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N. Khomasuridze, K. Ninidze, and Z. Siradze

A STEADY FLOW OF A VISCOUS MULTI-LAYER FLUID IN A TOROIDAL TUBE WITH SMALL RADIUS

Abstract. The steady flow of a multi-layer viscous incompressible fluid is considered in a toroidal tube. Each layer has a viscosity coefficient of its own. The velocity vector has one nonzero circular component not depending on the circular coordinate. The inertial terms in the Navier-Stokes equations are neglected or, in other words, the system of Stokes equations is considered under the following boundary contact- conditions: the nonslip conditions are given on the toroidal surface, hydrostatic pressure values are given at the tube ends, and the contact conditions are given on the interface between the layers. An inhomogeneous equation is obtained for the rate component and its analytic solution is found. The results obtained are used to study the blood flow in narrow curvilinear vessels, to determine the blood flow parameters, the distribution of erythrocytes over the vessel cross-section and also to determine the hydrodynamic resistance to the flow.

[illegible]

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In a living organism there exists no physiological or pathological process that is not related to hemorheological problems. In the physiological conditions the blood flow regulation takes place in the system of minute vessels such as arterioles, capillaries and venules. The lumen of these vessels varies from 10 to 100 mcm. Since capillaries are not surrounded by the muscular tissue, their walls can be assumed to be absolutely rigid. The plasma and erythrocytes are treated as different fluids. In view of the fact that the distribution of erythrocytes over the capillary cross-section is not uniform (they are amassed near the vessel axis), it makes sense to consider the flow in a microvessel as the flow of a multi-layer fluid whose every layer has its own dynamic viscosity coefficient. The entire process is viewed as a steady flow of a multi-layer viscous incompressible fluid. The viscosity of a certain part of layers is taken to be the plasma viscosity, while the viscosity of the remaining layers is taken to be the hypothetic viscosity of the erythrocyte mass. An analogous approach was taken by M. Sharan and L. Popel in [1] and by E. Damiano et al. in [2]. E. Damiano studied the flow of a fluid of continuous viscosity. N. Khomasuridze prefers the model which makes it possible to investigate the blood flow in capillaries with non-circular cross-section and in curved capillaries. In that case, if the flow rate profile has been established experimentally, then we can estimate the content of erythrocytes in different layers. This is the problem of delocalization in the sense that in the considered multi-layer flow it is possible – through the choice of a number of layers, their distribution and through the choice of erythrocyte viscosity and concentration – to obtain a more obtuse profile as compared with the parabolic Poiseuille profile. Then the flow resistance diminishes. Such a pattern has been established experimentally. Since in narrow vessels the shear rate is small and the Reynolds number does not exceed $0.01 - 0.1$, nonlinear inertial terms in the Navier-Stokes system of equations can be neglected. This paper deals with the steady flow of a viscous incompressible multi-layer fluid in a toroidal tube. Let us consider the toroidal system of coordinates ρ, α, β ($0 \le \rho < \infty$, $0 \le \alpha < 2\pi$, $0 < \beta < 2\pi$) with the Lamé parameters

$$h_\rho = h_\beta = h = \frac{m_0}{\operatorname{ch}\rho - \cos\beta}\,, \quad H = \frac{m_0}{\operatorname{ch}\rho - \cos\beta}\,,$$

where m_0 is the scale factor. It is assumed that the displacement velocity vector $\vec{U}(u, v, w)$, where u, v, w are the projections of the vector $\vec{U}$ on the normals to the coordinate surfaces $\rho = const$, $\alpha = const$, $\beta = const$, contains only the projection $v(\rho, \beta)$ and therefore $u = 0$, $w = 0$. In that case, the continuity condition is fulfilled identically and the Stokes system of equations takes the form

$$\frac{\partial p}{\partial \alpha} = 0, \quad \frac{\partial p}{\partial \beta} = 0, \quad \frac{\partial}{\partial \rho}\Big(H\,\frac{\partial v}{\partial \rho}\Big) + \frac{\partial}{\partial \beta}\Big(H\,\frac{\partial v}{\partial \beta}\Big) - \frac{h^2}{H} = \frac{h^2}{\mu}\,\frac{\partial p}{\partial \alpha}\,, \qquad (1)$$

where p is the hydrostatic pressure, μ is the dynamic viscosity coefficient, p_{ij} are normal and shear stresses, $p_{11} = p_{22} = p_{33} = -p$.

By the substitution $v = \sqrt{2(\operatorname{ch}\rho - \cos\beta)}\,\widetilde{v}$, the third equation in (1) is reduced to the form

$$\frac{\partial^2 \widetilde{v}}{\partial \rho^2} + \frac{\partial^2 \widetilde{v}}{\partial \beta^2} + \operatorname{cth}\rho\,\frac{\partial \widetilde{v}}{\partial \rho} + \Big(\frac{1}{4} - \frac{1}{\operatorname{sh}\rho}\Big)\widetilde{v} = \frac{h^{3/2}}{\sqrt{2}\,\mu\operatorname{sh}\rho}\,\frac{\partial p}{\partial \alpha}\,. \qquad (2)$$

The first two equations in (1) imply that $p = p(\alpha)$, while from the equation (2) it follows that $\frac{\partial p}{\partial \alpha} = const$.

We denote the viscosity coefficient of the external layer by μ_1, and the coefficients of the internal layers by μ_i $(i = 2, \ldots, m)$. Let us state the boundary-contact problem. On the surface of a toroidal tube $\rho = \rho_1$ the nonslip condition is fulfilled, i.e. $v = 0$. At the tube ends we are given the pressure values $\alpha = \alpha_0$ and $\alpha = \alpha_1$. Therefore the flow occurs due the pressure drop. The following contact conditions are given at the interface of the layers $\rho = \rho_1$:

$$v_i = v_{i+1}, \quad p_{23}^{(i)} = p_{23}^{(i+1)}, \quad p_{11}^{(i)} = p_{11}^{(i+1)}, \quad i = 1, \ldots, m. \tag{3}$$

Applying the method of separation of variables, we obtain the general solution of the equation (2)

$$\widetilde{v}_i = \sum_{n=1}^{\infty} \Big[A_{ni} P^1_{n-\frac{1}{2}}(\operatorname{ch} \rho) + B_{ni} Q^1_{n-\frac{1}{2}}(\operatorname{ch} \rho) \Big] \cos n\beta + \widetilde{v}_i^*, \quad i = 1, \ldots, m,$$

where $P^1_{n-\frac{1}{2}}$ and $Q^1_{n-\frac{1}{2}}$ are toroidal functions and $\widetilde{v}^*$ is some partial solution that can be found by the standard method. Note that the right-hand side of (2) is expanded into a Fourier series in terms of Legendre polynomials

$$\frac{1}{\operatorname{sh} \rho (\operatorname{ch} \rho - \cos \beta)} = \frac{\sqrt{2}}{\operatorname{sh}^2 \rho} \sum_{n=1}^{\infty} e^{-(n+\frac{1}{2})\rho} P_n(\cos \beta).$$

In the sequel we will consider the flow in the tube with a very small lumen. Then, taking into account the behavior of hyperbolic and toroidal functions for large values of the argument, we obtain the solution

$$v_1 = \sqrt{2(\operatorname{ch} \rho - \cos \beta)} \Big[C_1 e^{-\frac{\rho}{2}} + \frac{k}{4\mu_1} e^{-\frac{5}{2}\rho} \Big],$$

$$v_i = \sqrt{2(\operatorname{ch} \rho - \cos \beta)} \Big[A_{1i} e^{-\frac{\rho}{2}} + \frac{k}{4\mu_i} e^{-\frac{5}{2}\rho} \Big],$$

where C_1 and A_{1i} are calculated from the combined system of algebraic equations corresponding to the boundary-contact conditions, $k = (\frac{\partial p}{\partial \alpha}) \frac{m_0}{\sqrt{2}}$. The solution takes into account the fact that for $\rho \to \infty$ the velocity remains finite. Note that from the third condition in (3) it follows that the hydrostatic pressure is the same in all the fluid layers.

We conclude by the following observation. If the scale factor m_0 is sufficiently large, then the considered toroidal tube differs little from the straight cylindrical tube. By changing the value m_0, the tube curvature can be increased.

References

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Author's address:

N. Khomasuridze and Z. Siradze
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0143
Georgia
E-mail: surab_siradze@yahoo.com

K. Ninidze
N. Muskhelishvili Institute of Computational Mathematics
8, Akuri St., Tbilisi 0193
Georgia