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Ivan Kiguradze

SOME BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS FOR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. For nonlinear functional differential systems optimal sufficient conditions for the solvability and well-posedness of boundary value problems on infinite intervals are established.

[illegible]

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In the present paper on the infinite interval I we consider the nonlinear functional differential system

$$x'(t) = f_1(x,y)(t), \quad y' = f_2(x,y)(t), \tag{1}$$

where f_1 and f_2 are the operators acting from the space $C_{loc}(I;\mathbb{R}^{n_1+n_2})$ to the spaces $L_{loc}(I;\mathbb{R}^{n_1})$ and $L_{loc}(I;\mathbb{R}^{n_2})$. In the case $I = \mathbb{R}_+$, for this system we investigate the problem

$$x(0) = c, \quad \sup\big\{\|x(t)\| + \|y(t)\| : \ t \in \mathbb{R}_+\big\} < +\infty, \tag{2}$$

and in the case $I = \mathbb{R}$ the problem

$$\sup\big\{\|x(t)\| + \|y(t)\| : \ t \in \mathbb{R}\big\} < +\infty. \tag{3}$$

Earlier, these problems were studied only in the cases, where f_1 and f_2 are either the Nemytski's operators ([3], [4], [5]), or the linear operators ([1], [2], [6]). Below, we will present new, and in a certain sense, unimprovable conditions which guarantee, respectively, the solvability and well-posedness of $(1), (2)$ and $(1), (3)$.

Throughout the paper, the following notation will be used;

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$, $\mathbb{R}_- =]-\infty, 0]$.

$\mathbb{R}^n$ is the space of n-dimensional vectors $x = (x_i)_{i=1}^n$ with components $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^n |x_i|.$$

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$x \cdot y$ is the scalar product of the vectors x and $y \in \mathbb{R}^n$.

If $x = (x_i)_{i=1}^m \in \mathbb{R}^m$ and $y = (y_i)_{i=1}^n \in \mathbb{R}^n$, then $z = (x, y)$ is the $(m+n)$-dimensional vector with components $z_i = x_i$ $(i = 1, \dots, m)$ and $z_{m+i} = y_i$ $(i = 1, \dots, n)$.

If $x = (x_i)_{i=1}^n$, then $\operatorname{sgn} x = (\operatorname{sgn} x_i)_{i=1}^n$.

$X = (x_{ik})_{i,k=1}^n$ is the $n \times n$-matrix with components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$.

$r(X)$ is the spectral radius of X.

$C(I; \mathbb{R}^n)$ is the space of continuous and bounded on I vector functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\|_{C(I;\mathbb{R}^n)} = \sup\big\{\|x(t)\| : \ t \in I\big\}.$$

$C_{loc}(I; \mathbb{R}^n)$ is the space of continuous vector functions $x : I \to \mathbb{R}^n$ with topology of uniform convergence on every compact interval contained in I.

$L_{loc}(I; \mathbb{R}^n)$ is the space of locally Lebesgue integrable vector functions $x : I \to \mathbb{R}$ with topology of mean convergence on every compact interval contained in I.

We say that the operator $f : C_{loc}(I; \mathbb{R}^n) \to L_{loc}(I; \mathbb{R}^m)$ satisfies the local Carathéodory conditions if it is continuous and for every $\rho > 0$ there exists a nonnegative function $f_\rho^* \in L_{loc}(I; \mathbb{R})$, such that

$$\|f(x)(t)\| \le f_\rho^*(t) \ \text{ for } \ t \in I, \ \ x \in C(I; \mathbb{R}^n), \ \ \|x\|_{C(I;\mathbb{R}^n)} \le \rho.$$

The vector function $g : I \times \mathbb{R}^n \to \mathbb{R}^m$ satisfies the local Carathéodory conditions if $g(\cdot, x) : I \to \mathbb{R}^m$ is measurable for every $x \in \mathbb{R}^n$, $g(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is continuous for almost all $t \in I$ and for every $\rho > 0$ there exists a nonnegative function $g_\rho^* \in L_{loc}(I; \mathbb{R})$, such that

$$\|g(t, x)\| \le g_\rho^*(t) \ \text{ for } \ t \in I, \ \ x \in \mathbb{R}^n, \ \ \|x\| \le \rho.$$

A particular case (1) is the differential system with deviating arguments

$$x_i'(t) = g_i\big(t, x(t), x(\tau_1(t)), y(t), y(\tau_i(t))\big) \ \ (i = 1, \dots, n). \tag{4}$$

Everywhere below, when we will be concerned with the problem (1), (2) (with the problem (1), (3)) it will be assumed that $c \in \mathbb{R}^{n_1}$ and the operators

$$f_i : C_{loc}(I; \mathbb{R}^{n_1+n_2}) \to L_{loc}(I; \mathbb{R}^{n_i}) \ \ (i = 1, 2),$$

where $I = \mathbb{R}_+$ $(I = \mathbb{R})$ satisfy the local Carathéodory conditions.

Analogously, the problem (4), (2) (the problem (4), (3)) is considered under the assumption that $c \in \mathbb{R}^{n_1}$ and the functions

$$g_i : I \times \mathbb{R}^{2n_1+2n_2} \to R^{n_i} \ \ (i = 1, 2),$$

where $I = \mathbb{R}_+$ $(I = \mathbb{R})$ satisfy the local Carathéodory conditions.

Under the solution of the system (1) (of the system (4)) on I is meant the function $(x, y) : I \to \mathbb{R}^{n_1+n_2}$ with locally absolutely continuous components $x : I \to \mathbb{R}^{n_1}$ and $y : I \to \mathbb{R}^{n_2}$, which almost everywhere on I satisfies this system.

Theorem 1. *Let $I = \mathbb{R}_+$ $(I = \mathbb{R})$ and there exist operators $p_i : C(I;\mathbb{R}^{n_1+n_2}) \to L_{loc}(I;\mathbb{R}_+)$ $(i = 1,2)$, a nonnegative constant h_0, and a nonnegative constant matrix $H = (h_{ik})_{i,k=1}^2$, such that*

$$r(H) < 1 \tag{5}$$

and for any $(x,y) \in C(I;\mathbb{R}^{n_1+n_2})$ almost everywhere on I the inequalities

$$\begin{aligned}
&f_1(x,y)(t) \cdot \operatorname{sgn} x(t) \le \\
&\qquad \le p_1(x,y)(t)\big(- \|x(t)\| + h_{11}\|x\|_{C(I;\mathbb{R}^{n_1})} + h_{12}\|y\|_{C(I;\mathbb{R}^{n_2})} + h_0\big), \\
&f_2(x,y)(t) \cdot \operatorname{sgn} y(t) \le \\
&\qquad\qquad \le p_2(x,y)(t)\big(\|y(t)\| - h_{11}\|x\|_{C(I;\mathbb{R}^{n_1})} - h_{12}\|y\|_{C(I;\mathbb{R}^{n_2})} - h_0\big)
\end{aligned}$$

hold. The problem (1), (2) (*the problem* (1), (3)) *has at least one solution.*

Remark 1. For the condition (5) to be fulfilled, it is necessary and sufficient that

$$h_{11} + h_{22} < 2, \quad h_{11} + h_{22} - h_{11}h_{22} + h_{12}h_{21} < 1.$$

Remark 2. In the above-formulated theorem the condition (5) is unimprovable and it cannot be replaced by the condition $r(H) \le 1$.

Corollary 1. *Let for $I = \mathbb{R}_+$ (for $I = \mathbb{R}$) all the conditions of Theorem* 1 *be fulfilled and*

$$\int_0^{+\infty} p_2(x,y)(s)\,ds = +\infty \quad \left(\int_{-\infty}^0 p_1(x,y)\,ds = \int_0^{+\infty} p_2(x,y)(s)\,ds = +\infty \right) \tag{6}$$

for any $(x,y) \in C(I;\mathbb{R}^{n_1+n_2})$. Then every solution of the problem (1), (2) (*of the problem* (1), (3)) *admits the estimate*

$$\begin{gathered}
\|x\|_{C(\mathbb{R}_+;\mathbb{R}^{n_1})} + \|y\|_{C(\mathbb{R}_+;\mathbb{R}^{n_2})} \le \rho\big(\|c\| + h_0\big) \\
\Big(\|x\|_{C(\mathbb{R};\mathbb{R}^{n_1})} + \|y\|_{C(\mathbb{R};\mathbb{R}^{n_2})} \le \rho h_0 \Big),
\end{gathered} \tag{7}$$

where ρ is a positive constant depending only on H.

Remark 3. The condition (6) in Corollary 1 is essential and it cannot be omitted.

For the system (4), Theorem 1 and Corollary 1 yield the following propositions.

Corollary 2. *Let $I = \mathbb{R}_+$ $(I = \mathbb{R})$, and there exist functions $p_i : I \times \mathbb{R}^{2n_1+2n_2} \to \mathbb{R}_+$ $(i = 1,2)$, satisfying the local Carathéodory conditions, and nonnegative constants h_{ik} $(i,k = 1,2)$, h_0, h_1, h_2 such that the matrix*

$$H = \begin{pmatrix} h_{11} & h_1 + h_{12} \\ h_2 + h_{21} & h_{22} \end{pmatrix} \tag{8}$$

satisfies the condition (5) *and on the set* $I \times \mathbb{R}^{2n_1+2n_2}$ *the inequalities*

$$g_1(t,x,\overline{x},y,\overline{y}) \cdot \operatorname{sgn} x \leq$$
$$\leq p_1(t,x,\overline{x},y,\overline{y})(-\|x\| + h_{11}\|\overline{x}\| + h_1\|y\| + h_{12}\|\overline{y}\| + h_0),$$
$$g_2(t,x,\overline{x},y,\overline{y}) \cdot \operatorname{sgn} y \geq$$
$$\geq p_2(t,x,\overline{x},y,\overline{y})(\|y\| - h_2\|x\| - h_{21}\|\overline{x}\| - h_{22}\|\overline{y}\| + h_0)$$

hold. Then the problem (4), (2) (*the problem* (4), (3)) *has at least one solution.*

Corollary 3. *Let for* $I = \mathbb{R}_+$ (*for* $I = \mathbb{R}$) *all the conditions of Corollary* 2 *be fulfilled, and*

$$\int\limits_0^{+\infty} p_{02}(s)\,ds = +\infty \quad \Bigg(\int\limits_{-\infty}^{0} p_{01}(s)\,ds = \int\limits_0^{+\infty} p_{02}(s)\,ds = +\infty \Bigg), \tag{9}$$

where

$$p_{0i}(t) = \inf\big\{p_i(t,x,\overline{x},y,\overline{y}) : (x,\overline{x}) \in \mathbb{R}^{2n_1},\ (y,\overline{y}) \in \mathbb{R}^{2n_2}\big\} \ \ (i=1,2). \tag{10}$$

Then every solution of the problem (4), (2) (*of the problem* (4), (3)) *admits the estimate* (7), *where* ρ *is a positive constant depending only on* H.

Now along with the functional differential system (1) consider the perturbed system

$$x'(t) = f_1(x,y)(t) + q_1(x,y)(t), \quad y'(t) = f_2(x,y)(t) + q_2(x,y)(t) \tag{1$'$}$$

with the boundary conditions

$$x(a) = \tilde{c}, \quad \sup\big\{\|x(t)\| + \|y(t)\| : t \in \mathbb{R}_+\big\} < +\infty \tag{2$'$}$$

and (3).

Let us introduce the following

Definition. Let $I = \mathbb{R}_+$ ($I = \mathbb{R}$) and $p_i : C_{loc}(I;\mathbb{R}^{n_1+n_2}) \to L_{loc}(I;\mathbb{R}_+)$ ($i=1,2$). The problem (1), (2) (the problem (1), (3)) is said to be well-posed with the weight (p_1,p_2) if it has a unique solution (x_0,y_0) and there exists a positive constant ρ such that for arbitrary $\tilde{c} \in \mathbb{R}^{n_1}$, $q_0 \in \mathbb{R}_+$, and for any operators $q_i : C_{loc}(\mathbb{R}_+;\mathbb{R}^{n_1+n_2}) \to L_{loc}(I;\mathbb{R}^{n_i})$ ($i=1,2$), satisfying the local Carathéodory conditions and the inequalities

$$|q_i(x,y)(t)| \leq p_i(x,y)(t)q_0 \ \ (i=1,2),$$

the problem (1$'$), (2$'$) (the problem (1$'$), (3)) is solvable and its arbitrary solution admits the estimate

$$\|x - x_0\|_{C(\mathbb{R}_+;\mathbb{R}^{n_1})} + \|y - y_0\|_{C(\mathbb{R}_+;\mathbb{R}^{n_2})} \leq \rho(\|c - \tilde{c}\| + q_0)$$
$$\Big(\|x - x_0\|_{C(\mathbb{R};\mathbb{R}^{n_1})} + \|y - y_0\|_{C(\mathbb{R};\mathbb{R}^{n_2})} \leq \rho q_0\Big).$$

Theorem 2. *Let $I=\mathbb{R}_+$ $(I=\mathbb{R})$, $c=0$, $f_i(0,0)(t)\equiv 0$ $(i=1,2)$, and let there exist operators $p_i : C_{loc}(I;\mathbb{R}^{n_1+n_2})\to L_{loc}(I;\mathbb{R}_+)$ $(i=1,2)$ and a nonnegative constant matrix $H=(h_{ik})_{i,k=1}^2$, satisfying the conditions* (5) *and* (6)*, such that for any $(x,y)\in C(I;\mathbb{R}^{n_1+n_2})$ the inequalities*

$$f_1(x,y)(t)\cdot \operatorname{sgn} x(t)\le$$
$$\le p_1(x,y)(t)\big(-\|x(t)\|+h_{11}\|x\|_{C(I;\mathbb{R}^{n_1})}+h_{12}\|y\|_{C(I;\mathbb{R}^{n_2})}\big),$$
$$f_2(x,y)(t)\cdot \operatorname{sgn} y(t)\ge$$
$$\ge p_2(x,y)(t)\big(\|y(t)\|-h_{21}\|x\|_{C(I;\mathbb{R}^{n_1})}-h_{21}\|y\|_{C(I;\mathbb{R}^{n_2})}\big)$$

hold almost everywhere on I. Then the problem (1), (2) (*the problem* (1), (3)) *is well-posed with the weight (p_1,p_2).*

Corollary 4. *Let $I=\mathbb{R}_+$ $(I=\mathbb{R})$, $c=0$, $g_i(t,0,0,0,0)\equiv 0$ $(i=1,2)$, and on the set $I\times\mathbb{R}^{2n_1+2n_2}$ the inequalities*

$$g_1(t,x,\overline{x},y,\overline{y})\cdot \operatorname{sgn} x\le$$
$$\le p_1(t,x,\overline{x},y,\overline{y})\big(-\|x\|+h_{11}\|\overline{x}\|+h_1\|y\|+h_{12}\|\overline{y}\|\big),$$
$$g_2(t,x,\overline{x},y,\overline{y})\cdot \operatorname{sgn} y\ge$$
$$\ge p_2(t,x,\overline{x},y,\overline{y})\big(\|y\|-h_2\|x\|-h_{21}\|\overline{x}\|-h_{22}\|\overline{y}\|\big)$$

hold, where h_i, h_{ik} $(i,k=1,2)$ are nonnegative constants, and $p_i : I\times\mathbb{R}^{2n_1+2n_2}\to\mathbb{R}_+$ $(i=1,2)$ are functions, satisfying the local Carathéodory conditions. Let, moreover, the matrix H and the functions p_{0i} $(i=1,2)$, given by the equalities (8) *and* (10)*, satisfy the conditions* (5) *and* (9)*. Then the problem* (4), (2) (*the problem* (4), (3)) *is well-posed with the weight (p_1,p_2).*

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Author's address:
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: kig@rmi.acnet.ge