_4}{\partial x_1},$$
$$\tau''_{12}(u)=(\mu_3-\lambda_5)\,\frac{\partial u_1}{\partial x_2}+(\mu_3+\lambda_5)\,\frac{\partial u_2}{\partial x_1}+(\mu_2+\lambda_5)\,\frac{\partial u_3}{\partial x_2}+(\mu_2-\lambda_5)\,\frac{\partial u_4}{\partial x_1},$$
$$\tau''_{22}(u)=\Big(\lambda_4+\frac{\alpha_2\rho_2}{\rho}\Big)\operatorname{div}u'+\Big(\lambda_2+\frac{\alpha_2\rho_1}{\rho}\Big)\operatorname{div}u''+2\mu_3\,\frac{\partial u_2}{\partial x_2}+2\mu_2\,\frac{\partial u_4}{\partial x_2},$$

where $u := (u', u'')^\top \equiv (u_1, u_2, u_3, u_4)^\top$.

The partial stress vectors $\mathcal{T}' = (\mathcal{T}'_1, \mathcal{T}'_2)^\top$ and $\mathcal{T}'' = (\mathcal{T}''_1, \mathcal{T}''_2)^\top$ acting on an arc element with the normal $n = (n_1, n_2)$ are calculated then as

$$\begin{aligned}\mathcal{T}'_1 &\equiv (Tu)_1 = \tau'_{11}n_1 + \tau'_{21}n_2, \quad \mathcal{T}'_2 \equiv (Tu)_2 = \tau'_{12}n_1 + \tau'_{22}n_2,\\ \mathcal{T}''_1 &\equiv (Tu)_3 = \tau''_{11}n_1 + \tau''_{21}n_2, \quad \mathcal{T}''_2 \equiv (Tu)_4 = \tau''_{12}n_1 + \tau''_{22}n_2.\end{aligned} \tag{2.5}$$

The operator $T = T(\partial, n) = [T_{kj}(\partial, n)]_{4\times4}$ is called the *stress operator* in the linear theory of elastic mixtures [11].

It is convenient to introduce the following notation

$$Tu := (\mathcal{T}', \mathcal{T}'')^\top \equiv \big((Tu)_1, (Tu)_2, (Tu)_3, (Tu)_4\big)^\top. \tag{2.6}$$

Throughout the paper we assume that $n(x) = (n_1(x), n_2(x))$ is the outward normal vector on S and on S_0 at the point $x \in S \cup S_0$. This uniquely defines the positive direction of the normal vector on Σ.

A vector $u = (u', u'')^\top$ is said to be *regular* in the region Ω^+_Σ if it satisfies the following conditions:

(i) $u_l \in C^2(\Omega^+_\Sigma)$ and u_l are continuously extendable on S and on $\overline{\Sigma}$ for $l = \overline{1,4}$;

(ii) the components of the vector Tu are continuously extendable on S and on $\overline{\Sigma}$ except possibly at the singular points e_1, e_2, c_j, $j = 1, 2, \dots, 2p$; in a neighbourhood of a singular point $x_0 \in \{e_1, e_2, c_1, \dots, c_{2p}\}$ the functions $(Tu(x))_l$ admit the estimate

$$[Tu(x)]_l = \mathcal{O}\left(|x - x_0|^{-\beta}\right), \ \ x \in \Omega^+_\Sigma, \ \ 0 \le \beta < 1, \ \ l = \overline{1,4},$$

where $|x - x_0|$ is the Euclidian distance between the points x and x_0.

Now we are in a position to formulate the mixed boundary value problem: find a vector $u = (u', u'')^\top$ satisfying the system of differential equations (2.1) in Ω^+_Σ and the following boundary conditions:

$$[u]^+ = f^{(1)}(t), \ \ t \in S_D, \tag{2.7}$$
$$[Tu]^+ = f^{(2)}(t), \ \ t \in S_T, \tag{2.8}$$
$$[Tu]^+ = f^+(t), \ \ [Tu]^- = f^-(t), \ \ t \in \Sigma, \tag{2.9}$$

where $f^{(k)} = (f^{(k)}_1, f^{(k)}_2, f^{(k)}_3, f^{(k)}_4)^\top$, $k = 1, 2$, and $f^\pm = (f^\pm_1, f^\pm_2, f^\pm_3, f^\pm_4)^\top$ are given vector functions with the following smoothness properties:

$$f^{(1)}_l \in H(S_D), \ \ \partial_\tau f^{(1)}_l \in H^*(S_D), \ \ f^{(2)}_l \in H^*(S_T), \ \ f^\pm_l \in H^*(\Sigma); \tag{2.10}$$

here ∂_τ denotes the tangential derivative (or the derivative with respect to the arc parameter), while H and H^* stand for the well known Muskhelishvili spaces (see [9], [10]). Recall that for an open simple arc $\mathcal{M}$ with the end points t_1 and t_2, the symbol $H(\mathcal{M})$ denotes the set of Hölder continuous functions on $\overline{\mathcal{M}}$ with some exponent $0<\alpha<1$ and $H^*(\mathcal{M})$ denotes the set of functions which belong to $H(\mathcal{M}_0)$ for arbitrary $\overline{\mathcal{M}_0}\subset\mathcal{M}$ and near the end points t_1 and t_2 can be represented as $\varphi(t)|t-t_j|^{-\kappa}$ with $\varphi\in H(\mathcal{M})$ and $0\le\kappa<1$. For open disjoint arcs $\mathcal{M}_j$, $j=\overline{1,q}$, and $\mathcal{M}=\mathcal{M}_1\cup\cdots\cup\mathcal{M}_q$ we define by $H(\mathcal{M})$ and $H^*(\mathcal{M})$ the set of functions whose restrictions on $\mathcal{M}_j$ belong to $H(\mathcal{M}_j)$ and $H^*(\mathcal{M}_j)$, respectively.

The symbols $[\,\cdot\,]^+_{S\cup\Sigma}$ and $[\,\cdot\,]^-_{S\cup\Sigma}$ denote the one-sided limits on $S\cup\Sigma$ from the left and from the right, respectively, in accordance with the positive direction chosen above.

Let us introduce the complex vectors

$$\begin{aligned}
&U=(U_1,U_2):=(u_1+iu_2,u_3+iu_4)^\top,\\
&\mathcal{F}U=\big((\mathcal{F}U)_1,(\mathcal{F}U)_2\big)^\top:=\big((Tu)_2-i(Tu)_1,\ (Tu)_4-i(Tu)_3\big)^\top,
\end{aligned}\tag{2.11}$$

where u_j, $j=\overline{1,4}$, are real functions, the components of the partial displacement vectors.

The boundary conditions (2.7)–(2.9) can be rewritten then as

$$[U]^+=F^{(1)}(t),\quad t\in S_D,\tag{2.12}$$

$$[\mathcal{F}U]^+=-iF^{(2)}(t),\quad t\in S_T,\tag{2.13}$$

$$[\mathcal{F}U]^\pm=-iF^\pm(t),\quad t\in\Sigma,\tag{2.14}$$

where

$$\begin{aligned}
F^{(k)}&=\big(f_1^{(k)}+if_2^{(k)},f_3^{(k)}+if_4^{(k)}\big)^\top,\quad k=1,2,\\
F^\pm&=\big(f_1^\pm+if_2^\pm,\ f_3^\pm+if_4^\pm\big)^\top.
\end{aligned}\tag{2.15}$$

The Kolosov–Muskhelishvili type representation formula for a solution of the system (2.1) obtained in [2] has the form

$$U(z)\equiv U(x_1,x_2)=m\varphi(z)+\ell z\overline{\varphi'(z)}+\overline{\psi(z)},\quad z=x_1+ix_2,\tag{2.16}$$

where $\varphi=(\varphi_1,\varphi_2)^\top$ and $\psi=(\psi_1,\psi_2)^\top$ are arbitrary holomorphic vector functions in Ω^+_Σ, $\varphi'(z)=\varphi'_z(z)$ is the derivative with respect to the complex variable z, the over-bar denotes complex conjugation, m and ℓ are real matrices,

$$m=\begin{bmatrix}m_1 & m_2\\ m_2 & m_3\end{bmatrix},\quad \ell=\frac12[\begin{bmatrix}l_4 & l_5\\ l_5 & l_6\end{bmatrix}$$

with

$$m_1=l_1+\frac{l_4}{2}=\frac12\Big(\frac{a_2}{d_2}+\frac{a_2+b_2}{d_1}\Big)>0,\quad m_2=l_2+\frac{l_5}{2}=-\frac12\Big(\frac{c}{d_2}+\frac{c+d}{d_1}\Big),$$

$$m_3=l_3+\frac{l_6}{2}=\frac12\Big(\frac{a_1}{d_2}+\frac{a_1+b_1}{d_1}\Big)>0,$$

$$l_1 = \frac{a_2}{d_2}, \quad l_2 = -\frac{c}{d_2}, \quad l_3 = \frac{a_1}{d_2}\,,$$
$$d_2 = a_1 a_2 - c^2 = \Delta_1 - \lambda_5 a_0 > 0,$$
$$d_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2 = \Delta_1 + a + b > 0,$$
$$\Delta_1 = \mu_1 \mu_2 - \mu_3^2 > 0,$$
$$a = \mu_1(b_2 - \lambda_5) + \mu_2(b_1 - \lambda_5) - 2\mu_3(d + \lambda_5) > 0,$$
$$b = (b_1 - \lambda_5)(b_2 - \lambda_5) - (d + \lambda_5)^2 > 0,$$
$$a_0 = \mu_1 + \mu_2 + 2\mu_3 \equiv a_1 + a_2 + 2c > 0,$$
$$b_0 = b_1 + b_2 + 2d \equiv b_1 - \lambda_5 + b_2 - \lambda_5 + 2(d + \lambda_5) > 0,$$
$$\Delta_0 := \det m = \frac{4\Delta_1 + 2a + b - \lambda_5(2a_0 + b_0)}{4d_1 d_2} > 0.$$

For the generalized stress vector $\mathcal{F}U$ we have (see (2.11))

$$\mathcal{F}U(z) = \frac{\partial}{\partial \tau(z)} \Big[(A - 2I)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)} \Big], \tag{2.17}$$

where I stands for the unit 2×2 matrix,

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = 2\mu m, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = 2\mu \ell, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}.$$

Here

$$\frac{\partial}{\partial \tau(z)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}\,,$$

where $n = (n_1, n_2)$ is a unit vector. It is evident that for $z \in S \cup S_0$ this operator is a tangential differentiation operator at the point z.

We can easily see that

$$\begin{aligned}
A_1 &= 2 + B_1 + 2\lambda_5 \frac{a_2 + c}{d_2} = \\
&= \frac{d_1 + d_2 + a_1 b_2 - cd}{d_1} + \lambda_5 \Big(\frac{a_2 + c}{d_2} + \frac{a_2 + b_2 + c + d}{d_1} \Big), \\
A_4 &= 2 + B_4 + 2\lambda_5 \frac{a_1 + c}{d_2} = \\
&= \frac{d_1 + d_2 + a_2 b_1 - cd}{d_1} + \lambda_5 \Big(\frac{a_1 + c}{d_2} + \frac{a_1 + b_1 + c + d}{d_1} \Big), \\
A_2 &= B_2 - 2\lambda_5 \frac{a_1 + c}{d_2} = \frac{cb_1 - da_1}{d_1} - \lambda_5 \Big(\frac{a_1 + c}{d_2} + \frac{a_1 + b_1 + c + d}{d_1} \Big), \\
A_3 &= B_3 - 2\lambda_5 \frac{a_2 + c}{d_2} = \frac{cb_2 - da_2}{d_1} - \lambda_5 \Big(\frac{a_2 + c}{d_2} + \frac{a_2 + b_2 + c + d}{d_1} \Big).
\end{aligned}$$

Further, $\det A = 4\Delta_0 \Delta_1 > 0$ since $\det \mu = \Delta_1 > 0$ and $\det m = \Delta_0 > 0$.

With the help of the formulas (2.16) and (2.17), we derive

$$2\mu U(z) = A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}, \tag{2.18}$$

and

$$\mathcal{Q}(z) + D = (A - 2I)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}, \tag{2.19}$$

where $D = (D_1, D_2)^\top$ is an arbitrary constant complex vector and

$$\begin{aligned} \mathcal{Q}(z) : &= \int_{z_0}^{z} \mathcal{F}U\, ds = \Big(Q_2(z) - iQ_1(z), Q_4(z) - iQ_3(z)\Big)^\top, \\ Q_k(z) &= \int_{z_0}^{z} (Tu)_k\, ds, \quad k = 1, 2, 3, 4; \end{aligned} \tag{2.20}$$

here the integration is performed with respect to the arc parameter s along an arbitrary smooth arc which lies in the domain Ω_Σ^+ and connects some fixed point z_0 to the reference point z.

From the representations (2.16) and (2.17) we can derive the following results:

Conclusion (i). If $\mathcal{F}U(z) = 0$ in Ω_Σ^+, then $\psi(z) = \delta$, where $\delta = (\delta_1, \delta_2)^\top$ is an arbitrary complex constant vector, while $\varphi(z) = i\widetilde{\varepsilon}Rz + \gamma$, where R is an arbitrary real scalar constant, $\gamma = (\gamma_1, \gamma_2)^\top$ is an arbitrary complex constant vector, and $\widetilde{\varepsilon} = (\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2)^\top$ is a real constant vector with

$$\begin{gathered} \widetilde{\varepsilon}_1 = \Delta_2^{-1}[H_1A_2 - H_2(2 - A_4)], \quad \widetilde{\varepsilon}_2 = \Delta_2^{-1}[H_1(2 - A_1) - H_2A_3], \\ H_1 = (\mu_2 + \mu_3)(2 - A_4) - (\mu_1 + \mu_3)A_3, \\ H_2 = (\mu_2 + \mu_3)A_2 - (\mu_1 + \mu_3)(2 - A_1), \\ \Delta_2 = \det(A - 2I) > 0. \end{gathered}$$

Note that $|H_1| + |H_2| \neq 0$ and $H_1 \neq H_2$. Therefore, without loss of generality, in what follows we assume that $H_1 \neq 0$. This gives us possibility to represent the vector φ in the form $\varphi(z) = i\varepsilon Rz + \gamma$ in Ω_Σ^+, where R is again an arbitrary real scalar constant, $\gamma = (\gamma_1, \gamma_2)^\top$ is an arbitrary complex constant vector, and $\varepsilon = (\varepsilon_1, \varepsilon_2)^\top$ with

$$\varepsilon_1 = \frac{1}{\Delta_2}\,[A_2 - H_0(2 - A_4)], \quad \varepsilon_2 = \frac{1}{\Delta_2}\,[2 - A_1 - H_0A_3], \quad H_0 = H_2[H_1]^{-1}.$$

Thus we have

$$\varphi(z) = i\varepsilon Rz + \gamma, \quad \psi(z) = \delta. \tag{2.21}$$

Note that the vector $U(z)$ constructed by the formula (2.16) with $\varphi(z)$ and $\psi(z)$ as in (2.21) corresponds to the so called “rigid motion” vector in the theory of elastic mixtures.

Conclusion (ii). If $U(z) = 0$ for $z \in \Omega_\Sigma^+$, then $\varphi(z) = \gamma$ and $\psi(z) = -2^{-1}\mu^{-1}A\overline{\gamma}$, where γ is an arbitrary complex constant vector. Therefore, if $\varphi(z_0) = 0$ for some point $z_0 \in \Omega_\Sigma^+$, then φ and ψ vanish identically in Ω_Σ^+.

3. Green's Formulas and the Uniqueness Results

Let $\Omega \subset \mathbb{R}^2$ be a bounded region with a smooth boundary $\partial\Omega$ and a real vector-function $u = (u', u'')^\top$ be a regular solution to the system (2.1) in Ω. Then there holds the following Green's identity

$$\int\limits_{\Omega} W(u,u)\,dx = \int\limits_{\partial\Omega} [u]^+ \cdot [Tu]^+\,ds, \tag{3.1}$$

where $W(u,u) \geq 0$ is the so called density of the potential energy [11], [1]. Note that here and in what follows we employ the notation

$$a \cdot b := \sum_{j=1}^{p} a_j b_j \ \text{ for } \ a,\, b \in \mathbb{R}^p \ \text{ or } \ a,\, b \in \mathbb{C}^p.$$

The general solution of the equation $W(u,u) = 0$ is written as

$$u = (u', u'')^\top, \quad u' = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} + b_0' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad u'' = \begin{pmatrix} a_1'' \\ a_2'' \end{pmatrix} + b_0' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{3.2}$$

where a_j', a_j'', and b_0' are arbitrary constants (for details see [11], [1]). The vector $u = (u', u'')^\top$ defined by (3.2) is called a *generalized rigid displacement vector*. It is evident that if the vector (3.2) vanishes at two points, then the constants a_j', a_j'', and b_0' are equal to zero.

It can easily be shown that Green's formula (3.1) holds also for regular vector functions in Ω_Σ^+,

$$\int\limits_{\Omega_\Sigma^+} W(u,u)\,dx = \int\limits_{S} [u]^+ \cdot [Tu]^+\,ds + \int\limits_{\Sigma} \big\{[u]^+ \cdot [Tu]^+ - [u]^- \cdot [Tu]^-\big\}\,ds. \tag{3.3}$$

Note that $u \cdot Tu = \Im\{U \cdot \overline{\mathcal{F}U}\}$ due to the equalities (2.11). Therefore, (3.3) implies

$$\begin{aligned}
\int\limits_{\Omega_\Sigma^+} W(u,u)\,dx = \Im \int\limits_{S} \Big[2^{-1}\mu^{-1}\big(A\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big)\Big]^+ \times& \\
\times\, d\big[(A-2I)\overline{\varphi(t)} + B\bar{t}\varphi'(t) + 2\mu\psi(t)\big]^+& \\
+ \Im \int\limits_{\Sigma} \Big\{ \Big[2^{-1}\mu^{-1}\big(A\varphi(t) + B\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big)\Big]^+ \times& \\
\times\, d\big[(A-2I)\overline{\varphi(t)} + B\bar{t}\varphi'(t) + 2\mu\psi(t)\big]^+& \\
- \Big[2^{-1}\mu^{-1}\big(A\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big)\Big]^- \times& \\
\times\, d\big[(A-2I)\overline{\varphi(t)} + B\bar{t}\varphi'(t) + 2\mu\psi(t)\big]^- \Big\},&
\end{aligned} \tag{3.4}$$

where here and in what follows the differential $d[\cdot]$ is taken with respect to the arc parameter (length) s.

By standard arguments, from Green's formula (3.1) we derive the following uniqueness theorem.

Theorem 3.1. *The homogeneous boundary value problem* (2.1), (2.7)–(2.9) ($f^{(1)} = f^{(2)} = f^+ = f^- = 0$) *has only the trivial solution.*

Remark 3.2. Let a pair of functions $u' = (u_1, u_2)^\top$ and $u'' = (u_3, u_4)^\top$ be a solution to the system (2.1) in the exterior domain $\Omega^- := \mathbb{R}^2 \setminus \overline{\Omega^+}$. Moreover, let they be bounded at infinity, while their first order derivatives decay as $\mathcal{O}(|x|^{-2})$. Then there holds Green's formula

$$\int\limits_{\Omega^-} W(u,u)\,dx = -\int\limits_{\partial\Omega^-} [u]^- \cdot [Tu]^-\,ds, \tag{3.5}$$

which implies that the homogeneous exterior Dirichlet and Neumann boundary value problems (with given displacements and stresses on S, respectively) possess only the trivial solutions.

Note that if the holomorphic vector functions φ and ψ corresponding to the vectors $u' = (u_1, u_2)^\top$ and $u'' = (u_3, u_4)^\top$ are bounded at infinity and their derivatives decay as $\mathcal{O}(|z|^{-2})$, then the formula similar to (3.4) still holds

$$\begin{aligned}\int\limits_{\Omega^-} W(u,u)\,dx = -\Im \int\limits_{S} \Big[2^{-1}\mu^{-1}\big(A\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big)\Big]^- \times \\ \times d\big[(A-2I)\overline{\varphi(t)} + B\bar{t}\varphi'(t) + 2\mu\psi(t)\big]^-. \end{aligned}\tag{3.6}$$

4. Reduction to a System of Integral Equations

With the help of the representation formulas (2.18) and (2.19) and the boundary conditions (2.7)–(2.9) (see also (2.12)–(2.14)), the original BVP for $u = (u', u'')^\top$ is reduced to the following problem for the holomorphic vectors φ and ψ:

$$\big[A\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big]^+ = 2\mu F^{(1)}(t), \ \ t \in S_D, \tag{4.1}$$

$$\begin{aligned}\big[(A-2I)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big]^+ = \\ = -i\int\limits_{c_{2j}}^{t} F^{(2)}(\tau)\,ds + D^{(j)}, \ \ t \in S_{2j}, \ \ j = \overline{1,p}, \end{aligned}\tag{4.2}$$

$$\big[(A-2I)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big]^+ = -i\int\limits_{e_1}^{t} F^+(\tau)\,ds + C^+, \ \ t \in \Sigma, \tag{4.3}$$

$$\big[(A-2I)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}\big]^- = -i\int\limits_{e_1}^{t} F^-(\tau)\,ds + C^-, \ \ t \in \Sigma, \tag{4.4}$$

where $C^{\pm} = (C_1^{\pm}, C_2^{\pm})^{\top}$ and $D^{(j)} = (D_1^{(j)}, D_2^{(j)})^{\top}$, $j = 1, 2, \dots, p$, are arbitrary complex constant vectors.

We look for the unknown holomorphic vectors (complex potentials) φ and ψ in the form of Cauchy type integrals:

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_S \frac{g(t)}{t-z}\,dt + \frac{1}{2\pi i}\int\limits_\Sigma \frac{h(t)}{t-z}\,dt + \frac{M}{2\pi i}[1+\Omega(z)], \quad z \in \Omega_\Sigma^+, \tag{4.5}$$

$$\begin{aligned}2\mu\psi(z) &= \frac{1}{2\pi i}\int\limits_S \frac{A\overline{g(t)} - B\overline{t}g'(t)}{t-z}\,dt - \frac{1}{2\pi i}\int\limits_\Sigma \frac{(A-2I)\overline{h(t)} + B\overline{t}h'(t)}{t-z}\,dt + \\ &+\frac{1}{2\pi i}\int\limits_\Sigma \frac{\overline{\omega(t)}}{t-z}\,dt - \frac{\overline{\chi}}{2\pi i}\int\limits_\Sigma \frac{(\overline{t}-\overline{e}_1)}{t-z}\,dt - \frac{BM\overline{e}_1}{2\pi i(e_2-e_1)}\,\ln\frac{z-e_2}{z-e_1} + \\ &+\frac{A\overline{M}}{2\pi i}\,\ln(z-e_2), \quad z \in \Omega_\Sigma^+,\end{aligned} \tag{4.6}$$

where the densities $g = (g_1, g_2)^{\top}$, $h = (h_1, h_2)^{\top}$, and $\omega = (\omega_1, \omega_2)^{\top}$ are Hölder continuous vector functions and have the first order derivatives of the class H^* on S and Σ with the nodal (singular) points $\{e_1, e_2, c_1, \dots, c_{2p}\}$. In addition, we assume that

$$h(e_1) = h(e_2) = 0, \tag{4.7}$$

$$\omega(e_1) = 0, \quad \omega(e_2) = -2M, \tag{4.8}$$

with

$$M = (M_1, M_2)^{\top} = \frac{i}{2}\int\limits_\Sigma \big[F^+(t) - F^-(t)\big]\,ds. \tag{4.9}$$

The vector function $\Omega(z)$ involved in (4.5) has the form

$$\Omega(z) = \frac{(z-e_2)\ln(z-e_2) - (z-e_1)\ln(z-e_1)}{e_2-e_1}\,, \tag{4.10}$$

while the vector function $\chi = (\chi_1, \chi_2)^{\top}$ involved in (4.6) reads as follows

$$\chi = \frac{(A-2I)M}{e_2-e_1} + \frac{B\overline{M}}{\overline{e}_2 - \overline{e}_1}\,. \tag{4.11}$$

Substituting the expressions (4.5) and (4.6) into the representations (2.18) and (2.19), we arrive at the relations

$$\begin{aligned}&A\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)} = \\ &= \frac{A}{2\pi i}\int\limits_S g(t)\,d_t\ln\frac{t-z}{\overline{t}-\overline{z}} - \frac{A-2I}{2\pi i}\int\limits_\Sigma h(t)\,d_t\ln\frac{t-z}{\overline{t}-\overline{z}} + \\ &+\frac{A-I}{\pi i}\int\limits_\Sigma \frac{h(t)}{t-z}\,dt - \frac{B}{2\pi i}\bigg[\int\limits_S \overline{g(t)}\,d_t\Big(\frac{t-z}{\overline{t}-\overline{z}}\Big) + \int\limits_\Sigma \overline{h(t)}\,d_t\Big(\frac{t-z}{\overline{t}-\overline{z}}\Big)\bigg] - \\ &-\frac{1}{2\pi i}\int\limits_\Sigma \frac{\omega(t)}{\overline{t}-\overline{z}}\,d\overline{t} + \frac{\chi}{2\pi i}\int\limits_\Sigma \frac{(t-e_1)}{\overline{t}-\overline{z}}\,d\overline{t} -\end{aligned}$$

$$-\frac{B\overline{M}(z-e_1)}{2\pi i(\overline{e}_2-\overline{e}_1)}\ln\frac{\overline{z}-\overline{e}_2}{\overline{z}-\overline{e}_1}+\frac{AM}{2\pi i}\big[1+\Omega(z)-\ln(\overline{z}-\overline{e}_2)\big], \tag{4.12}$$

$$(A-2I)\varphi(z)+Bz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}=$$

$$-\frac{1}{\pi i}\int\limits_S\frac{g(t)}{t-z}\,dt+\frac{A}{2\pi i}\int\limits_S g(t)\,d_t\ln\frac{t-z}{\overline{t}-\overline{z}}+\frac{A-2I}{\pi i}\int\limits_\Sigma\frac{h(t)}{t-z}\,dt-$$

$$-\frac{A-2I}{2\pi i}\int\limits_\Sigma h(t)\,d_t\ln\frac{t-z}{\overline{t}-\overline{z}}-$$

$$-\frac{B}{2\pi i}\bigg[\int\limits_S\overline{g(t)}\,d_t\Big(\frac{t-z}{\overline{t}-\overline{z}}\Big)+\int\limits_\Sigma\overline{h(t)}\,d_t\Big(\frac{t-z}{\overline{t}-\overline{z}}\Big)\bigg]-$$

$$-\frac{1}{2\pi i}\int\limits_\Sigma\frac{\omega(t)\,d\overline{t}}{\overline{t}-\overline{z}}+\frac{\chi}{2\pi i}\int\limits_\Sigma\frac{(t-e_1)\,d\overline{t}}{\overline{t}-\overline{z}}-\frac{AM}{2\pi i}\ln(\overline{z}-\overline{e}_2)-$$

$$-\frac{B\overline{M}(z-e_1)}{2\pi i(\overline{e}_2-\overline{e}_1)}\ln\frac{\overline{z}-\overline{e}_2}{\overline{z}-\overline{e}_1}+\frac{(A-2I)M}{2\pi i}\,[1+\Omega(z)],\quad z\in\Omega_\Sigma^+. \tag{4.13}$$

Here and in what follows we employ the notation

$$\frac{d\overline{t}}{dt}=\frac{d\overline{t}}{ds}:\frac{dt}{ds}\,.$$

Using the properties of Cauchy type integrals and applying the restrictions (4.7) and (4.8), we can easily establish that the vector U given by (2.18) is single valued and continuous in Ω_Σ^+.

From the boundary conditions (4.3) and (4.4) along with the formula (4.13), we get

$$\big[(A-2I)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}\big]_\Sigma^+-\big[(A-2I)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}\big]_\Sigma^-=$$

$$=\omega(t)=-i\int\limits_{e_1}^{t}\big[F^+(\tau)-F^-(\tau)\big]\,ds+C^+-C^-,\quad t\in\Sigma.$$

In view of the equality (4.9), it is evident that the conditions (4.8) will be satisfied if $C^+=C^-$. Thus, the vector function ω is defined explicitly as

$$\omega(t)=-i\int\limits_{e_1}^{t}\big[F^+(\tau)-F^-(\tau)\big]\,ds,\quad t\in\Sigma. \tag{4.14}$$

Due to the relations (2.15) and (2.10), it follows from (4.14) that

$$\omega\in[H(\Sigma)]^2,\quad \omega'\in[H^*(\Sigma)]^2. \tag{4.15}$$

Further, upon summing of the boundary conditions (4.3) and (4.4), we get

$$\big[(A-2I)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}\big]_\Sigma^++\big[(A-2I)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}\big]_\Sigma^-=$$

$$= -i\int\limits_{e_1}^{t} \big[F^+(\tau)+F^-(\tau)\big]\,ds + 2C^+.$$

Applying the representation (4.13) and transferring the known terms to the right-hand side, we arrive at the equation for the unknown vector functions g and h:

$$\frac{A-I}{\pi i}\int\limits_{\Sigma}\frac{h(t)}{t-t_0}\,dt - \frac{1}{\pi i}\int\limits_{S}\frac{g(t)}{t-t_0}\,dt + \frac{A}{2\pi i}\int\limits_{S} g(t)\,d_t \ln\frac{t-t_0}{\overline{t}-\overline{t}_0} -$$
$$-\frac{A-2I}{2\pi i}\int\limits_{\Sigma} h(t)\,d_t \ln\frac{t-t_0}{\overline{t}-\overline{t}_0} -$$
$$-\frac{B}{2\pi i}\bigg[\int\limits_{S}\overline{g(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big) + \int\limits_{\Sigma}\overline{h(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)\bigg] =$$
$$= \Phi^{(1)}(t_0) + C^+, \quad t_0\in\Sigma, \tag{4.16}$$

where

$$\Phi^{(1)}(t_0) = -\frac{i}{2}\int\limits_{e_1}^{t_0}\big[F^+(\tau)+F^-(\tau)\big]\,ds + \frac{1}{2\pi i}\int\limits_{\Sigma}\frac{\omega(t)}{\overline{t}-\overline{t}_0}\,d\overline{t} -$$
$$-\frac{\chi}{2\pi i}\int\limits_{\Sigma}\frac{(t-e_1)}{\overline{t}-\overline{t}_0}\,d\overline{t} - \frac{\chi}{2}(t_0-e_1) +$$
$$+\frac{B\overline{M}(t_0-e_1)}{2\pi i(\overline{e}_2-\overline{e}_1)}\ln\frac{\overline{t}_0-\overline{e}_2}{\overline{t}_0-\overline{e}_1} - \frac{(A-2I)M}{2\pi i}\,[1+\Omega(t_0)] +$$
$$+\frac{AM}{2\pi i}\ln(\overline{t}_0-\overline{e}_2), \quad t_0\in\Sigma.$$

Here $\ln(\overline{t}_0-\overline{e}_2)$ and $\ln(\overline{t}_0-\overline{e}_1)$ are the limits of the functions $\ln(\overline{z}-\overline{e}_2)$ and $\ln(\overline{z}-\overline{e}_1)$, respectively, at the point $t_0\in\Sigma$ from the left.

Let us show that

$$\Phi^{(1)}\in[H(\Sigma)]^2, \quad [\Phi^{(1)}]'\in[H^*(\Sigma)]^2. \tag{4.17}$$

Taking into consideration the conditions (4.7), (4.8) and (4.15), we can show that the function $\Phi^{(1)}$ is continuous at the end point e_1, while in a neighbourhood of the point e_2 we have

$$\Phi^{(1)}(t_0) = \frac{1}{2\pi i}\Big[\omega(e_2) - \chi(e_2-e_1) +$$
$$+\frac{B\overline{M}(e_2-e_1)}{\overline{e}_2-\overline{e}_1} + AM\Big]\ln(\overline{t}_0-\overline{e}_2) + O(1) = O(1).$$

On the other hand,

$$\frac{d\Phi^{(1)}(t_0)}{dt_0} = -\frac{i}{2}\big[F^+(t_0)+F^-(t_0)\big] + \bigg\{\frac{1}{2\pi i}\int\limits_{\Sigma}\frac{\omega'(t)}{\overline{t}-\overline{t}_0}\,dt - \frac{\chi}{2\pi i}\int\limits_{\Sigma}\frac{dt}{\overline{t}-\overline{t}_0} +$$

$$+\frac{B\overline{M}}{2\pi i(\overline{e}_2-\overline{e}_1)}\Big[\frac{t_0-e_2}{\overline{t}_0-\overline{e}_2}-\frac{t_0-e_1}{\overline{t}_0-\overline{e}_1}\Big]\bigg\}\,\frac{d\overline{t}_0}{ds}\,\Big[\frac{dt_0}{ds}\Big]^{-1}-$$

$$-\frac{\chi}{2}+\frac{B\overline{M}}{2\pi i(\overline{e}_2-\overline{e}_1)}\,\ln\frac{\overline{t}_0-\overline{e}_2}{\overline{t}_0-\overline{e}_1}-\frac{(A-2I)M}{2\pi i(e_2-e_1)}\,\ln\frac{t_0-e_2}{t_0-e_1}\,, \quad (4.18)$$

whence (4.17) follows immediately.

We can easily show that the relations (4.16) and (4.17) imply the conditions (4.7).

Rewrite the equation (4.16) in the form

$$\frac{A-2I}{2\pi i}\Big[\int\limits_S\frac{g(t)}{t-t_0}\,dt+\int\limits_\Sigma\frac{h(t)}{t-t_0}\,dt\Big]+\frac{I-A}{\pi i}\int\limits_S\frac{g(t)}{t-t_0}\,dt+$$

$$+\frac{A}{2\pi i}\int\limits_S g(t)\,d_t\ln\frac{t-t_0}{\overline{t}-\overline{t}_0}-\frac{A-2I}{2\pi i}\int\limits_\Sigma h(t)\,d_t\ln\frac{t-t_0}{\overline{t}-\overline{t}_0}-$$

$$-\frac{B}{2\pi i}\left[\int\limits_S\overline{g(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)+\int\limits_\Sigma\overline{h(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)\right]=$$

$$=\Phi^{(1)}(t_0)+C^+,\ \ t_0\in\Sigma. \quad (4.19)$$

The boundary conditions (4.1) and (4.2) along with the representations (4.12) and (4.13) lead to the equations

$$Ag(t_0)+\frac{A}{2\pi i}\int\limits_S g(t)\,d_t\ln\frac{t-t_0}{\overline{t}-\overline{t}_0}+\frac{A-I}{\pi i}\int\limits_\Sigma\frac{h(t)}{t-t_0}\,dt-$$

$$-\frac{A-2I}{2\pi i}\int\limits_\Sigma h(t)\,d_t\ln\frac{t-t_0}{\overline{t}-\overline{t}_0}-$$

$$-\frac{B}{2\pi i}\left[\int\limits_S\overline{g(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)+\int\limits_\Sigma\overline{h(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)\right]=$$

$$=\Phi^{(2)}(t_0),\ \ t_0\in S_D, \quad (4.20)$$

$$(A-I)g(t_0)-\frac{1}{\pi i}\left[\int\limits_S\frac{g(t)}{t-t_0}\,dt+\int\limits_\Sigma\frac{h(t)}{t-t_0}\,dt\right]+$$

$$+\frac{A}{2\pi i}\int\limits_S g(t)\,d_t\ln\frac{t-t_0}{\overline{t}-\overline{t}_0}+\frac{A-I}{\pi i}\int\limits_\Sigma\frac{h(t)}{t-t_0}\,dt-$$

$$-\frac{A-2I}{2\pi i}\int\limits_\Sigma h(t)\,d_t\ln\frac{t-t_0}{\overline{t}-\overline{t}_0}-$$

$$-\frac{B}{2\pi i}\left[\int\limits_S\overline{g(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)+\int\limits_\Sigma\overline{h(t)}\,d_t\Big(\frac{t-t_0}{\overline{t}-\overline{t}_0}\Big)\right]=$$

$$=\Phi^{(3)}(t_0)+D(t_0),\ \ t_0\in S_T, \quad (4.21)$$

where

$$
\begin{aligned}
\Phi^{(2)}(t_0) &= 2\mu F^{(1)}(t_0) + \frac{1}{2\pi i}\int\limits_{\Sigma} \frac{\omega(t)}{\overline{t}-\overline{t}_0}\,d\overline{t} - \frac{\chi}{2\pi i}\int\limits_{\Sigma} \frac{(t-e_1)}{\overline{t}-\overline{t}_0}\,d\overline{t} + \\
&\quad + \frac{B\overline{M}(t_0-e_1)}{2\pi i(\overline{e}_2-\overline{e}_1)}\,\ln\frac{\overline{t}_0-\overline{e}_2}{\overline{t}_0-\overline{e}_1} - \\
&\quad - \frac{AM}{2\pi i}\big[1+\Omega(t_0)-\ln(\overline{t}_0-\overline{e}_2)\big], \ \ t_0\in S_D, \\
\Phi^{(3)}(t_0)\,d\overline{t} &= -i\int\limits_{c_{2j}}^{t_0} F^{(2)}(\tau)\,ds + \frac{1}{2\pi i}\int\limits_{\Sigma} \frac{\omega(t)}{\overline{t}-\overline{t}_0}\,d\overline{t} - \frac{\chi}{2\pi i}\int\limits_{\Sigma} \frac{(t-e_1)}{\overline{t}-\overline{t}_0}\,d\overline{t} + \\
&\quad + \frac{B\overline{M}(t_0-e_1)}{2\pi i(\overline{e}_2-\overline{e}_1)}\,\ln\frac{\overline{t}_0-\overline{e}_2}{\overline{t}_0-\overline{e}_1} - \frac{(A-2I)M}{2\pi i}\big[1+\Omega(t_0)\big] - \\
&\quad - \frac{AM}{2\pi i}\,\ln(\overline{t}_0-\overline{e}_2), \ \ t_0\in S_{2j}, \ \ j=1,2,\dots,p, \\
D(t_0) &= D^{(j)}, \ \ t_0\in S_{2j}, \ \ j=1,2,\dots,p.
\end{aligned}
$$

It is evident that

$$
\Phi^{(2)}\in[H(S_D)]^2, \ \ \Phi^{(3)}\in[H(S_T)]^2, \ \ [\Phi^{(2)}]'\in[H^*(S_D)]^2, \ \ [\Phi^{(3)}]'\in[H^*(S_T)]^2.
$$

Thus, we have obtained the following system of singular integral equations with discontinuous coefficients

$$
\begin{aligned}
A^*(t_0)\sigma(t_0) + \frac{B^*(t_0)}{\pi i}\int\limits_{\Lambda}\frac{\sigma(t)}{t-t_0}\,dt + \int\limits_{\Lambda} K_1(t_0,t)\sigma(t)\,dt + \\
+ \int\limits_{\Lambda}\overline{K_2(t_0,t)\sigma(t)}\,d\overline{t} = \Psi(t_0)+D^*(t_0), \ \ t_0\in\Lambda,
\end{aligned} \tag{4.22}
$$

where $\Lambda = S\cup\Sigma$,

$$
\begin{aligned}
A^*(t_0) &= \begin{cases} A & \text{for } \ t_0\in S_D, \\ A-I & \text{for } \ t_0\in S_T, \\ 0 & \text{for } \ t_0\in\Sigma, \end{cases} \\
B^*(t_0) &= \begin{cases} 0 & \text{for } \ t_0\in S_D, \\ -I & \text{for } \ t_0\in S_T, \\ A-2I & \text{for } \ t_0\in\Sigma, \end{cases}
\end{aligned} \tag{4.23}
$$

$$
\begin{aligned}
\sigma(t_0) &= \begin{cases} g(t_0) & \text{for } \ t_0\in S, \\ h(t_0) & \text{for } \ t_0\in\Sigma, \end{cases} \\
K_2(t_0,t) &= \frac{B}{2\pi i}\,\frac{\partial}{\partial t}\Big(\frac{\overline{t}-\overline{t}_0}{t-t_0}\Big), \ \ t,\, t_0\in\Lambda,
\end{aligned} \tag{4.24}
$$

$$K_1(t_0,t)=\begin{cases}\dfrac{A}{2\pi i}\,\dfrac{\partial}{\partial t}\,\ln\dfrac{t-t_0}{\overline{t}-\overline{t}_0} & \text{for } t_0,\,t\in S,\\ \dfrac{A-I}{\pi i}\,\dfrac{1}{t-t_0}-\dfrac{A-2I}{2\pi i}\,\dfrac{\partial}{\partial t}\,\ln\dfrac{t-t_0}{\overline{t}-\overline{t}_0} & \text{for } t_0\in S,\ t\in\Sigma,\\ \dfrac{I-A}{\pi i}\,\dfrac{1}{t-t_0}+\dfrac{A}{2\pi i}\,\dfrac{\partial}{\partial t}\,\ln\dfrac{t-t_0}{\overline{t}-\overline{t}_0} & \text{for } t_0\in\Sigma,\ t\in S,\\ -\dfrac{A-2I}{2\pi i}\,\dfrac{\partial}{\partial t}\,\ln\dfrac{t-t_0}{\overline{t}-\overline{t}_0} & \text{for } t_0,\,t\in\Sigma,\end{cases} \tag{4.25}$$

$$\Psi(t_0)=\begin{cases}\Phi^{(1)}(t_0) & \text{for } t_0\in\Sigma,\\ \Phi^{(2)}(t_0) & \text{for } t_0\in S_D,\\ \Phi^{(3)}(t_0) & \text{for } t_0\in S_T,\end{cases} \tag{4.26}$$

$$D^*(t_0)=\begin{cases}0 & \text{for } t_0\in S_D,\\ D^{(j)} & \text{for } t_0\in S_{2j},\quad j=1,\dots,p,\\ C^+ & \text{for } t_0\in\Sigma.\end{cases} \tag{4.27}$$

The theory of singular integral equations with discontinuous coefficients on closed, smooth, simple curves is developed in the reference [7]. To apply this theory in our case, we proceed as follows. First we extend the system (4.22) from $\Lambda=S\cup\Sigma$ to the closed curve $\Lambda_0=S\cup S_0$. We recall that Σ is a part of the closed, simple, $C^{2,\alpha}$-smooth curve S_0 which lies inside the region Ω^+_Σ (see Section 2). The extended equation reads as

$$\mathcal{K}\sigma(t_0)\equiv A^*(t_0)\sigma(t_0)+\frac{B^*(t_0)}{\pi i}\int\limits_{\Lambda_0}\frac{\sigma(t)}{t-t_0}\,dt+\int\limits_{\Lambda_0}K_1(t_0,t)\sigma(t)\,dt+$$
$$+\int\limits_{\Lambda_0}\overline{K_2(t_0,t)\sigma(t)}\,d\overline{t}=\Psi(t_0)+D^*(t_0),\quad t_0\in\Lambda_0, \tag{4.28}$$

where A^*, B^*, K_1, K_2, Ψ, and D^* are defined by the formulas (4.23)–(4.27) and the relations

$$\begin{gathered}A^*(t_0)=I,\quad B^*(t_0)=0,\quad \Psi(t_0)=0,\\ D^*(t_0)=0\ \text{ for } t_0\in\Sigma_0:=S_0\setminus\overline{\Sigma},\end{gathered} \tag{4.29}$$
$$K_1(t_0,t)=K_2(t_0,t)=0\ \text{ for } t_0\in\Sigma_0,\ \ t\in\Lambda_0\ \text{ or }\ t_0\in\Lambda,\ \ t\in\Sigma_0.$$

Let

$$\mathcal{S}(t_0):=A^*(t_0)+B^*(t_0),\quad \mathcal{D}(t_0):=A^*(t_0)-B^*(t_0),\quad t_0\in\Lambda_0.$$

We easily derive

$$\det\mathcal{S}(t_0)=\begin{cases}\det A=4\Delta_0\Delta_1>0 & \text{for } t_0\in S_D,\\ \det(A-2I)=\Delta_2>0 & \text{for } t_0\in S_T,\\ \Delta_2>0 & \text{for } t_0\in\Sigma,\\ 1>0 & \text{for } t_0\in\Sigma_0,\end{cases}$$

$$\det \mathcal{D}(t_0) = \begin{cases} 4\Delta_0\Delta_1 > 0 & \text{for } \ t_0 \in S, \\ \Delta_2 > 0 & \text{for } \ t_0 \in \Sigma, \\ 1 > 0 & \text{for } \ t_0 \in \Sigma_0. \end{cases}$$

From these relations it follows that the equation (4.28) is of normal type (see [7]).

Further we have to characterize the points of discontinuity. To this end, let us construct the characteristic equation for the unknown ν,

$$\det \big[G^{-1}(t+0)G(t-0) - \nu I\big] = 0 \ \text{ for } \ t \in \{e_1, e_2, c_j, \ j = \overline{1,2p}\}, \tag{4.30}$$

where

$$G(t) = \mathcal{D}^{-1}(t)\,\mathcal{D}(t) = \begin{cases} I & \text{for } \ t \in S_D, \\ \Delta_2^{-1}(4\Delta_0\Delta_1 - 2A) & \text{for } \ t \in S_T, \\ -I & \text{for } \ t \in \Sigma, \\ I & \text{for } \ t \in \Sigma_0. \end{cases}$$

From the equation (4.30) we get

$$\det \big[G^{-1}(e_1+0)G(e_1-0) - \nu I\big] = \\ = \det \big[G^{-1}(e_2+0)G(e_2-0) - \nu I\big] = (\nu+1)^2 = 0, \tag{4.31}$$

$$\det \big[G^{-1}(c_{2j-1}+0)G(c_{2j-1}-0) - \nu I\big] = \\ = \nu^2 - \frac{2}{\Delta_2}\,(4\Delta_0\Delta_1 - A_1 - A_4)\nu + \frac{4\Delta_0\Delta_1}{\Delta_2} = 0, \ \ j = \overline{1,p}, \tag{4.32}$$

$$\det \big[G^{-1}(c_{2j}+0)G(c_{2j}-0) - \nu I\big] = \\ = \nu^2 - \frac{4\Delta_0\Delta_1 - A_1 - A_4}{2\Delta_0\Delta_1}\,\nu + \frac{\Delta_2}{4\Delta_0\Delta_1} = 0, \ \ j = \overline{1,p}. \tag{4.33}$$

The roots of the equation (4.31) are $\nu_1 = \nu_2 = -1$. The roots ν_3 and ν_4 of the quadratic equation (4.32) are negative since the discriminant and the free term are positive and the second coefficient is negative in accordance with the following inequalities

$$\Delta_2^{-2}\big[(A_1+A_4)^2 - 16\Delta_0\Delta_1\big] > 0, \ \ 4\Delta_0\Delta_1\Delta_2^{-1} > 0, \ \ 4\Delta_0\Delta_1 - A_1 - A_4 < 0.$$

Thus $\nu_3 < 0$ and $\nu_4 < 0$, and they are different from -1, in general. It is easy to see that the roots ν_5 and ν_6 of the quadratic equation (4.33) are inverses of the roots ν_3 and ν_4, i.e., $\nu_5 = 1/\nu_3 < 0$ and $\nu_6 = 1/\nu_4 < 0$. Further, let

$$\varkappa_q = \frac{1}{2\pi i}\,\ln \nu_q, \ \ q = \overline{1,6}.$$

Here the branch of the logarithmic function is chosen in such a way that

$$\Re \varkappa_q = \frac{1}{2}\,, \ \ q = \overline{1,6}.$$

We then have

$$\varkappa_1 = \varkappa_2 = \frac{1}{2}\,, \ \ \varkappa_3 = \frac{1}{2} - i\beta_3, \ \ \varkappa_4 = \frac{1}{2} - i\beta_4, \ \ \varkappa_5 = \frac{1}{2} + i\beta_3, \ \ \varkappa_6 = \frac{1}{2} + i\beta_4, \tag{4.34}$$

with

$$\beta_3 = \frac{1}{2\pi}\ln|\nu_3|, \quad \beta_4 = \frac{1}{2\pi}\ln|\nu_4|.$$

We remark that $\beta_3 \neq 0$ and $\beta_4 \neq 0$, in general. Thus, due to the general theory developed in [13], all the points of discontinuity of the coefficients of the integral equation (4.28), the nodal points e_1, e_2, and c_j, $j = \overline{1, 2p}$, are *non-special*, i.e., the corresponding numbers $\Re\varkappa_q$ are not integers.

Denote by $\mathbf{h}_{2p+2} := \mathbf{h}(c_1, c_2, \dots, c_{2p}, e_1, e_2)$ the subclass of vector functions with components from $H^*(\Lambda_0)$ which are bounded at the nodal points $c_1, c_2, \dots, c_{2p}, e_1, e_2$ (for details see [10], [13]).

Applying the embedding results obtained in [7] for solutions of singular integral equations with discontinuous coefficients, we conclude that if the equation (4.28) has a solution σ of the class $\mathbf{h}_{2p+2}$, then

$$\sigma \in [H(\Lambda_0)]^2, \ \ \partial_t\sigma \in [H^*(\Lambda_0)]^2. \tag{4.35}$$

Evidently, we have the similar inclusions for the vectors g and h in view of (4.24) (see also (4.15)). Therefore, we can show that the vector functions

$$A\varphi(z)+Bz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}, \ \ (A-2I)\varphi(z)+Bz\overline{\varphi'(z)}+2\mu\overline{\psi(z)}, \ \ z\in\Omega^+_\Sigma,$$

where $\varphi(z)$ and $\psi(z)$ are constructed by means of the densities g, h and ω in accordance with the formulas (4.5) and (4.6), are continuously extendable on $S \cup \overline{\Sigma}$ due to (4.12) and (4.13). Moreover, the corresponding partial displacement vectors $u' = (u_1, u_2)^\top$ and $u'' = (u_3, u_4)^\top$ defined with the help of (2.11) are *regular* in Ω^+_Σ.

Thus, the main problem now is to show the solvability of the integral equation (4.28) in the space $\mathbf{h}_{2p+2}$.

In the next section we will study some properties of the integral operator $\mathcal{K}$ defined by (4.28) and establish the corresponding existence and regularity results for the original boundary value problem.

5. Existence Results

Here we show that for an arbitrary vector function $\Psi(t_0)$ we can chose a piecewise constant vector $D^*(t_0)$ in such a way that the equation (4.28) becomes solvable.

To this end, first we investigate the null spaces of the operator $\mathcal{K}$ in $\mathbf{h}_{2p+2}$ and its adjoint one, $\mathcal{K}'$. Due to the general theory it is well known that the index of the operator $\mathcal{K}$ is

$$\operatorname{ind}\mathcal{K} := q - q' = -4(p+1), \tag{5.1}$$

where $q = \dim\ker\mathcal{K}$ in $\mathbf{h}_{2p+2}$ and $q' = \dim\ker\mathcal{K}'$ in the space of vector functions $\mathbf{h}'_{2p+2}$ adjoint to $\mathbf{h}_{2p+2}$ (see [7], [8]).

Now we prove that $q = 0$, i.e., the homogeneous equation $\mathcal{K}\sigma = 0$ possesses only the trivial solution. Let $\sigma_0 \in \mathbf{h}_{2p+2}$ be a solution to this equation,

$$\sigma_0(t) = \begin{cases} g_0 & \text{for } \ t \in S, \\ h_0 & \text{for } \ t \in \Sigma, \\ 0 & \text{for } \ t \in \Sigma_0. \end{cases} \tag{5.2}$$

Remark that actually we have a higher regularity property for the vectors g_0 and h_0 due to (4.35).

Further we construct the complex potentials $\varphi_0(z)$ and $\psi_0(z)$ by means of the formulas (4.5) and (4.6) with $\omega = 0$, and g_0 and h_0 for g and h, respectively,

$$\varphi_0(z) = \frac{1}{2\pi i}\int\limits_S \frac{g_0(t)}{t-z}\,dt + \frac{1}{2\pi i}\int\limits_\Sigma \frac{h_0(t)}{t-z}\,dt, \quad z \in \Omega^+_\Sigma, \tag{5.3}$$

$$\begin{aligned} 2\mu\psi_0(z) = {} & \frac{1}{2\pi i}\int\limits_S \frac{A\overline{g_0(t)} - B\bar{t}g_0'(t)}{t-z}\,dt - \\ & -\frac{1}{2\pi i}\int\limits_\Sigma \frac{(A-2I)\overline{h_0(t)} + B\bar{t}h_0'(t)}{t-z}\,dt, \quad z \in \Omega^+_\Sigma. \end{aligned} \tag{5.4}$$

We easily verify that there hold the following boundary conditions

$$\begin{aligned} \big[A\varphi_0(t) + Bt\overline{\varphi_0'(t)} + 2\mu\overline{\psi_0(t)}\big]^+ &= 0, \quad t \in S_D, \\ \big[(A-2I)\varphi_0(t) + Bt\overline{\varphi_0'(t)} + 2\mu\overline{\psi_0(t)}\big]^+ &= 0, \quad t \in S_T, \\ \big[(A-2I)\varphi_0(t) + Bt\overline{\varphi_0'(t)} + 2\mu\overline{\psi_0(t)}\big]^\pm &= 0, \quad t \in \Sigma. \end{aligned} \tag{5.5}$$

By (2.18) and (2.19) along with (2.11), (2.17) and (2.20), we conclude that the partial displacement vectors $u' = (u_1, u_2)^\top$ and $u'' = (u_3, u_4)^\top$, corresponding to the complex potentials φ_0 and ψ_0, are *regular* and consequently Green's formulas (3.3) and (3.4) hold. Therefore $W(u, u) = 0$ in Ω^+_Σ, whence

$$u = (u', u'')^\top, \ \ u' = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} + b_0' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \ \ u'' = \begin{pmatrix} a_1'' \\ a_2'' \end{pmatrix} + b_0' \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{5.6}$$

and due to the homogeneous Dirichlet condition on S_D we get $u' = u'' = 0$ in Ω^+_Σ.

In accordance with the remarks at the end of Section 2 (see Conclusion (ii)), we have $\varphi_0(z) = \gamma$ and $\psi_0(z) = -2^{-1}\mu^{-1}A\overline{\gamma}$ in Ω^+_Σ, where γ is an arbitrary complex constant vector. But then from the second equation in (5.5) we conclude that $\gamma = 0$ and, consequently, $\varphi_0(z) = 0$ and $\psi_0(z) = 0$ in Ω^+_Σ.

Therefore

$$[\varphi_0(t)]^+ - [\varphi_0(t)]^- = h_0(t) = 0 \ \text{ for } \ t \in \Sigma, \tag{5.7}$$

due to the Plemelj–Sokhotski formula [10]. In view of (5.3) and (5.4) we arrive at the equations

$$\varphi_0(z)=\frac{1}{2\pi i}\int\limits_S\frac{g_0(t)}{t-z}\,dt=0,\quad z\in\Omega^+,\tag{5.8}$$

$$2\mu\psi_0(z)=\frac{1}{2\pi i}\int\limits_S\frac{A\overline{g_0(t)}-B\overline{t}g_0'(t)}{t-z}\,dt=0,\quad z\in\Omega^+.\tag{5.9}$$

It is evident that the Cauchy type integrals in (5.8) and (5.9) define holomorphic vector functions in the exterior domain $\Omega^-=\mathbb{R}^2\setminus\overline{\Omega^+}$ which vanish at infinity. In what follows we show that these functions identically vanish in Ω^- as well. To this end, let us introduce the vector functions

$$\varphi_0^*(z)=-\frac{1}{2\pi}\int\limits_S\frac{g_0(t)}{t-z}\,dt,\quad z\in\Omega^-,\tag{5.10}$$

$$2\mu\psi_0^*(z)=-\frac{1}{2\pi}\int\limits_S\frac{A\overline{g_0(t)}-B\overline{t}g_0'(t)}{t-z}\,dt,\quad z\in\Omega^-.\tag{5.11}$$

Evidently, $\varphi_0^*(\infty)=0$ and $\psi_0^*(\infty)=0$. Moreover, the Plemelj–Sokhotski formula and the equalities (5.8) and (5.9) yield that

$$\begin{aligned}&[\varphi_0^*(t)]^-=ig_0(t)\ \text{ for }\ t\in S,\\&[2\mu\psi_0^*(t)]^-=i\big[A\overline{g_0(t)}-B\overline{t}g_0'(t)\big]\ \text{ for }\ t\in S.\end{aligned}\tag{5.12}$$

From these relations we see that

$$\big[A\varphi_0^*(t)+Bt\overline{[\varphi_0^*(t)]'}+2\mu\overline{\psi_0^*(t)}\big]^-=0,\quad t\in S.\tag{5.13}$$

Further, we construct the vectors $u'=(u_1,u_2)^\top$ and $u''=(u_3,u_4)^\top$ in accordance with the formulas (2.11) and the representation (2.16) with φ_0^* and ψ_0^* for φ and ψ. Evidently, u' and u'' solve the system (2.1), vanish at infinity along with their first order derivatives as $\mathcal{O}(|x|^{-1})$ and $\mathcal{O}(|x|^{-2})$, respectively, and satisfy the homogeneous exterior Dirichlet boundary condition on S in view of (5.13). Therefore by the uniqueness theorem (see Remark 3.2) u' and u'' vanish in Ω^-. Consequently, $\varphi_0^*(z)=0$ and $\psi_0^*(z)=0$ for $z\in\Omega^-$ due to Conclusion (ii) in Section 2. By (5.12) this yields $g_0(t)=0$ for $t\in S$. This proves that the homogeneous equation $K\sigma=0$ has only the trivial solution. Therefore, $q'=\dim\ker\mathcal{K}'=4p+4$, i.e., the homogeneous equation $\mathcal{K}'\zeta=0$ has $4p+4$ linearly independent solutions in the space $\mathbf{h}'_{2p+2}$. We denote them by $\{\zeta^{(j)}\}_{j=1}^{4p+4}$.

A sufficient condition of solvability of the equation (4.28) in the space $\mathbf{h}_{2p+2}$ reads then as follows (see [7]):

$$\Re\int\limits_{\Lambda_0}\big[\Psi(t)+D^*(t)\big]\zeta^{(j)}(t)\,dt=0,\quad j=1,2,\dots,4(p+1).\tag{5.14}$$

This can be treated as a system of linear algebraic equations with respect to the real and imaginary parts of the arbitrary complex constant vectors C^+ and $D^{(q)}$, $q=\overline{1,p}$ (see (4.27) and (4.29)). Thus, (5.14) represents $4p+4$ simultaneous equations with the $4p+4$ unknowns.

Let us show that the determinant of the system is different from zero, i.e., the homogeneous system

$$\Re \int\limits_{\Lambda} D^*(t)\zeta^{(j)}(t)\,dt \equiv$$

$$=\equiv \Re\Big[\sum_{q=1}^{p} D^{(q)} \int\limits_{S_{2q}} \zeta^{(j)}(t)\,dt + C^+ \int\limits_{\Sigma} \zeta^{(j)}(t)\,dt\Big] = 0, \quad j=\overline{1,4(p+1)}, \tag{5.15}$$

has only the trivial solution.

Let $D^{(q)}$ and C^+ be some solution of the homogeneous system (5.15). Then the integral equation (4.28) with the right-hand side defined by (4.27) and (4.29) is solvable in $\mathbf{h}_{2p+2}$. Note that we assume $\Psi=0$ on Λ_0. Denote this solution by σ_0 which can be written again in the form (5.2).

Further, as above we construct the complex potentials $\varphi_0(z)$ and $\psi_0(z)$ by means of the formulas (4.5) and (4.6) with $\omega=0$, and g_0 and h_0 for g and h, respectively. Thus we have again the formulas (5.3) and (5.4) for $\varphi_0(z)$ and $\psi_0(z)$ which along with the integral equation $\mathcal{K}\sigma = D^*$ now lead to the boundary conditions

$$\begin{aligned}
\big[A\varphi_0(t) + Bt\overline{\varphi_0'(t)} + 2\mu\overline{\psi_0(t)}\big]^+ &= 0, \quad t\in S_D,\\
\big[(A-2I)\varphi_0(t) + Bt\overline{\varphi_0'(t)} + 2\mu\overline{\psi_0(t)}\big]^+ &= D^{(q)}, \quad t\in S_{2q}, \quad q=\overline{1,p},\\
\big[(A-2I)\varphi_0(t) + Bt\overline{\varphi_0'(t)} + 2\mu\overline{\psi_0(t)}\big]^\pm &= C^+, \quad t\in \Sigma.
\end{aligned} \tag{5.16}$$

By the same arguments as above and with the help of Green's formula, we derive that the partial displacement vectors $u'=(u_1,u_2)^\top$ and $u''=(u_3,u_4)^\top$ corresponding to the complex potentials φ_0 and ψ_0 vanish in Ω_Σ^+. Therefore due to Conclusion (ii) in Section 2 we have $\varphi_0(z)=\gamma$ and $\psi_0(z) = -2^{-1}\mu^{-1}A\overline{\gamma}$, where $\gamma=(\gamma_1,\gamma_2)^\top$ is an arbitrary complex constant vector. From the third and second equations in (5.16) we have

$$\gamma = -\frac{1}{2}\,C^+, \quad D^{(q)} = C^+, \quad q=\overline{1,p}. \tag{5.17}$$

Thus

$$\varphi_0(z) = \gamma = -\frac{1}{2}\,C^+, \quad 2\mu\overline{\psi_0(z)} = -A\gamma = \frac{1}{2}AC^+, \quad z\in\Omega_\Sigma^+. \tag{5.18}$$

From (5.3) we derive

$$[\varphi_0(t)]^+ - [\varphi_0(t)]^- = h_0(t) = 0 \ \text{ for } \ t\in\Sigma. \tag{5.19}$$

Therefore finally we get from (5.3) and (5.4)

$$\varphi_0(z)=\frac{1}{2\pi i}\int\limits_S\frac{g_0(t)}{t-z}\,dt=-\frac{1}{2}\,C^+,\quad z\in\Omega^+,\tag{5.20}$$

$$2\mu\psi_0(z)=\frac{1}{2\pi i}\int\limits_S\frac{A\overline{g_0(t)}-B\bar t\overline{g_0'(t)}}{t-z}\,dt=\frac{1}{2}\,A\overline{C^+},\quad z\in\Omega^+.\tag{5.21}$$

Let us introduce holomorphic vector functions φ^* and ψ^* in Ω^- defined by the formulas

$$\varphi_0^*(z)=-\frac{1}{2\pi}\int\limits_S\frac{g_0(t)}{t-z}\,dt-\frac{i}{2}\,C^+,\quad z\in\Omega^-,\tag{5.22}$$

$$2\mu\psi_0^*(z)=-\frac{1}{2\pi}\int\limits_S\frac{A\overline{g_0(t)}-B\bar t\overline{g_0'(t)}}{t-z}\,dt+\frac{i}{2}\,A\overline{C^+},\quad z\in\Omega^-.\tag{5.23}$$

Evidently,

$$\varphi_0^*(\infty)=-\frac{i}{2}\,C^+,\quad 2\mu\psi_0^*(\infty)=\frac{i}{2}\,A\overline{C^+}.\tag{5.24}$$

Moreover, the Plemelj–Sokhotski formula and the equalities (5.20) and (5.21) yield that

$$\begin{aligned}[\varphi_0^*(t)]^-&=ig_0(t)\ \text{ for }\ t\in S,\\ [2\mu\psi_0^*(t)]^-&=i\big[A\overline{g_0(t)}-B\bar t\overline{g_0'(t)}\big]\ \text{ for }\ t\in S,\end{aligned}$$

whence we get

$$\big[A\varphi_0^*(t)+Bt\overline{\varphi_0^{*\prime}(t)}+2\mu\overline{\psi_0^*(t)}\big]^-=0,\quad t\in S.\tag{5.25}$$

Taking into consideration the behaviour of the vectors φ_0^* and ψ_0^*, using word for word the arguments applied after the formula (5.13) and with the help of Remark 3.2 we conclude that

$$\varphi_0^*(z)=-\frac{i}{2}\,C^+,\quad 2\mu\psi_0^*(z)=\frac{i}{2}\,A\overline{C^+},\quad z\in\Omega^-.\tag{5.26}$$

But then from (5.25) we derive $C^+=0$ since $\det A>0$. Consequently, by (5.17) we have $D^{(q)}=0$ for $q=1,2,\dots,p$. Thus we have shown that the homogeneous system (5.15) possesses only the trivial solution and therefore the non-homogeneous system (5.14) is uniquely solvable for arbitrary vector function Ψ. Thus, for arbitrary Ψ we can chose the constant complex vectors C^+ and $D^{(q)}$, $q=1,2,\dots,p$, in such a way that the integral equations (4.28) would be solvable. Finally we can formulate the following existence result for the original mixed BVP.

Theorem 5.1. *The mixed boundary value problem* (2.1), (2.7)–(2.9) *is uniquely solvable in the class of regular vector functions if the conditions* (2.10) *hold.*

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