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**BOUNDARY VALUE PROBLEMS
FOR DIFFERENTIAL EQUATIONS
OF FRACTIONAL ORDER**

Abstract. We carry out spectral analysis of a class of integral operators associated with fractional order differential equations arising in mechanics. We establish a connection between the eigenvalues of these operators and the zeros of Mittag-Leffler type functions. We give sufficient conditions for complete nonselfadjointness and completeness of the systems of the eigenfunctions. We prove the existence and uniqueness of solutions for several kinds of two-point boundary value problems for fractional differential equations with Caputo or Riemann–Liouville derivatives, and design single shooting methods to solve them numerically.

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რეზიუმე. ნაშრომში ჩატარებულია წილადი რიგის დიფერენციალურ განტოლებებთან ასოცირებული ინტეგრალური ოპერატორების ერთი კლასის სპექტრალური ანალიზი. დადგენილია კავშირი ამ ოპერატორების საკუთრივ მნიშვნელობებსა და მიტაგ-ლეფლერის ტიპის ფუნქციების ნულებს შორის. მოყვანილია საკუთრივ ფუნქციების სისტემების არათვითშეუღლებულობისა და სისრულის საკმარისი პირობები. დამტკიცებულია კაპუტოსა და რიმან-ლიუვილის წარმოებულებიანი წილადი რიგის დიფერენციალური განტოლებებისათვის სხვადასხვა სახის ორწერტილოვანი სასაზღვრო ამოცანების ამონახსნის არსებობა და ერთადერთობა. ამ ამოცანის რიცხვითი ამოხსნისათვის შემუშავებულია ერთი გასროლის მეთოდები.

Boundary Value Problems for Differential Equations of Fractional Order

1. The Basic Concepts

The spectral analysis of operators of the form

$$A_{\gamma}^{[\alpha, \beta]} u(x) = c_{\alpha} \int_0^x (x-t)^{\frac{1}{\alpha}-1} u(t) dt + c_{\beta, \gamma} \int_0^1 x^{\frac{1}{\beta}-1} (1-t)^{\frac{1}{\gamma}-1} u(t) dt$$

was carried out in [1] (similar operators were considered by G. M. Gubreev in the paper [2]). Here $\alpha, \beta, \gamma, c_{\alpha}, c_{\beta, \gamma}$ are real numbers, and α, β, γ are positive. These operators arise in the study of boundary value problems for differential equations of fractional order (see [3] and references therein, where the corresponding Green functions are constructed).

The present paper is devoted to studying boundary value problems for differential equations of fractional order and the accompanying integral operators of the form $A_{\gamma}^{[\alpha; \beta]}$.

In order to state the problems in concern we must mention some concepts from fractional calculus.

Let $f(x) \in L_1(0, 1)$. Then the function

$$\frac{d^{-\alpha}}{dx^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt \in L_1(0, 1)$$

is called the fractional integral of order $\alpha > 0$ with starting point $x = 0$, and the function

$$\frac{d^{-\alpha}}{d(1-x)^{-\alpha}} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1} f(t) dt \in L_1(0, 1)$$

is called the fractional integral of order $\alpha > 0$ with ending point $x = 1$ (refer to [6]). Here $\Gamma(\alpha)$ is Euler's Gamma-function. It is clear that when $\alpha = 0$, we identify both fractional integrals with the function $f(x)$. As we know (see [6]), the function $g(x) \in L_1(0, 1)$ is called the fractional derivative of the function $f(x) \in L_1(0, 1)$ of order $\alpha > 0$ with starting point $x = 0$ if

$$f(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} g(x).$$

Then denoting

$$g(x) = \frac{d^\alpha}{dx^\alpha} f(x),$$

in the future we will mean by

$$\frac{d^\alpha}{dx^\alpha}$$

the fractional integral when $\alpha < 0$ and the fractional derivative when $\alpha > 0$. The fractional derivative

$$\frac{d^\alpha}{d(1-x)^\alpha}$$

of order $\alpha > 0$ of a function $f(x) \in L_1(0, 1)$, with ending point $x = 1$ is defined in the similar way.

Let $\{\gamma_k\}_0^n$ be any set of real numbers satisfying the condition $0 < \gamma_j \leq 1$ ($0 \leq j \leq n$). We denote

$$\sigma_k = \sum_{j=0}^k \gamma_j - 1; \quad \mu_k = \sigma_k + 1 = \sum_{j=0}^k \gamma_j \quad (0 \leq k \leq n),$$

and we assume that

$$\frac{1}{\rho} = \sum_{j=0}^n \gamma_j - 1 = \sigma_n = \mu_n - 1 > 0.$$

Following M. M. Dzhrbashyan (see [6]), we consider the integro-differential operators

$$\begin{aligned} D^{(\sigma_0)} f(x) &\equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} f(x), \\ D^{(\sigma_1)} f(x) &\equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x), \\ D^{(\sigma_2)} f(x) &\equiv \frac{d^{-(1-\gamma_2)}}{dx^{-(1-\gamma_2)}} \frac{d^{\gamma_1}}{dx^{\gamma_1}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x), \\ &\dots\dots\dots \\ D^{(\sigma_n)} f(x) &\equiv \frac{d^{-(1-\gamma_n)}}{dx^{-(1-\gamma_n)}} \frac{d^{\gamma_{n-1}}}{dx^{\gamma_{n-1}}} \dots \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x). \end{aligned}$$

Here we note that if $\gamma_0 = \gamma_1 = \dots = \gamma_n = 1$, then obviously

$$D^{(\sigma_k)} f(x) = f^{(k)}(x) \quad (k = 0, 1, 2, \dots, n).$$

The objects of our investigation are boundary value problems for the following equations:

$$D^{(\sigma_n)} u - [\lambda + q(x)] u = 0, \quad 0 < \sigma_n < \infty, \tag{1.1}$$

$$u'' + D_{0x}^\alpha u + q(x)u = \lambda u, \quad 0 < \alpha < 1. \tag{1.2}$$

We will consider various versions of the equation (1.1). For $\gamma_0 = \gamma_1 = 1$, $\gamma_3 = \gamma_4 = \dots = \gamma_n = 0$, the equation (1.1) turns into the equation

$$\frac{1}{\Gamma(1 - \gamma_2)} \int_0^x \frac{u''(t)}{(x-t)^{\gamma_2}} dt - [\lambda + q(x)]u(x) = 0, \quad (1.3)$$

which is called a fractional oscillating equation [5], and the operator $D^{(\sigma_2)}$ is called the operator of fractional differentiation in Caputo sense [5]. For $\gamma_0 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = \dots = \gamma_n = 0$, the equation (1.1) turns into the equation

$$\frac{1}{\Gamma(1 - \gamma_1)} \frac{d}{dx} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt - [\lambda + q(x)]u(x) = 0. \quad (1.4)$$

The equation (1.4) has been investigated as a model equation of fractional order $1 < \sigma < 2$ (see [3] and references therein). Further, if $\gamma_0 = \gamma_2 = \dots = \gamma_n = 1$, then the equation (1.1) will be written as

$$D^{(\sigma_n)}u = \frac{1}{\Gamma(1 - \gamma_1)} \frac{d^{n-1}}{dx^{n-1}} \int_0^x \frac{d(t)}{(x-t)^{\gamma_1}} dt - [\lambda + q(x)]u(x) = 0. \quad (1')$$

The two-point boundary value problem of Dirichlet $u(0) = 0$, $u(1) = 0$ for the fractional oscillatory equation was studied by one of the authors of this paper in [5]. Therein for the first time Green's function for similar boundary value problems has been constructed. In particular, it was proved that the two-point problem of Dirichlet $u(0) = 0$, $u(1) = 0$ for the fractional oscillatory equation with $q(x) = 0$ is equivalent to the equation

$$u(x) = \frac{\lambda}{\Gamma(1 - \gamma_2)} \left[\int_0^x (1-t)^{1-\gamma_2} u(t) dt - \int_0^1 x^{1-\gamma_2} (1-t)^{1-\gamma_2} u(t) dt \right].$$

The same problem for the model fractional differential equation of order $1 < \sigma < 2$ is equivalent to the equation [3]

$$u(x) = \frac{\lambda}{\Gamma(1 + \gamma_1)} \left[\int_0^x (1-t)^{\gamma_1} u(t) dt - \int_0^1 x(1-t)^{\gamma_1} u(t) dt \right].$$

The operator A inverse to the operator B induced by the differential expression (1') and natural boundary conditions

$$u(0) = 0; \quad D^{(\sigma_1)}u|_{x=0} = 0, \dots, D^{(\sigma_{n-2})}u|_{x=0} = 0, \quad u(1) = 0$$

looks like (see [3] and [6])

$$Au = \frac{1}{\Gamma(\rho-1)} \left[\int_0^x x(1-t)^{\frac{1}{\rho}-1} u(t) dt - \int_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t) dt \right].$$

Thus we will note that if $\gamma_0 = \gamma_1 = \dots = \gamma_n = 1$, then the operator B will look like

$$Bu = \begin{cases} u^{(n)}, \\ u(0) = 0, \quad u'(0) = 0, \dots, u^{(n-2)}(0) = 0, \quad u(1) = 0 \end{cases}$$

and Green's function $H(x, s)$ of the corresponding inverse operator will be rewritten as

$$H(x, s) = \begin{cases} \frac{(1-s)^{n-1}x^{n-1} - (x-s)^{n-1}}{(n-1)!}, & 0 \leq x \leq s \leq 1, \\ \frac{(1-s)^{n-1}x^{n-1}}{(n-1)!}. \end{cases}$$

It was also stated in [3] that

$$Au = \int_0^x (x-t)u(t) dt - \int_0^1 x(1-t)u(t) dt.$$

In a certain sense, the disturbance of the operator can be expressed as

$$A_\varepsilon u = \frac{1}{\Gamma(2+\varepsilon)} \left[\int_0^x (x-t)^{1+\varepsilon} u(t) dt - \int_0^1 x(1-t)^{1+\varepsilon} u(t) dt \right].$$

It has been proved that the eigenvalues of the operator A_ε are simple.

In [3] the expression for disturbance of the operator A has not been given explicitly. In the present work, the corresponding formula is obtained and it seems to the authors that this work contributes to including the theory of differential equations of fractional order in the general frame of the theory of disturbance.

2. Expression and Properties of Green's Function

Let's consider the operator

$$Au = \frac{1}{\Gamma(\rho-1)} \left[\int_0^x (x-t)^{\frac{1}{\rho}-1} u(t) dt - \int_0^1 x^{\frac{1}{\rho}-1} (1-t)^{\frac{1}{\rho}-1} u(t) dt \right], \quad 0 < \rho < \frac{1}{2}$$

in $L_2(0, 1)$. It is known [1] that the number λ is an eigenvalue of the operator A_ρ only if λ_0^{-1} is a zero of the function $E_\rho(\lambda; \rho^{-1})$ and the corresponding eigenfunctions look like $\varphi_n^\rho(x) = x^{\frac{1}{\rho}-1} E_\rho(\lambda_n^\rho x^{1/\rho}; \rho^{-1})$. Let λ_n^ρ be the n -th eigenvalue of the operator A_ρ , U_ρ be a bounded area with a rectifiable boundary dU_ρ such that $\lambda_n^\rho \in U_\rho$ and $(\sigma(A_\rho)\lambda_n^\rho) \cap \overline{U_\rho} = \emptyset$.

Let

$$P_{\lambda_n^\rho}(A_\rho) = -\frac{1}{2\pi i} \int_{dU_\rho} R_\lambda(A_\rho) d\lambda$$

be the Riesz projector for operator the A_ρ corresponding to the eigenvalue λ_n^ρ .

Theorem 1.1. *The projector $P_{\lambda_n^\rho}$ continuously depends on the parameter ρ .*

Proof. The operators A_ρ are sums of special one-dimensional operators and operators of fractional integration. It is known that the operators of fractional integration form in $L_P(0,1)$, $P \geq 1$, a semigroup continuous in the uniform topology for all $\alpha > 0$ and strongly continuous for all $\alpha = 0$. As A_ρ is continuous in the uniform (operator) topology (i.e. $\|A_\rho - A_{\rho_0}\| \rightarrow 0$ for $\rho \rightarrow \rho_0$), we have $\|P_{\lambda_n^\rho} - P_{\lambda_n^{\rho_0}}\|$ as $\rho \rightarrow \rho_0$, which proves the theorem. \square

Corollary 1.1. *Let $0 < \rho < 1/2$. Then*

$$\dim P_{\lambda_n^\rho}(A_\rho)L_2(0,1) = 1.$$

Proof. Since

$$\lim_{\rho \rightarrow \rho_0} \|A_\rho - A_{\rho_0}\| = 0,$$

for ρ and ρ_0 close enough, we have

$$\|P_{\lambda_n^\rho}(A_\rho) - P_{\lambda_n^{\rho_0}}(A_{\rho_0})\| < 1.$$

Therefore, according to Theorem 2.1, it follows that the spaces $P_{\lambda_n^\rho}(A_\rho)L_2(0,1)$ and $P_{\lambda_n^{\rho_0}}(A_{\rho_0})L_2(0,1)$ have the same dimension. Since $\dim P_{\lambda_n^{1/2}}(A_{1/2})L_2(0,1) = 1$, the function $\dim P_{\lambda_n^\rho}(A_\rho)L_2(0,1) = 1$ for all $\rho \in (0,1)$. It follows from Corollary 2.1 that the eigenvalues of the operator A_ρ are simple, and so are all the zero points of the function $E_\rho(\lambda; \rho^{-1})$. By the way, for the case $1/2 < \rho < 2$ this result was already stated in [3]. \square

3. The Basic Oscillatory Properties of the Operator A ($0 < \rho < 1/2$)

Theorem 1.2. *We have the representation*

$$A_\varepsilon u = A + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots + \varepsilon^n A_n + \cdots, \quad \varepsilon > 0, \quad (1.5)$$

where

$$Au = \int_0^x (x-t)u(t) dt - \int_0^1 x(1-t)u(t) dt,$$

$$A_n u = \frac{1}{n!} \left[\int_0^x (x-t) \ln^n(x-t) dt - \int_0^1 x(1-t) \ln^n(x-t) dt \right]$$

are operators with special kernels.

Proof. Let's rewrite the operator as

$$A_\varepsilon u = M_\varepsilon u + N_\varepsilon u,$$

where

$$M_\varepsilon u = \int_0^x K_\varepsilon(x, t)u(t) dt, \quad (1.6)$$

$$K_\varepsilon(x, t) = \begin{cases} (x-t)^{1+\varepsilon}, & t < x, \\ 0, & t \geq x \end{cases} \quad (1.7)$$

$$N_\varepsilon = \int_0^1 \tilde{K}_\varepsilon(x; t)u(t) dt, \quad (1.8)$$

$$\tilde{K}_\varepsilon(x, t) = \begin{cases} x^{1+\varepsilon}(1-t)^{1+\varepsilon}, & t \neq 1, \\ 0, & t = 1. \end{cases}$$

Considering the disturbance of the operator A_ε , we write

$$(A - A_\varepsilon)u = (M - M_\varepsilon)u - (N - N_\varepsilon)u.$$

First, we deal with $(M - M_\varepsilon)u$. Clearly,

$$(M - M_\varepsilon)u = \int_0^1 [K(x, t) - K_\varepsilon(x, t)]u(t) dt.$$

Since

$$K(x, t) - K_\varepsilon(x, t) = \begin{cases} (x-t)[1 - (x-t)^\varepsilon], & t < x \\ 0, & t \geq x \end{cases} =$$

$$= \begin{cases} (x-t) \left[\varepsilon \frac{\ln(x-t)}{1!} + \varepsilon^2 \frac{\ln^2(x-t)}{2!} + \dots + \varepsilon^n \frac{\ln^n(x-t)}{n!} + \dots \right], & t < x, \\ 0, & t \geq x, \end{cases}$$

we have

$$(M - M_\varepsilon)u = \varepsilon \int_0^1 K_1(x, t)u(t) dt + \dots + \varepsilon^n \int_0^1 K_n(x, t)u(t) dt + \dots,$$

where

$$K_n(x, t) = \begin{cases} \frac{(x-t) \ln^n(x-t)}{n!}, & t < x, \\ 0, & t \geq x. \end{cases}$$

Thus

$$M_\varepsilon u = \int_0^1 K(x, t)u(t) dt - \varepsilon \int_0^1 K_1(x, t)u(t) dt - \dots - \varepsilon^n \int_0^1 K_n(x, t)u(t) dt - \dots.$$

Similarly,

$$N_\varepsilon u = \int_0^1 \tilde{K}(x, t) u(t) dt - \varepsilon \int_0^1 \tilde{K}_1(x, t) u(t) dt - \cdots - \varepsilon^n \int_0^1 \tilde{K}_n(x, t) u(t) dt + \cdots,$$

where

$$\tilde{K}_n(x, t) = \begin{cases} \frac{x(1-t) \ln^n(x-xt)}{n!}, & t < 1, \\ 0, & t = 1, \end{cases}$$

which proves Theorem 1.2. \square

Theorem 1.3. *All eigenvalues $\lambda_n(\varepsilon)$ of the operator $A(\varepsilon)$ are real.*

Proof. We have

$$\lambda_n(\varepsilon) = \pi n^2 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots, \quad (1.9)$$

$$\varphi_n(\varepsilon) = \sin nx + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \cdots, \quad (1.10)$$

where

$$\lambda_n = \sum_{k=1}^n (A_k \varphi_{n-k}, \sin nx), \quad (1.11)$$

$$\varphi_n = R \sum_{k=1}^n (\lambda_k - A_k) \varphi_{n-k}. \quad (1.12)$$

Here R is the resolvent of the operator A , corresponding to the eigenvalue πn^2 . This resolvent is an integral operator with the kernel

$$S(x, y) = \left[-\frac{y}{n} \cos ny \sin nx + \frac{1-x}{n} \sin ny \cos nx + \frac{1}{2n^2} \sin ny \sin nx \right], \quad y \leq x$$

(if $y > x$ it is necessary to interchange y and x in the right part of this formula).

Clearly R transforms H_0 (H_0 is the orthogonal complement of the function $\sin \pi nx$) into itself and cancels $\sin \pi nx$.

From (1.11) it follows that $\lambda_1 = (A_1 \sin nx, \sin x)$. As the kernel of the operator A_1 assumes real values, we have $\sin \lambda_1 = 0$. From (1.12) it follows that $\varphi_1 = R(nk^2 - A_1) \sin nx$ and since the kernels of the operators R and A_1 assume real values, we have $\text{Im } \varphi_1 = 0$. So, successively it is possible to establish that all λ_i are real. Since ε is real, $\lambda u(\varepsilon)$ is real too. \square

Theorem 1.4. *For the eigenvalues $\lambda_n(\varepsilon)$ and eigenfunctions $\varphi_n(\varepsilon)$ of the operator $A(\varepsilon)$ there hold the estimates*

$$|\lambda_n(\varepsilon) - \pi n^2| < \frac{\pi(2n-1)}{2}, \quad |\varphi_n(\varepsilon) - \sin nx| < \frac{1}{2}.$$

Proof. From (1.11) and (1.12), assuming that

$$\|A_n u\| \leq p^{n-1} \{u\|u\| + b\|A_0 u\|\}, \quad m = \|A_0\|; \quad \frac{1}{d} = \|R\|, \quad |\varepsilon| < \frac{1}{c},$$

where

$$c = \max \left\{ \frac{8(a+mb)}{d}, 8p + 4 \frac{a+mb}{d} \right\},$$

we obtain the simple formulas [8]

$$|\lambda(\varepsilon) - \lambda_0 - \varepsilon \lambda_1 - \dots - \varepsilon^n \lambda_n| \leq \frac{d}{2} (|\varepsilon|c)^{n+1}, \quad (1.13)$$

$$|\varphi(\varepsilon) - \varphi_0 - \varepsilon \varphi_1 - \dots - \varepsilon^n \varphi_n| \leq \frac{1}{2} (|\varepsilon|c)^{n+1}. \quad (1.14)$$

Let's calculate the values of the parameters a, b, c, d, m . First we find m , $m = \|A\| = \sup A$, where $\sup A$ is the spectral radius of the operator A . As $\sup A = \pi^{-1}$, we have $m = \pi^{-1}$. Further, $d = \text{dist}(\pi n^2; \Sigma'')$ (d is an isolating distance), where Σ'' is the spectrum of the operator A^{-1} with the excluded point πn^2 . Clearly $d = \pi(2n-1)$. To find other parameters a, b, c , we will obtain an estimate of the norm of the operator A_n :

$$\|A_n \varphi\|_{L(0,1)} \leq \int_0^1 \int_0^1 |K_n(x,t)| |\varphi(t)| dt dx.$$

Here $K_n(x,t)$ is the kernel of the operator A_n

$$\begin{aligned} \int_0^1 \int_0^1 |K_n(x,t)| \cdot |\varphi(t)| dt dx &= \int_0^1 \int_0^x \left| \frac{(x-t) \ln^n(x-t)}{n!} \right| |\varphi(t)| dt dx = \\ &= \int_0^1 |\varphi(t)| \int_0^{1-t} \frac{z |\ln^n z|}{n!} dz dt \leq \frac{1}{n!} \int_0^1 z |\ln^n z| dz \cdot \|\varphi\|_{L_1(0,1)}. \end{aligned}$$

Let's calculate the integral $\int z \ln z^n dz$.

$$\begin{aligned} &\int z \ln z^n dz = \\ &= \frac{z^2 (\ln z)^n}{2} - \frac{nx^2 (\ln x)^{n-1}}{2^2} + \dots + \frac{(-1)^{n-1} n(n-1)(n-2) \dots 2}{2^{n-1}} \left(\frac{x^2}{2} - \frac{1}{2^2} \right). \end{aligned}$$

Hence $\|A_n\| \leq \frac{1}{2^{n+1}}$. Now we will take $a = 1/4$, $p = 1/2$, and $b = 0$. Since

$$c = \max \left\{ 8 \frac{a+mb}{d} + 8p + 4 \frac{a+mb}{d} \right\},$$

we have

$$c = \max \left\{ \frac{2}{d}; 4 + \frac{1}{d} \right\} = 4 + \frac{1}{d} = 5.$$

Then

$$|\lambda(\varepsilon) - \lambda| \leq \frac{1}{2} \frac{\pi(2n-1)}{2}.$$

Thus we have found $|\varphi_\varepsilon - \varphi_0| \leq 1/2$, and proved Theorem 1.4. \square

Theorem 1.5. *Let $u_0(x), u_1(x), \dots, u_n(x), \dots$ be the eigenfunctions of the operator A_ρ , and $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$ be the corresponding eigenvalues. Then the least frequency self-oscillation, i.e. $\varphi_0(x)$, has no units, i.e. $u_0(x) \neq 0, 0 < x < 1$.*

Proof. Let λ_0 be the least eigenvalue of the operator A_ρ . Then

$$u_0(x) = E_\rho(-\lambda_0 x; \rho^{-1}) = \sum_{k=0}^{\infty} \frac{(-\lambda_0 x)^k}{\Gamma(\rho^{-1} + \rho_k^{-1})}.$$

Show that the function $u_0(x)$ has no zero in the interval $(0, 1)$. Suppose that the function $u_0(x)$ in a point $x_0 \in (0, 1)$ equals to zero, that is,

$$u_0(x_0) = E_\rho(-\lambda_0 x_0; \rho^{-1}) = \sum_{k=0}^{\infty} \frac{(-\lambda_0 x_0)^k}{\Gamma(\rho^{-1} + \rho_k^{-1})} = 0.$$

That's to say, the number $-\lambda_0 x_0$ is a zero point of $E_\rho(z, \rho^{-1})$. However, $-\lambda_0 x_0 < \lambda_0$ as $x_0 \in (0, 1)$ while we have assumed that the least zero is λ_0 . The obtained contradiction proves Theorem 1.5. \square

Remark 1. It is also possible to show analogously that $u_1(x)$ in the interval has exactly one zero, etc.

Theorem of existence of the basis made of root spaces of the operator A_ρ ($0 < \rho < 1/2$) is connected with the problem of completeness of systems of eigenfunctions of the operator induced by the differential expression

$$l(D)_n = \begin{cases} \frac{1}{\Gamma(1 - \gamma_1)} \frac{d^{n-1}}{dx^{n-1}} \int_0^x \frac{u'(t)}{(x-t)^{\gamma_1}} dt - (\lambda + q(x))u, \\ u(0) = 0, \quad D^{\sigma_1} u|_{x=0} = 0, \dots, D^{\sigma_{n-2}} u|_{x=0} = 0, \quad u(1) = 0 \end{cases}$$

which is studied in case where $q(x)$ is a semi-function (see [3] and references therein). If completeness of system of eigenfunctions bounded is proved, then the question is: is it possible to make basis with the eigenfunctions of this operator. Let us give an answer to this question.

Lemma 1.1. *The operator A is dissipative*

Proof. We will consider the operator

$$Au = \int_0^x (x-t)^{1+\varepsilon} u(t) dt - \int_0^1 x^{1+\varepsilon} (1-t)^{1+\varepsilon} u(t) dt.$$

Denote

$$v(t) = \int_0^x (x-t)^{1+\varepsilon} u(t) dt - \int_0^1 x^{1+\varepsilon} (1-t)^{1+\varepsilon} u(t) dt.$$

From this expression it follows that $u(x) = Dv \in L_2(0, 1)$. Consider again the product

$$\begin{aligned} (Au, \bar{u}) & \left[\int_0^x (x-t)^{1+\varepsilon} u(t) dt - \int_0^1 x^{1+\varepsilon} (1-t)^{1+\varepsilon} u(t) dt \right] \bar{u}(x) = \\ & = \int_0^1 \left[\int_0^x (x-t)^{1+\varepsilon} u(t) dt - \int_0^1 x^{1+\varepsilon} (1-t)^{1+\varepsilon} u(t) dt \right] \bar{u}(x) dx = \\ & = \int_0^1 v(x) D_x \bar{v} dt = \int_0^1 v(x) \frac{d}{dx} \left(\int_0^x \frac{\bar{v}'(t)}{(x-t)^\varepsilon} dt \right) dx = \\ & = \int_0^1 v(x) \left(\int_0^x \frac{\bar{v}(t)}{(x-t)^\varepsilon} dt \right) dx = \\ & = v(x) \int_0^1 v(x) \left(\int_0^x \frac{v'(t)}{(x-t)^\varepsilon} dt \right) \Big|_0^1 - \int_0^1 \left(\int_0^x \frac{\bar{v}'(t)}{(x-t)^\varepsilon} dt \right) \overline{v'(x)} dx = \\ & = - \int_0^1 \left(\int_0^x \frac{\bar{v}'(t)}{(x-t)^\varepsilon} dt \right) \overline{v'(t)} dx, \end{aligned}$$

as $v(0) = v(1) = 0$. Define $v'(x) = z(x)$. Then $(Au, \bar{u}) = -(J^\varepsilon z, \bar{z})$. Now, by virtue of a theorem of Matsaev–Polant, it follows that the values of the form $(J^\varepsilon z, \bar{z})$ lay in the angle $|\arg z| < \frac{\pi\varepsilon}{2}$, which proves Lemma 1.1. \square

In papers [1] and [23], the dissipativity of the operator $l(D)$ is proved.

4. The Basis Problem for Systems of Eigenfunctions

Theorem 1.6. *The system of eigenfunctions of the operator A forms a basis of the closed linear hull.*

Proof. Remind that the system of vectors $\{u_1, u_2, \dots, u_n, \dots\}$ forms a basis of the closed linear hull $G \subset H$ if the inequality

$$m \sum_{k=1}^n |c_k|^2 \leq \left\| \sum_{k=1}^n c_k \varphi^2 \right\|^2 \leq M \sum_{k=1}^n |c_k|^2$$

holds, where m, M are positive constants independent of $c_1, c_2, c_3, \dots, c_n$ ($n = 1, 2, 3, \dots$). Any vector f ($f \in G$) in that case uniquely expands into

a series $f = \sum_{n=1}^{\infty} c_n u_n$ ($\sum_{n=1}^{\infty} |c_n|^2 < \infty$). It is known that if the spectrum of a dissipative operator with completely continuous imaginary component “is pressed enough” to the real axis, then it is possible to form a basis of the linear closed hull of eigenvectors of this operator.

We need a theorem of Glazman.

Theorem of Glazman. Let A be a linear bounded dissipative operator with completely continuous imaginary component, possessing an infinite system of eigenvectors $\{u_k\}_{k=1}^{\infty}$, normalized by the condition $(u_k, u_k) = 1$ ($k = 1, 2, 3, \dots$), and $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding sequence of different eigenvalues. If the condition

$$\sum \frac{\operatorname{Im} \lambda_j \operatorname{Im} \lambda_k}{|\lambda_j - \bar{\lambda}_k|} < \infty$$

is satisfied, then the system $\{\varphi_k\}_{k=1}^{\infty}$ is a Bari–Riesz’s basis of the closed linear hull. Since for $0 < \rho < 1/2$ all eigenvalues of the operator A_ρ are real, due to theorem of Glazman the proof of Theorem 1.6 follows. Analogously, it is possible to prove similar statements for more general problems. \square

5. Partial Problem of Eigenvalues for the Operator A_ρ

$$(0 < \rho < 1/2)$$

In [1], [3], there have been determined areas in the complex plane where there are no eigenvalues of the operator A_ρ for any ρ . Here, as above, we will assume that $0 < \rho < 1/2$. In applied problems the greatest interest represents usually determination of first eigenvalues, therefore we will solve this problem for eigenvalues of the operator A_ρ .

Let’s consider the operator

$$A = \frac{1}{\Gamma(\rho-1)} \left[\int_0^x (x-t)^{1/\rho} u(t) dt - \int_0^1 x^{1/\rho-1} (1-t)^{1/\rho-1} u(t) dt \right] = A_0 u + A_1 u.$$

As $0 < \rho < 1/\rho$, the operator A_0 and A_1 are invertible. Therefore $sp(A) = spA_0 + spA_1$.

Clearly

$$spA_0 = 0, \quad spA_1 = \frac{1}{\Gamma(2 - 1/\rho)}.$$

Therefore the determinant

$$D_A(\lambda) = \sum_{j=1}^{\infty} (1 - \lambda \mu_j), \quad \mu_k = \frac{1}{\lambda_k},$$

is meaningful.

Earlier it has been established that the number λ is an eigenvalue of the operator A only if λ^{-1} is a zero of the function $E_\rho(\lambda; \rho^{-1})$. Since the entire

functions $D_a(A)$ and $E_\rho(\lambda; \rho^{-1})$ are of genus zero, we will notice that

$$D_A(\lambda) = cE_\rho(\lambda; 2),$$

where c is a constant unknown as yet. Consider the logarithmic derivative

$$[\ln D_A(\lambda)]' = \frac{D_A'(\lambda)}{D_A(\lambda)} = -\sum_{j=1}^{\infty} \frac{\lambda_j}{1 - \lambda\lambda_j} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\lambda_j^k k) \lambda^{k-1} \quad (|\lambda| < \lambda_1^{-1}),$$

that is,

$$[\ln D_A(\lambda)]' = \frac{D_A'(\lambda)}{D_A(\lambda)} = -sp(A(J - \lambda A)^{-1}) = -\sum \chi_{n+1} \lambda^n,$$

where $\chi_n = spA^n$.

Let $D_A(A) = \sum_{k=1}^{\infty} a_k \lambda^k$ be the representation by Taylor's series of the function $D_A(A)$. Establish the interrelation between χ_k and a_k . As

$$D_A(\lambda) = E_\rho(\lambda; \rho^{-1}) = \sum \frac{\lambda^k}{\Gamma(\rho^{-1} + k\rho^{-1})},$$

we have $a_n = \frac{1}{\Gamma(\rho^{-1} + k\rho^{-1})}$. Further, since

$$\frac{D_A'(\lambda)}{D_A(\lambda)} = \frac{\sum (na_n \lambda^n)}{\sum a_n \lambda^n} = -\sum \chi_{n+1} \lambda^n,$$

we obtain the recurrent formula

$$\begin{aligned} \chi_{n+1} + \sum a_s \chi_{n+1-s} &= -(n+1)a_{n+1}, \\ \chi_{n+1} &= -\sum a_{n-k} \chi_k - na_n = \left[-\sum \frac{(-1)^{n-k}}{\Gamma(2-\beta(n-k))} \chi_k - n \frac{(-1)^n}{\Gamma(2-n\beta)} \right], \\ \chi_1 &= \frac{1}{\Gamma(2+1/\rho)}, \quad 2a_2 + a_1 \chi_1 = -\chi_2, \quad \chi_2 = a_1^2 - 2a_2, \\ -\chi_3 &= \sum_{s=1}^2 3a_3^2 \chi_{3-s}, \quad \chi_3 = a_2 a_1 + a_1(2a_2 - a_1^2) - 3a_2. \end{aligned}$$

As $1/\chi_1 < \lambda_1 < \chi_1/\chi_2$, we have

$$\frac{1}{\Gamma(2+1/\rho)} < \lambda_1 < \frac{2}{\Gamma(2+1/\rho)}.$$

Let the set $\{\gamma_0, \gamma_1, \gamma_2\}$ consist of three numbers $0 \leq \gamma_j \leq 1$ ($j = 0, 1, 2$). Following M. M. Dzhrbashyan [9], we denote

$$\sigma_k = \sum_{j=0}^k \gamma_j - 1, \quad \mu_k = \sigma_k + 1 = \sum_{j=0}^k \gamma_j \quad (k = 0, 1, 2).$$

Assume that

$$\frac{1}{\rho} = \sum_{j=0}^2 \gamma_j - 1 = T_2 = \mu_2 - 1 > 0.$$

Since $1 < \sigma_2 < 2$, we have $1 < \frac{1}{\lambda} < 2$, i.e. $1/2 < \rho < 1$. The last assumption is very important.

Introduce into consideration the differential operators [5]

$$\begin{aligned}\tilde{\mu}_k &= \tilde{\sigma}_k + 1 = \sum_{j=0}^k \gamma_{2-j} \quad (k = 0, 1, 2), \\ D_1^{(\tilde{\sigma}_0)} f(x) &\equiv \frac{d^{-(1-\gamma_2)}}{d(1-x)^{-(1-\gamma_2)}} f(x), \\ D_1^{(\tilde{\sigma}_1)} f(x) &\equiv -\frac{d^{-(1-\gamma_1)}}{d(1-x)^{-(1-\gamma_1)}} \frac{d^{\gamma_2}}{d(1-x)^{\gamma_2}} f(x), \\ D_1^{(\tilde{\sigma}_2)} f(x) &\equiv \frac{d^{-(1-\gamma_0)}}{d(1-x)^{-(1-\gamma_0)}} \frac{d^{\gamma_1}}{d(1-x)^{\gamma_1}} \frac{d^{\gamma_2}}{d(1-x)^{\gamma_2}} f(x).\end{aligned}$$

Now the problem (\tilde{A}) may be formulated as follows. In the class $L_2(0, 1)$ (or $L_1(0, 1)$), find a nontrivial solution of the equation

$$D^{(\tilde{\sigma}_2)} z - \{\lambda + q(x)\}z = 0, \quad x \in [0, 1),$$

satisfying the boundary conditions

$$\begin{aligned}D_1^{(\tilde{\sigma}_0)} z \Big|_{x=0} \cos \alpha + D_1^{(\tilde{\sigma}_1)} z \Big|_{x=0} \sin \alpha &= 0, \\ D_1^{(\tilde{\sigma}_0)} z \Big|_{x=0} \cos \beta + D_1^{(\tilde{\sigma}_1)} z \Big|_{x=0} \sin \beta &= 0.\end{aligned}$$

The associated problem gives essentially new results, in the case where the order of the fractional differential equation is less than one. We will devote a separate paper to this case. To show how to transfer the obtained results to the case of the differential equations of order higher than two, we consider the following problem.

6. Operators of Transformation

V. A. Marchenko [7] when solving a reverse problem for the equation

$$y'' - q(x)y + \lambda y = 0 \quad (0 \leq x \leq 1) \quad (1.15)$$

builds an operator of transformation, transforming a solution of the equation (1.15) into a solution of the equation

$$y'' + \lambda y = 0. \quad (1.16)$$

By means of Green's function, with the initial data $y(0) = 0$, (1.16) corresponds to the differential operator

$$l(y) = y'' - q(x)y$$

and the integral operator

$$Ay = \int_0^x G(x, \xi)y(\xi) d\xi \quad (0 \leq x \leq 1),$$

where

$$G(x, x) = 0, \quad \left. \frac{d}{dx} G(x, \xi) \right|_{\xi=x} = 1.$$

From Marchenko's results it follows that the operator A is linearly equivalent to the operator of repeated integration [9]

$$J^2 y = \int_0^x (x - \xi) y(\xi) d\xi,$$

where $y(\xi) \in L^2[0, 1]$.

In the present paper, similar results will be obtained for the differential equations of the fractional order σ ($1 < \sigma \leq 2$). In what follows, we will use the operators of fractional differentiation of M. M. Dzhrbashyan [9], which are defined as follows.

Consider a problem of Cauchy type

$$J\{y; \gamma_0, \gamma_1, \gamma_2\} = D^{\sigma_2} y - \{\lambda + q(x)\}y = 0, \quad x \in (0, 1], \quad (1.17)$$

$$\left. \frac{d}{dx} D^{\sigma_0} \right|_{x=0} = C_0, \quad \left. \frac{d}{dx} D^{\sigma_1} \right|_{x=0} = C_1, \quad (1.18)$$

where $q(x) \in C[0, 1]$.

Note that some results of this paper have been published earlier in [1], [7]. In those works, operator of transformation translating a solution of the equation (1.18) to a solution of the equation $D^{\sigma_2} y - \lambda y = 0$ has been constructed. Let $y(x, \lambda)$ be a solution of a problem of Cauchy type. Then it is known [2] that the identity

$$\begin{aligned} y(x; \lambda) &= C_0 x^{\mu_0 - 1} E_\rho(\lambda x^{1/\rho}; \mu_0) + C_1 x^{\mu_1 - 1} E_\rho(\lambda x^{1/\rho}; \mu_1) + \\ &+ \int_0^x (x - \tau)^{1/\rho} E_\rho\left(\lambda(x - \tau)^{1/\rho}; \frac{1}{\rho}\right) q(\tau) y(\tau; \lambda) d\tau \end{aligned} \quad (1.19)$$

holds, where

$$E_\rho(x, \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + \frac{k}{\rho})} \quad (1.20)$$

is a function of Mittag-Leffler type.

In what follows, we will consider the case where $C_0 = 0$, $C_1 = 1$, $\lambda = iz$, $\text{Im } z = 0$, $\mu_1 = 1$, $|q(x)| < 1$. As $\mu_1 = 1$, we have

$$|E_\rho(\lambda x^{1/\rho}; \mu_1)| \leq C x^{-1/\rho} \quad (1 \leq x < \infty)$$

(see [5]). In [7],

$$\begin{aligned} y(x; \lambda) &= x^{\mu_1 - 1} E_\rho(\lambda x^{1/\rho}; \mu_1) + \\ &+ \int_0^x (x - \tau)^{1/\rho - 1} E_\rho\left(\lambda(x - \tau)^{1/\rho}; \frac{1}{\rho}\right) q(\tau) y(\tau; \lambda) d\tau, \end{aligned} \quad (1.21)$$

From the general Volterra theory it follows that the equation (1.21) can be solved by a method of iterations. Since the function $q(x)$ is continuous, we have

$$y(x, \lambda) = E_\rho(\lambda x^{1/\rho}; \mu_1) + \varphi_1(\lambda, x) + \varphi_2(\lambda, x) + \cdots, \quad (1.22)$$

where

$$\begin{aligned} \varphi_n(\lambda, x) = & \int_0^x d\tau_1 \cdots \int_0^{\tau_{n-1}} d\tau_n \cdots q(\tau_n) E_\rho(\lambda(x - \tau_1)^{1/\rho}; \mu_1) \cdots \times \\ & \times E_\rho(\lambda(\tau_{n-1} - \tau_n)^{1/\rho}; \mu_1) E_\rho(\lambda \tau_n^{1/\rho}; \mu_1). \end{aligned}$$

Obviously the series (1.22) converges in regular intervals in each interval $(0; l)$, $l < 1$.

Theorem 1.7. *There is an operator of transformation V , transforming a solution $E_\rho(\lambda x^{1/\rho}; \mu_1)$ of the equation (1.19) into a solution $y(x; \lambda)$ of the equation (1.17).*

Lemma 1.2. *The equality*

$$(J - \lambda J^{1/\rho})^{-1} \frac{1}{\Gamma(\rho^{-1})} = E_\rho(\lambda x^{1/\rho}; 1) \quad (1.23)$$

holds.

Proof. Let $f(x) \in L_1(0, 1)$, $\rho > 0$ and λ be any complex parameter. Then the integral equation

$$u(x) = f(x) + \frac{\lambda}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{1/\rho} u(t) dt \quad (1.24)$$

has a unique solution

$$u(x) = f(x) + \lambda \int_0^x (x-t)^{1/\rho-1} E_\rho\left(\lambda(x-t)^{1/\rho}; \frac{x}{\rho}\right) f(t) dt \quad (1.25)$$

belonging to the class $L_1(0, 1)$.

In (1.24), let $f(x) = \frac{1}{\Gamma(\rho^{-1})}$. We obtain

$$u(x) = \frac{1}{\Gamma(\rho^{-1})} + \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t) u(t) dt. \quad (1.26)$$

Due to (1.25) and (1.26), we have

$$u(x) = \frac{1}{\Gamma(\rho^{-1})} + \frac{1}{\Gamma(\rho^{-1})} \int_0^x (x-t)^{1/\rho-1} E_\rho\left(\lambda(x-t)^{1/\rho}; \frac{x}{\rho}\right) dt. \quad (1.27)$$

Using the known formula of M. M. Dzhrbashyan

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} E_\rho(\lambda t^{1/\rho}; \mu) t^{\mu-1} dt &= \\ &= z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho}; \mu + \alpha) \quad (\mu > 0, \alpha > 0), \end{aligned}$$

we calculate the integral

$$\frac{1}{\Gamma(\rho-1)} \int_0^x (x-t)^{1/\rho-1} E_\rho\left(\lambda(x-t)^{1/\rho}; \frac{x}{\rho}\right) dt = \lambda x^{1/\rho} E_\rho\left(\lambda x^{1/\rho}; 1 + \frac{1}{\rho}\right). \quad (1.28)$$

From (1.27) and (1.28), it follows that

$$u(x) = \frac{1}{\Gamma(\rho-1)} + \lambda x^{1/\rho} E_\rho\left(\lambda x^{1/\rho}; 1 + \frac{1}{\rho}\right) = E_\rho(\lambda x^{1/\rho}; 1).$$

Thus Lemma 1.2 is proved. \square

Lemma 1.3. *Let $y(x, \lambda)$ be a solution of a problem of Cauchy type (1.17)–(1.18). Then*

$$y(x, \lambda) = (J + \lambda)KA^{-1}f,$$

where

$$Au = \int_0^x (x-t)^{1/\rho-1} u(t) dt, \quad K = (J + Aq(x)), \quad f = 1.$$

Proof. Obviously the solution of the equation

$$\begin{cases} D^\sigma y + q(x)y - \lambda y = 0, \\ D^{\sigma_0} y|_{x=0} = 0, \\ D^{\sigma_1} y|_{x=0} = -1 \end{cases}$$

coincides with the solution of the equation

$$y + A^{-1}q(x)y + \lambda Ay = f.$$

Denote $K = (J + A^{-1}q(x))$. Then we will obtain

$$y = (J + \lambda x A^{-1})^{-1} f,$$

which proves Lemma 1.3. \square

Lemma 1.4. *The operator B is linearly equivalent to the operator $J^{1/\rho}$.*

Proof. From Theorem 1.7 and Lemma 1.3 it follows that

$$VE_\rho(\lambda x^{1/\rho}; 1) = (I + \lambda KA^{-1})^{-1} f. \quad (1.29)$$

From (1.23) and (1.29) it follows that

$$(J + \lambda B)^{-1} f = V(J + J^{1/\rho})^{-1} + \frac{1}{\Gamma(\rho-1)}. \quad (1.30)$$

Dividing both parts of the equality (1.30) successively by degrees of λ , we obtain

$$\lambda^0 f = \lambda^0 V \frac{1}{\Gamma(\rho-1)}, \quad Kf = V J^{1/\rho} \frac{1}{\Gamma(\rho-1)}.$$

From these equalities it follows that $K = V J^{1/\rho} V^{-1}$, which proves Lemma 1.4. \square

Theorem 1.8. *The operator B is a monocell Volterra operator.*

Proof. As the operator $J^{1/\rho}$ is monocell, then by virtue of a Lemma 1.4, the operator B is monocell too. \square

Theorem 1.8 can be used for solution of inverse problems for the differential equations of the fractional order.

The Sturm–Liouville Problems for a Second Order Ordinary Differential Equation with Fractional Derivatives in the Lower Terms

1. On Some Problems from the Theory of Equations of Mixed Type Leading to Boundary Problems for the Differential Equations of the Second Order with Fractional Derivatives

Consider the equation

$$\begin{cases} u'' + a_0(x)u' + \sum_{i=1}^m a_i(x)D_{0x}^{\alpha_i}\omega_i(x)u + u_{m+1}(x)u = f(x), \\ u(0)\cos\alpha + u'(0)\sin\alpha = 0, \\ u(1)\cos\beta + u'(1)\sin\beta = 0, \end{cases} \quad (2.1)$$

where $0 < \alpha_m < \dots < \alpha_1 < 1$, D^α is the operator of fractional differentiation of order α

$$D^\sigma u = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{u(t)}{(x-t)^\sigma} dt, \quad 0 < \sigma < 1.$$

Many direct and inverse problems associated with a degenerating hyperbolic equation and equation of the mixed hyperbolic-parabolic type are reduced to equations of the type (2.1). In [3] it is shown that to a problem (2.1) reduces an analogue of a problem of Tricomi type for the hyperbolic-parabolic equation with Gellerstendt operator in the body.

On Euclid planes with the cartesian orthogonal coordinates x and y we will consider the model equation in partial derivatives of the mixed (parabolic-hyperbolic) type

$$|y|^{mH(-1)} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^{1+H(-y)} u}{\partial y^{1+H(-y)}}, \quad (2.2)$$

where $m = \text{const} > 0$, $H(y)$ is the Heavyside function, $u = u(x, y)$. The equation (2.1) in the upper half-plane coincides with Fourier's equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \quad (2.3)$$

and in the lower half-plane it coincides with

$$(-y)^m \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \quad (2.4)$$

which for $y = 0$ transforms to an equation of parabolic type.

Let Ω be an area bounded by the segments of straight lines

$$\begin{aligned} AA_0 &: x = 0, \\ A_0B_0 &: y = y_0, \\ B_0B &: x = r, \end{aligned}$$

and the characteristics

$$AC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$$

and

$$BC : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = r$$

of the equation (2.4); $\Omega^+ = \{(x, y) : 0 < x < r, 0 < y < y_0\}$ be the parabolic part of the mixed area Ω ; Ω^- be the part of the area Ω , laying in the lower half-plane $y < 0$ and bounded by the characteristics AC , $0 \leq x \leq \frac{r}{2}$, BC , $\frac{r}{2} \leq x \leq r$ and the segment AB ; $]0, r[= \{(x, 0) : 0 < x < r\}$;

$$u = \begin{cases} u^+(x, y), & \forall (x, y) \in \Omega^+, \\ u^-(x, y), & \forall (x, y) \in \Omega^-, \end{cases} \quad (2.5)$$

where D_{0x}^l is the operator of fractional integrodifferentiation of Sturm–Liouville of order $|l|$ starting at the point 0.

Consider a problem of Tricomi type for the equation (2.1) in Ω , with nonlocal condition of linear conjugation.

Problem 1. Find a regular solution of the equation (2.1) in the areas Ω^+ , Ω^- , with the following properties

$$u^+ \in C(\overline{\Omega}^+) \cap C^1(\Omega^+ \cup]0, r[), \quad u^- \in C(\overline{\Omega}^-) \cap C^1(\Omega^- \cup]0, r[), \quad (2.6)$$

$$u^+(x, 0) = u^-(x, 0) - \lambda D_{0x}^{-2} u^-(t, 0), \quad 0 \leq x \leq r, \quad (2.7)$$

$$\frac{\partial(u^+ - u^-)}{\partial y} \Big|_{y=0} = 0, \quad 0 < x < r, \quad (2.8)$$

$$u^+(0, y) = \varphi_0(y), \quad u^-(r, y) = \varphi_r(y), \quad 0 \leq y \leq y_0, \quad (2.9)$$

$$u^- \Big|_{AC} = \psi(0), \quad 0 < x < r, \quad (2.10)$$

where $\overline{\Omega}$ is the closure of Ω , λ is the spectral parameter, $\varphi_0(y)$ and $\varphi_r(y)$ are given functions of the class $C^1[0, r]$, and

$$\psi(x) = u \left[\frac{x}{2}, - \left(\frac{m+2}{4} x \right)^{2/(m+2)} \right]$$

is a given function of the class $C^3[0, r]$.

Nonlocal condition of conjugation set by the equation (2.5), and local boundary conditions (2.6), (2.7) coincide with Tricomi conditions. For $\lambda = 0$, the problem (2.1) is an analogue of Tricomi type problem.

From (2.5) and (2.8) for $x = 0$ and (2.7) for $y = 0$, it follows that the equality $\varphi_0(0) = \psi(0)$ is a necessary condition for the consistency of boundary data.

We have the following lemma

Lemma 2.1. *Let (2.3) be a solution of Problem 1*

$$\tau(x) = u^-(x, 0), \quad \nu(x) = \left. \frac{\partial u^-}{\partial y} \right|_{y=0} = 0; \quad x^{-\beta} \nu(x) \in L[0, r].$$

Then

$$\tau''(x) - \lambda \tau(x) = \nu(x), \quad (2.11)$$

$$\Gamma(2\beta) D_{0x}^{1-2\beta} \tau(t) = [\beta_m \nu(x) + x^\beta D_{0x}^{1-\beta} \psi(t)] \Gamma(\beta), \quad (2.12)$$

$$\tau(0) = \varphi_0(0), \quad \tau(r) - \lambda \int_0^r (r-t) \tau(t) dt = \varphi_r(0), \quad (2.13)$$

where $\Gamma(z)$ is Eulers gamma-function, and

$$\beta = \frac{m}{2m+4}, \quad \beta_m = \frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)} (2-4\beta)^{2\beta-1}.$$

In fact, as $u = u^+(x, y)$ is a solution of the problem (2.1) in the area Ω^+ , from (2.2) by (2.4) we have $u^{+''}(x, 0) = \nu(x)$. On the other hand, according to the nonlocal condition of conjunction (2.5) $u^{+''}(x, 0) = r''(x) - \lambda \tau(x)$. From this equality the equality (2.10) follows between $\tau(x)$ and $\nu(x)$. The equation (2.11) represents another form of the known equation of theories of the relation between $\tau(x)$ and $\nu(x)$ known from the theory of mixed type equations, brought from the area Ω^- to the line of parabolic degeneration for Cellersted's equations. The condition (2.13) is a consequence of (2.5) and (2.7).

Excluding from the system (2.10)–(2.11), we have

$$L_{\beta\tau(x)}(x) = \lambda \tau + \psi_\beta(x), \quad (2.14)$$

where

$$L_{\beta\tau(x)}(x) = \tau''(x) - \mu_\beta D_{0x}^{1-2\beta} \tau(t), \quad (2.15)$$

$$\mu_\beta = \frac{\Gamma(2\beta)}{\beta_m \Gamma(\beta)}, \quad \psi_\beta = -\frac{1}{\beta_m} x^\beta D_{0x}^{1-\beta} \psi(t). \quad (2.16)$$

Hence, due to the condition of Lemma 2.1 the function $\tau(x) = u^-(x, 0)$ should be a solution of the following nonlocal problem.

Problem 2. Find a solution $\tau(x)$ of the equation (2.13) of class $C^2]0, r[\cap C^1]0, r[$, satisfying the condition (2.12).

Passing to investigation of structural and qualitative properties of the solution of the Problem 2, we will notice that any solution of the equation (2.13) of the class $C^2[0, r] \cap C^1[0, r] \cap [0, r]$ will be a solution of the equation

$$\tau(x) = \tau'(0)x + \tau(0) + \mu_\beta D_{0x}^{-2} D_{0t}^{1-2\beta} \tau(\xi) + \lambda D_{0x}^{-2} \tau(t) + D_{0t}^{1-2\beta} \psi_\beta(t). \quad (2.17)$$

The equation (2.16) is deduced from the equality (2.13) after application of the operator $D_{0t}^{1-2\beta}$ to both its parts.

It is easy to see that

$$D_{0x}^{-2} D_{0t}^{1-2\beta} \tau(\xi) = \int_0^x (x-t) \frac{\partial}{\partial t} D_{0t}^{-2\beta} \tau(\xi) dt = D_{0x}^{-1} D_{0t}^{-2\beta} \tau(\xi) = D_{0x}^{-1} D_{0t}^{-2\beta} \tau(t).$$

In view of the latter from (2.16) by virtue of (2.12) we have

$$\varphi_r(0) - \varphi_0(0) = rr'(0) + \mu_\beta D_{0r}^{-1-2\beta} \tau(t) + D_{0r}^{-2} \psi_\beta(t). \quad (2.18)$$

Substituting the value of $r'(0)$ (2.17) into the equation (2.16), we have

$$\tau(x) = \mu_\beta D_{0x}^{-\alpha} \tau(t) + a_1 x D_{0x}^{-\alpha} \tau(t) + \lambda D_{0x}^{-2} \tau(t) + f_0(x), \quad (2.19)$$

where $a_1 = \frac{-\mu_\beta}{r}$, $\alpha = 1 + 2\beta = 2(m+1)/(m+2)$,

$$f_\beta(x) = D_{0x}^{-2} \psi_\beta(t) + \varphi_0(0) + \frac{x}{r} [\varphi_r(0) - \varphi_0(0) - D_{0x}^{-2} \psi_\beta(t)]. \quad (2.20)$$

The equation (2.18) is an integral Fredholm's equation of the second kind and it is equivalent to Problem 1.

2. Formulas for Calculation of Eigenvalues of a Boundary Value Problem

Introduce some formulas from the theory of disturbance which we need in the further. Let $T(\chi, \varepsilon)$ be a linear operator bunch,

$$T(\chi, \varepsilon) = T + \chi T' + \varepsilon T'',$$

where T is a complete self-adjointed operator all eigenvalues of which are isolated and have multiplicity equal to 1; T' and T'' are defined in the same Hilbert space as T ; T is bounded.

$$T(\chi, \varepsilon) - \zeta = T - \zeta + \chi T' + \varepsilon T'' = [J + (\chi T' + \varepsilon T'')R(\zeta, T)](T - \zeta)$$

if ζ is not anigenvalue of $T(\chi, \varepsilon)$. Then, the resolvent

$$R(\zeta, x, \varepsilon) = R(\zeta, T) [J - (\chi T' - \varepsilon T'')R(\zeta, T)]^{-1}$$

exists if the term

$$[J - (\chi T' - \varepsilon T'')R(\zeta, T)]^{-1}$$

may be determined as a Neumann's series

$$[J - (\chi T' - \varepsilon T'')R(\zeta, T)]^{-1} = \sum_{n=0}^{\infty} [- (\chi T' - \varepsilon T'')R(\zeta, T)]^n,$$

i.e. the operator A , inverse to the operator $[J + (\chi T' + \varepsilon T'')R(\zeta, T)]^{-1}$, exists and equals to

$$\sum [- (\chi T' + \varepsilon T'')R(\zeta, T)]^n,$$

and for this a sufficient condition is the condition

$$\| \chi T' + \varepsilon T'' \| \leq \| R(\zeta, T) \|. \quad (*)$$

So, if the equality $(*)$ is fulfilled, then

$$\begin{aligned} R(\zeta, T) [J + (\chi T' + \varepsilon T'')R(\zeta, T)]^{-1} &= \\ &= R(\zeta, T) \sum_{p=0}^{\infty} (- (\chi T' + \varepsilon T'')R(\zeta, T))^p = R(\zeta, \chi, \varepsilon). \end{aligned} \quad (2.21)$$

Let λ be an eigenvalue of the operator $T = T(0, 0)$ of multiplicity $m = 1$, Γ be a closed positively oriented contour contained in the resolvent set and containing only eigenvalues λ of T . Consider the projector

$$P(\chi, \varepsilon) = -\frac{1}{2i\pi} \int_{\Gamma} R(\zeta, \chi, \varepsilon).$$

Assertion 2.1.

$$\dim P(\chi, \varepsilon) = \dim P(0, 0) = 1.$$

Proof of Assertion 2.1 follows from Lemma 1.4.10 [16].

Assertion 2.2. *Let λ be an eigenvalue of the operator $T(\chi, \varepsilon)$ corresponding to the eigenvalue λ of the operator T . Then*

$$\lambda(\chi, \varepsilon) = \text{tr}(T(\chi, \varepsilon)P(\chi, \varepsilon)) = \lambda + \text{tr}(T(\chi, \varepsilon) - \lambda)P(\chi, \varepsilon).$$

Proof of this assertion follows from the definition and from

$$\text{tr}P(\chi, \varepsilon) = \dim P(\chi, \varepsilon) = 1.$$

This formula gives a complete solution of the eigenvalues problem.

Assertion 2.3.

$$\lambda(\chi, \varepsilon) = -\frac{1}{2i\pi} \text{tr} \int_{\Gamma} (\zeta - \lambda)R(\zeta, \chi, \varepsilon) d\zeta. \quad (2.22)$$

In fact,

$$\begin{aligned} (T(\chi, \varepsilon) - \lambda)R(\zeta, \chi, \varepsilon) &= (T(\chi, \varepsilon) - \zeta + \zeta)R(\zeta, \chi, \varepsilon) = \\ &= [(T(\chi, \varepsilon) - \zeta) + (\zeta - \lambda)]R(\zeta, \chi, \varepsilon) = J + (\zeta - \lambda)R(\zeta, \chi, \varepsilon). \end{aligned}$$

Integrating both parts of the last inequality over ζ , we obtain

$$-\frac{1}{2i\pi} \int_{\Gamma} R(\zeta, \chi, \varepsilon) d\zeta = -\frac{1}{2i\pi} \int_{\Gamma} d\zeta - \frac{1}{2i\pi} \int_{\Gamma} (\zeta - \lambda)R(\zeta, \chi, \varepsilon) d\zeta.$$

As

$$\int_{\Gamma} R(\zeta, \chi, \varepsilon) d\zeta = P(\chi, \varepsilon), \quad \int_{\Gamma} d\zeta = 0,$$

Assertion 2.3 is proved.

Assertion 2.4. *The following equality holds*

$$\lambda(\chi, \varepsilon) - \lambda = -\frac{1}{2i\pi} \int_{\Gamma} \log [1 + (\chi T' + \varepsilon T'')] R(\zeta, T) d\zeta.$$

Proof. From the formula (2.22) we have

$$\lambda(\chi, \varepsilon) - \lambda = -\frac{1}{2i\pi} \operatorname{tr} \int_{\Gamma} \sum_{p=0}^{\infty} \frac{1}{p} (\zeta - \lambda) \frac{d}{d\zeta} [-(\chi T' + \varepsilon T'') R(\zeta, T)]^p d\zeta.$$

Substitution of (2.21) instead of the resolvent $R(\zeta, \chi, \varepsilon)$ gives us

$$\begin{aligned} \lambda(\chi, \varepsilon) - \lambda &= -\frac{1}{2i\pi} \operatorname{tr} \int_{\Gamma} (\zeta - \lambda) R(\zeta, \chi, \varepsilon) d\zeta = \\ &= \frac{1}{2i\pi} \int_{\Gamma} \log (1 + [(\chi T' + \varepsilon T'') R(\zeta, T)]) d\zeta. \end{aligned}$$

Until now we have considered various power series of χ and ε and did not specify explicitly a condition of their convergence. Now we investigate such conditions.

The series (2.21) obviously converges if the equality

$$\|(\chi T' + \varepsilon T'')\| < 1$$

holds and this condition takes place if

$$|\chi| < \frac{1}{2} (\|T' R(\zeta, T)\|), \quad |\varepsilon| < \frac{1}{2} (\|T'' R(\zeta, T)\|).$$

Denote by $r_0(\chi)$ and $r_1(\varepsilon)$ the values of χ and ε , respectively for which we have

$$(\|T' R(\zeta, T)\|)^{-1} = 2|\chi|, \quad (\|T'' R(\zeta, T)\|)^{-1} = 2\varepsilon.$$

Clearly the series (2.21) converges in regular intervals on $\zeta \in \Gamma$ if

$$|\chi| \leq r_0 = \min_{\zeta \in \Gamma} r_0(\zeta), \quad |\varepsilon| < r_1 = \min_{\zeta \in \Gamma} r_1(\zeta). \quad (2.23)$$

Hence it is clear that the radii of convergence r_0 and r_1 depend on the contour Γ . \square

Assertion 2.5. *Let $\rho = \max_{\zeta \in \Gamma} r_1 |\zeta - \lambda|$. Then*

$$|\lambda(\chi, \varepsilon) - \lambda| < \frac{\beta}{2\pi}.$$

In fact, from (2.22) we have

$$\lambda(\chi, \varepsilon) - \lambda = \operatorname{tr}[T(\chi, \varepsilon)P(\chi, \varepsilon)] = -\frac{1}{2\pi i} \operatorname{tr} \int_{\Gamma} (\zeta - \lambda)R(\zeta, \chi, \varepsilon) d\zeta.$$

Then

$$\begin{aligned} |\lambda(\chi, \varepsilon) - \lambda| &= \frac{1}{2\pi} \operatorname{tr} \int_{\Gamma} (\zeta - \lambda)R(\zeta, \chi, \varepsilon) d\zeta \leq \\ &\leq \frac{\rho}{2\pi} \operatorname{tr} \int_{\Gamma} R(\zeta, \chi, \varepsilon) d\zeta = \frac{\rho}{2\pi} \operatorname{tr} P(\chi, \varepsilon). \end{aligned}$$

Since

$$\frac{\rho}{2\pi} \operatorname{tr} P(\chi, \varepsilon) = \dim P(\chi, \varepsilon) = 1,$$

we have

$$|\lambda(\chi, \varepsilon) - \lambda| < \rho(2\pi)^{-1}. \quad (2.24)$$

A simple analysis of the formula (2.24) shows that the less is ρ , the closer gets $\lambda(\chi, \varepsilon)$ to λ as expected. Moreover, the less are χ, ε , the less should be $|\chi|, |\varepsilon|$ in general, for $\lambda(\chi, \varepsilon)$ to get within Γ .

Assertion 2.6. Let $T'R(\zeta, T) \in G_F$:

$$\|(\chi T' + \varepsilon T'')R(\zeta, T)\| < 1, \quad (2.25)$$

$$\{T'R(\zeta, T) + T''R(\zeta, T)\}v = \Sigma[a_k(v_i\varphi_i)\varphi_i + b_k(v_i\varphi_i)\psi_i], \quad (2.26)$$

where $(\varphi_i\varphi_j) = 0$, $(\varphi_i\varphi_i) = 1$, $(\varphi_i\psi_j) = 0$. Then

$$\begin{aligned} (T'R + T''R)^{2n} &= (T'R)^{2n-1} + (T''R)^{2n-1}, \\ (T'R + T''R)^{2n+1} &= (T'R)^{2n+1} + T'(T'')^{2n}R^{2n+1}. \end{aligned}$$

Proof. Consider the double power series

$$\begin{aligned} &1 + (-\chi T'R(\zeta, T)) + (-\chi T'R(\zeta, T))^2 + (-\chi T'R(\zeta, T))^3 + \\ &+ \dots + (-\varepsilon T''R(\zeta, T)) + (-\chi T'R(\zeta, T))(-\varepsilon T''R(\zeta, T)) + \\ &+ (-\varepsilon T''R(\zeta, T))(-\chi T'R(\zeta, T)) + \\ &+ \dots + (-\chi T''R(\zeta, T)) + (-\varepsilon T''R(\zeta, T))^2 + \dots \end{aligned}$$

The general member of this series can be rewritten as

$$[(\chi T' + \varepsilon T'')(R(\zeta, T))]^p.$$

Rewrite the formula (2.26):

$$\begin{aligned} &[\chi T'R(\zeta, T) + \varepsilon T''R(\zeta, T)]v^2 = \\ &= \left\{ [\chi T'R(\zeta, T)]^2 + \chi\varepsilon T'R(\zeta, T)T''R(\zeta, T) + \chi\varepsilon T''R(\zeta, T)T'R(\zeta, T) \right\}v = \\ &= \left\{ \chi^2 [T'R(\zeta, T)]^2 + \chi\varepsilon [T''R(\zeta, T)T'R(\zeta, T)]v \right\}, \end{aligned}$$

$$\begin{aligned}
[\chi T'R(\zeta, T) + \varepsilon T''R(\zeta, T)]^{2n}v &= [\chi^{2n}(T'R)^{2n} + \varepsilon\chi^{2n-1}T''R(T'R)^{2n-1}]v, \\
&= (T'R(\zeta, T) + T''R(\zeta, T))^3v = \\
&= (T'R(\zeta, T) + T''R(\zeta, T))^2(T'R(\zeta, T) + T''R(\zeta, T))v = \\
&= [(T'R)^3 + T''R(T'R)^2]v, \\
[T'R + T''R]^{2n+1}v &= [(T'R)^{2n} + T''(T')^{2n}(R(\zeta, T))^{2n+1}]v
\end{aligned}$$

which proves Assertion 2.6. \square

As a consequence, we note the following interesting fact.

Corollary 2.1. *Let $T(x) = \chi T' + T$, $\lambda_n(x)$ be the eigenvalues of $T(x)$, λ_n be the eigenvalues of T , where $T'R(\zeta, T)$ and $R(\zeta, T)$ are completely continuous operators, χ be a complex parameter, and*

$$R(\zeta, T)T' = T'R(\zeta, T).$$

Then

$$\lambda_n(\chi) - \lambda_n = \chi \operatorname{tr} T' P_n, \quad (2.27)$$

where [16]

$$P_n = -\frac{1}{2i\pi} \int_{\Gamma_n} R(\zeta, T) d\zeta.$$

Proof. From the equality

$$\lambda_n(\chi) - \lambda_n = \frac{1}{2i\pi} \int_{\Gamma_n} \operatorname{tr} \left(\sum \frac{(-1)^p}{p!} (\chi T'R)^p \right) d\zeta,$$

considering

$$\frac{d^{n+1}}{d\zeta^{n+1}} (R(\zeta, T)) = n!R(\zeta, T)$$

and

$$n! [T'R(\zeta, T)]^n = [R(\zeta, T)]^n n!(T')^n,$$

we obtain

$$\lambda_n(x) - \lambda_n = \frac{1}{2i\pi} \operatorname{tr} \sum \frac{(-\chi)^n}{nn!} (T')^n \frac{d^{n+1}}{d\zeta^{n+1}} R(\zeta, T) d\zeta.$$

Calculate the integrals

$$\int_{\Gamma_n} \frac{d}{d\zeta} (R(\zeta, T)) d\zeta = \int_{\Gamma_n} d(R(\zeta, T)) = 0, \quad n > 1.$$

Therefore

$$\lambda_n(x) - \lambda_n = -\frac{1}{2i\pi} \operatorname{tr} \int_{\Gamma_n} \chi T'R(\zeta, T) d\zeta = \chi \operatorname{tr} T' p_n.$$

Corollary 2.1 is proved. \square

If T coincides with a projector, the statement of Corollary 2.1 is obvious without planimetric integration. As

$$\lambda_n(x) - \lambda_n = -\frac{1}{2i\pi} \int_{\Gamma_n} \text{tr}[(\zeta - \lambda)R(\zeta, \chi, \varepsilon)] d\zeta,$$

we have

$$\begin{aligned} \lambda_n(x) - \lambda_n &= \frac{1}{2i\pi} \int_{\Gamma_n} \text{tr} \left(\sum [(\chi T' + \varepsilon T'')R(\zeta, T)]^P \frac{1}{P!} \right) d\zeta = \\ &= \frac{1}{2i\pi} \int_{\Gamma_n} \text{tr} \left[\chi T' R(\zeta, T) + \varepsilon T'' R(\zeta, T) + \chi^2 (T')^2 (R(\zeta, T))^2 + \right. \\ &\left. + \chi \varepsilon T' R(\zeta, T) T'' R(\zeta, T) + \chi \varepsilon T'' R(\zeta, T) T' R(\zeta, T) + \varepsilon^2 (T'' R(\zeta, T))^2 + \dots \right] d\zeta. \end{aligned}$$

Theorem 2.1. *Let the operator T' be permutable with the operator T , and the operator $T''R(\zeta, T)$ be a nilpotent operator with parameter of nilpotence equal to 2, $T'R, T''R \in G_F$. Then*

$$\lambda_n(x) - \lambda_n = \chi \text{sp} T' P_n + \varepsilon \text{sp} T'' P_n,$$

where P_n is a Riesz's projector of the operator T corresponding to λ_n :

$$P_n = \frac{1}{2i\pi} \int_{\Gamma_n} R(\zeta, T) d\zeta$$

Proof. From (2.27),

$$\lambda_n(\chi, \varepsilon) - \lambda_n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \chi^m \varepsilon^n \widehat{\lambda}_{nm},$$

where

$$\widehat{\lambda}_{nm} = \frac{(-1)^{m+n+1}}{2i\pi} \int_{\Gamma_n} a_{nm} \text{tr} (T' R(\zeta, T))^m (T'' R(\zeta, T))^n d\zeta.$$

As the operator $T''R(\zeta, T)$ is nilpotent, we have

$$\begin{aligned} \widehat{\lambda}_{0m} &= \frac{(-1)^{m+1}}{2i\pi} C_{mk}^k \int_{\Gamma_n} (T')^m (R(\zeta, T))^m, \\ \widehat{\lambda}_{1m} &= \frac{(-1)^{m+2}}{2i\pi} C_{mk}^m \int_{\Gamma_n} (T'')^m R(\zeta, T) (T') R^m, \\ \widehat{\lambda}_{jm} &= \frac{(-1)^{m+1}}{2i\pi} \int_{\Gamma_n} a_{jm} (T'' R(\zeta, T))^m T' R. \end{aligned}$$

We obtain $\widehat{\lambda}_{0m} = \chi \operatorname{tr} T' P_n$. Now as in proof of Corollary 2.1 we have $\lambda_n(\chi, \varepsilon) = \chi \widehat{\lambda}_{0m} + \varepsilon \widehat{\lambda}_{1m}$. For calculation of $\widehat{\lambda}_{1m}$ we will use

$$\int_{\Gamma_n} d(R^2(\zeta, T)) = 0.$$

Thus

$$\begin{aligned} \widehat{\lambda} &= \frac{(-1)^{m+2}}{2i\pi} a_{1m} \int_{\Gamma_n} T'(R(\zeta, T))(T''R(\zeta, T)) d\zeta = \\ &= \int_{\Gamma_n} R(\zeta, T) T' T'' R(\zeta, T) d\zeta = 0. \end{aligned}$$

Taking into account that

$$\left(\frac{d}{d\zeta}\right)^n R(\zeta, T) = n! [R(\zeta, T)]^{n+1} \quad (n = 1, 2, 3, \dots),$$

we obtain

$$\int_{\Gamma_n} T' T'' R^{k+1}(\zeta, T) d\zeta = \begin{cases} T' P_n, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Thus $\widehat{\lambda}_{0m} = \chi \operatorname{tr} T' P_n$, $\widehat{\lambda}_{1m} = \varepsilon \operatorname{tr} T'' P_n$, which proves Theorem 2.1. \square

Now using Theorem 2.1 we will calculate the eigenvalues of the problem

$$u'' + cD^{1-\alpha}u + \lambda u = 0, \quad u(0) = 0, \quad u(\pi) = 0, \quad 0 < \alpha < 1, \quad (2.28)$$

where $D^{1-\alpha}$ is the operator of fractional order $1 - \alpha$ in Weil sense, i.e.

$$\begin{aligned} D^{1-\alpha}u &= \frac{1}{2\pi} \frac{d}{dx} \int_0^{2\pi} u(x-t) \psi^{1-\alpha}(t) dt, \\ \psi^{1-\alpha}(t) &= 2 \sum_{k=1}^{\infty} \frac{\cos [kt - (1-\alpha)\pi/2]}{k^{1-\alpha}}. \end{aligned}$$

Suppose first that $c(x) \equiv \text{const}$. Represent the operator $D^{1-\alpha}R(\zeta, T)$ as the sum

$$D^{1-\alpha}R(\zeta, T) = \chi T' R(\zeta, T) + \varepsilon T'' R(\zeta, T)$$

so that T' and T'' satisfy the conditions of Theorem 2.1. Since the operator

$$Tu = \begin{cases} -u'', \\ u(0) = 0, \quad u(\pi) = 0, \end{cases}$$

is a full self-adjointed operator,

$$R(\zeta, T)v = \sum_{k=1}^{\infty} \frac{2}{\pi} \frac{(v, \sin kx)}{k^2 - \zeta} \sin kx.$$

Thus

$$D^{1-\alpha}R(\zeta, T)v = \frac{2}{\pi} D^{1-\alpha} \left(\sum_{k=1}^{\infty} \frac{2}{\pi} \frac{(v, \sin kx)}{k^2 - \zeta} \right) \sin kx.$$

As

$$\begin{aligned} D^{1-\alpha} \sin kx &= \frac{d}{dx} [D^{1-\alpha} \sin kx] = \\ &= \frac{d}{dx} \left[n^{-\alpha} \sin \left(nx - \frac{\alpha\pi}{2} \right) \right] = n^{1-\alpha} \left[\cos nx \cos \frac{\alpha\pi}{2} + \sin \frac{\alpha\pi}{2} \sin nx \right], \end{aligned}$$

we obtain

$$\begin{aligned} D^{1-\alpha}R(\zeta, T)v &= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{n^{1-\alpha}(v, \sin kx)}{k} \left[\cos kx \cos \frac{\alpha\pi}{2} + \sin \frac{\alpha\pi}{2} \right] = \\ &= \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} \frac{(v, \sin kx)}{k^2 - \zeta} \cos kx + \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \frac{(v, \sin kx)}{k^2 - \zeta}. \end{aligned}$$

Thus

$$D^{1-\alpha}R(\zeta, T)v = A\tilde{R}(\zeta, T)v + B\tilde{R}(\zeta, T)v,$$

where

$$\begin{aligned} Av &= \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} (v, \sin kx) \sin kx, \\ Bv &= \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} (v, \sin kx) \cos kx. \end{aligned}$$

Indeed,

$$\begin{aligned} AR(\zeta, T)v &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \left[\sum_{k=1}^{\infty} \frac{(v, \sin kx)}{k^2 - \zeta} \sin jx \sin kx \right] \sin kx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \frac{(v, \sin kx)}{k^2 - \zeta}, \\ BR(\zeta, T)v &= \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \left(\sum_{k=1}^{\infty} \frac{(v, \sin jx)}{k^2 - \zeta} \sin jx, \sin jx \right) \cos kx = \\ &= \frac{2}{\pi} \cos \frac{\alpha\pi}{2}, \end{aligned}$$

$$[AR+BR]^{2n+1}v = [(AR)^{2n+1} + BA^{2n}R^{2n+1}]v.$$

The fact that the operators A , $R(\zeta, T)$ are permutable, is checked directly:

$$\begin{aligned} AR(\zeta, T)v &= \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \left(\sum_{j=1}^{\infty} \frac{(v, \sin jx)}{j^2 - \zeta} \sin jx, \sin kx \right) \sin kx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \frac{(v, \sin kx)}{k^2 - \zeta} \sin kx, \end{aligned}$$

$$\begin{aligned}
R(\zeta, T)Av &= \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{n=1}^{\infty} \frac{(Av, \sin nx)}{n^2 - \zeta} \sin nx = \\
&= \frac{2}{\pi} \sin \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{\infty} n^{1-\alpha} (v, \sin kx) \sin nx}{n^2 - \zeta} \sin nx = \\
&= \frac{2}{\pi} \frac{\sin \alpha\pi}{2} \sum_{n=1}^{\infty} \frac{n^{1-\alpha} (v, \sin kx)}{n^2 - \zeta} \sin nx = \\
&= AR(\zeta, T)v,
\end{aligned}$$

i.e.,

$$AR(\zeta, T)v = R(\zeta, T)Av.$$

Now show the nilpotence of the operator $BR(\zeta, T)$:

$$\begin{aligned}
[BR(\zeta, T)]^2 v &= BR(\zeta, T)BR(\zeta, T)v = \\
&= \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} k^{1-\alpha} \frac{(BR(\zeta, T)v, \sin kx)}{k^2 - \zeta} \cos kx = \\
&= \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} \frac{\left(\frac{2}{\pi} \cos \frac{\alpha\pi}{2} \sum_{j=1}^{\infty} \frac{(v, \sin jx)}{j^2 - \zeta} \cos jx, \sin kx \right)}{k^2 - \zeta} \cos kx = \\
&= \left[\frac{2}{\pi} \cos \frac{\alpha\pi}{2} \right]^2 \sum_{k=1}^{\infty} k^{1-\alpha} \frac{\left(\sum_{j=1}^{\infty} \frac{(v, \sin jx)}{j^2 - \zeta} \cos jx, \sin kx \right) \cos kx}{k^2 - \zeta} = 0.
\end{aligned}$$

According to Theorem 1.4.1 [16], for the eigenvalues of the problem we have

$$\mu_k = n^2 + \widehat{\lambda}_{0m} + \widehat{\lambda}_{1m}.$$

Calculate $\widehat{\lambda}_{0m} = \text{tr} AP_n$. As $P_n = \frac{1}{2i\pi} \int_{\Gamma_n} R(\zeta, T) d\zeta$, p_n is an integral operator with the kernel

$$P_n(x, y) = \frac{2}{\pi} \sin nx \sin ny,$$

i.e.

$$P_n v = \frac{2}{\pi} \int_0^\pi \sin nx \sin ny v(y) dy = \frac{2}{\pi} (v, \sin nx) \sin nx.$$

We obtain

$$\begin{aligned}
Ap_n v &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum k^{1-\alpha} (v, \sin nx) (\sin nx, \sin kx) \sin kx = \\
&= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} (v, \sin nx) n^{1-\alpha} \sin nx.
\end{aligned}$$

Find the eigenvalues of the operator AP_n . $AP_n v = \lambda v$, that is,

$$\frac{2}{\pi} \sin \frac{\alpha\pi}{2} n^{1-\alpha} (v, \sin nx) = v\lambda.$$

Thus $\lambda_1 = n^{1-\alpha} \frac{2}{\pi} \sin \frac{\alpha\pi}{2}$, then ${}^A P_n = \lambda_1 = \frac{2}{\pi} n^{1-\alpha} \sin \frac{\alpha\pi}{2}$. Further

$$\begin{aligned} BP_n &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum (p_n v, \sin nx) \sin nx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \sum_{k=1}^{\infty} n^{1-\alpha} \left(\frac{2}{\pi} (v, \sin kx) \sin kx, \sin nx \right) \cos nx = \\ &= \frac{2}{\pi} \sin \frac{\alpha\pi}{2} (v, \sin kx) \cos nx = \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \cos nx \int_0^{\pi} v(t) \sin nt dt, \end{aligned}$$

i.e.,

$$BP_n v = \frac{2}{\pi} \sin \frac{\alpha\pi}{2} \cos nx \int_0^{\pi} v(t) \sin nt dt.$$

Calculate the trace of the operator BP_n

$$BP_n = \sum \lambda_n(BP_n).$$

To find the eigenvalues of the operator BP_n , we will solve the equation

$$BP_n v = v\lambda,$$

that is,

$$\frac{2}{\pi} \sin \frac{\alpha\pi}{2} \cos nx \int_0^{\pi} v(t) \sin nt dt = \lambda v(t).$$

Clearly ${}^B P_n = 0$. Thus

$$\mu_n = n^2 + n^{1-\alpha} \sin \frac{\pi\alpha}{2}.$$

For solution of the problem (2.28) in case $c(x) = const$ we use the following theorem being certainly of an independent interest as well.

3. Estimation of Eigenvalues

Theorem 2.2. *Let the eigenvalues of the self-adjoint operator A_0 with discrete spectrum be $\lambda_n^{(0)} = n^q$ ($n = 1, 2, 3, \dots, q$), the normed eigenvectors be φ_n and let B be a closed A_0 -bounded operator with*

$$D_{A_0} \subset D_B, \quad \|B\varphi_n\| = O(n^p)$$

Then for the eigenvalues λ_n of the operator $A = A_0 + B$ we have

$$\lambda_n - n^q = O(n^p).$$

Proof. Let Γ_n be the circle with radius 1 and with the center at the point n^q . Let n be so enough that inside of Γ_n there are no eigennumbers except λ_n . Let the operator A be defined by the formula $A = A_0 + B$. We need the following known result. \square

Lemma 2.2. Let A_0 be an operator with compact resolvent $R(\zeta, A_0)$, operator B be compact with respect to A_0 . If Δ is a bounded area with rectifiable boundary Γ and $\sigma(A_0 + tB) \cap \Gamma = \emptyset$ for all $t \in [0, 1]$, then the operator A also has a compact resolvent and

$$N(\Delta, A_0 + B) = N(\Delta, A_0),$$

where $N(A)$ is the number of eigenvalues of the operator laying inside the area.

Proof. Let $R(\zeta, A_0)$ be the resolvent of the operator A_0 . Then

$$R(A_0, \zeta) = \sum_{n=1}^{\infty} \frac{(v, \varphi_n) \varphi_n}{n^q - \zeta}.$$

By Lemma 2.1 it follows that if $\|BR(\zeta, A_0)\|_{\Gamma_n} < 1$, then $\sigma(A_0 + tB) \cap \Gamma = \emptyset$ for all $t \in [0, 1]$ and

$$\begin{aligned} \|BR(\zeta, A_0)\|_{\Gamma_n} &= \left\| \sum_{n=1}^{\infty} \frac{(v, \varphi_n) \varphi_n}{k^q - \zeta} B\varphi_k \right\|_{\Gamma_n} \leq \\ &\leq \sum_{n=1}^{\infty} \left\| \frac{(v, \varphi_n) \varphi_n}{k^q - \zeta} \right\|_{\Gamma_n} \|B\varphi_k\| \leq \frac{\|(v, \varphi_n) \varphi_n\|}{|k^q - \zeta|} n^\rho \leq \sum_{n=1}^{\infty} \frac{k^\rho}{k^\rho - (n^q - 1/3)}. \end{aligned}$$

Further,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{k^\rho}{k^\rho - (n^q - 1/3)} = \\ &= \sum_{n=1}^{n-1} \frac{k^\rho}{k^\rho - (n^q - 1/3)} + \frac{n^\rho}{n^\rho - n^q + 1/3} + \sum_{k=n+1}^{n-1} \frac{k^\rho}{k^\rho - (n^q - 1/3)}. \end{aligned}$$

The last sum will be less than 1 if by means of finite disturbance the operator A_0 is replaced by the operator $\widetilde{A}_0 v = A_0 v - 3n^\rho (v, \varphi_n) \varphi_n$. \square

Theorem 2.3 essentially strengthens and generalizes a known result of M. K. Gavurin.

Theorem of Gavurin. Let the eigenvalues of a self-adjoint operator with discrete spectrum be $\lambda_n^{(0)} = n^q$ ($n = 1, 2, 3, \dots; q > 1$), the normed eigenfunctions φ_n and B be a closed symmetric operator with

$$D_{A_0} \subset D_B, \quad \|B\varphi_n\| = O(n^\alpha).$$

Then, for the eigenvalues of the operator $A v = A_0 v + B v$ we have

$$\lambda_n = n^q + \sum_{k \neq n} \frac{(B\varphi_k, \varphi_n) |(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2} - \frac{\sum_{k \neq n} \frac{|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2}}{1 + \sum_{k \neq n} \frac{|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2}} +$$

$$+ \frac{\sum_{k,l \neq n} \frac{(B\varphi_k, \varphi_n)(\overline{B\varphi_n, \varphi_l})}{(k^q - n^q)(l^q - n^q)}}{1 + \sum_{k \neq n} \frac{|(B\varphi_k, \varphi_n)|^2}{|k^q - n^q|^2}} + O(n^{q-5}).$$

Here we will note that from Theorem 2.2 follows the known formula of I. M. Lifshits, having greater applications in quantum statistics.

Theorem of I. M. Lifshits. *Let H be a self-adjoint hypermaximal operator, T be a self-adjoint finite operator. Then*

$$\sum (\mu_n - \lambda_n) = spT,$$

where μ_n are the eigenvalues of the operator $H + T$, $\lambda_n = \lambda_n(H)$.

From Theorem 2.2 it follows a theorem generalizing Lifshits's theorem.

Theorem 2.4. *Under the conditions of Theorem 2.2 the formula*

$$\sum_{n=1}^{\infty} (\lambda_n(\chi, \varepsilon) - \lambda_n) = \varepsilon spT' + \chi spT''$$

holds.

Now in the case $c(x) \neq const$ from Theorem 2.2 follows

Theorem 2.5. *For the eigenvalues λ_n of the problem the following estimates hold*

$$\lambda_n - n^2 = O(n^\alpha).$$

Proof of this theorem follows from the proof of Theorem 2.3 for $q = 2$. \square

4. Cauchy Problem for Differential Equations with Fractional Derivatives

Consider the problem

$$-u'' + D_{0x}^\alpha u + \lambda u = 0, \quad (2.29)$$

$$u(0) = 0, \quad u'(0) = 1. \quad (2.30)$$

We know [63] that the problem (2.29)–(2.30) has a unique solution $u(x, \lambda)$ for any λ . For the further it will be important to know whether the function $u(x, \lambda)$ will be an entire function of genus zero.

Theorem 2.6. *The solution $u(x, \lambda)$ of the problem (2.29)–(2.30) is an entire function of genus zero of the parameter λ .*

Proof. Assume that $u_0 = x$ and

$$u_n(x, \lambda) = u_0(x, \lambda) + \int_0^x \left\{ (x-t)^{1-\alpha} u_{n-1}(t, \lambda) + \lambda(x-t) u_{n-1}(t, \lambda) \right\} dt.$$

Let $|\lambda| \leq N$. Then

$$|u_1(x, \lambda) - u_0(x, \lambda)| \leq \frac{x^{2-\alpha}}{2-\alpha} + N \frac{x^2}{2}.$$

For $n \geq 2$ we have

$$\begin{aligned} & |u_2(x, \lambda) - u_0(x, \lambda)| \leq \\ & \leq \int_0^x \left\{ (x-t)^{1-\alpha} u(t) + \lambda(x-t)u_1(t) - (x-t)^{1-\alpha} t - \lambda(x-t)t \right\} dt. \end{aligned}$$

Hence

$$\begin{aligned} & |u_2(x, \lambda) - u_1(x, \lambda)| \leq \\ & \leq \frac{x^{3-\alpha}}{(2-\alpha)(3-\alpha)} + N \frac{x^3}{3!} + N \left(\frac{x^{3-\alpha}}{(2-\alpha)(3-\alpha)} \right) = (N+1) \left[\frac{x^{3-\alpha}}{(2-\alpha)(3-\alpha)} \right], \end{aligned}$$

and in general

$$|u_n(x, \lambda) - u_{n-1}(x, \lambda)| \leq (N+M)^n \left[\frac{x^{n-\alpha}}{(2-\alpha)(3-\alpha)\cdots(n-\alpha)} + N \frac{x^n}{n!} \right].$$

Hence the series

$$u(x, \lambda) = u_0(x, \lambda) + \sum_{n=1}^{\infty} \{u_n(x, \lambda) - u_{n-1}(x, \lambda)\} \quad (2.31)$$

converges in regular intervals of λ , for $|\lambda| \leq N$ and for $0 \leq x \leq 1$. As for any $n \geq 2$

$$\begin{aligned} & u'_n(x, \lambda) - u'_{n-1}(x, \lambda) = \\ & = \int_0^x \left\{ (x-t)^{-\alpha} (u_{n-1}(t, \lambda) - u_{n-2}(t, \lambda)) + \lambda(u_{n-1}(t, \lambda) - u_{n-2}(t, \lambda)) \right\} dt, \\ & u''_n(x, \lambda) - u''_{n-1}(x, \lambda) = \{D_{0x}^\alpha + \lambda\} (u_{n-1}(x, \lambda) - u_{n-2}(x, \lambda)), \end{aligned}$$

we have that the series obtained by once and twice differentiation of the series (2.31) also converge in regular intervals of x . Thus

$$\begin{aligned} u''(x, \lambda) &= \sum_{n=1}^{\infty} \{u''_n(x, \lambda) - u''_{n-1}(x, \lambda)\} = \\ &= u''_1(x, \lambda) - u''_0(x, \lambda) + \sum_{n=2}^{\infty} \{u''_n(x, \lambda) - u''_{n-1}(x, \lambda)\} = \\ &= \{D_{0x}^\alpha + \lambda\} \{u_0(x, \lambda)\} + \sum_{n=2}^{\infty} \{u_n(x, \lambda) - u_{n-1}(x, \lambda)\} = \\ &= \{D_{0x}^\alpha + \lambda\} \{u_0(x, \lambda)\} + \lambda u_0(x, \lambda), \end{aligned}$$

and $u(x, \lambda)$ satisfies the equation (2.29). Clearly $u(x, \lambda)$ satisfies the conditions (2.30).

Now, along with the problem (2.29)–(2.30), we will consider the following problem

$$v'' - v'(x) = \lambda v(x), \quad v(0) = 0, \quad v'(0) = 1.$$

It is known that the solution $v(x, \lambda)$ of this problem is an entire function of genus zero. Take a $v_0(x, \lambda)$ and let

$$v_n(x, \lambda) = v_0(x, \lambda) + \int_0^x \{1 + \lambda(x-t)v_{n-1}(t, \lambda)\} dt.$$

Write $u_n(x, \lambda)$ and $v_n(x, \lambda)$ as

$$\begin{aligned} u_n(x, \lambda) &= a_0(x) + \lambda a_1(x) + \cdots + \lambda^n a_n(x), \\ v_n(x, \lambda) &= b_0(x) + \lambda b_1(x) + \cdots + \lambda^n b_n(x). \end{aligned}$$

Clearly $|a_i(x)| < b_i(x)$, $x \in (0, 1)$. Then, from the theorem of Hadamard follows that $u(x, \lambda)$ is an entire function of genus zero. \square

5. On a Method of Estimation of First Eigenvalues

Consider the problem

$$-u'' + D_{0x}^\alpha u + \lambda u = 0, \quad (2.32)$$

$$u(0) = 0, \quad u'(0) = 1. \quad (2.33)$$

Let $u(x, \lambda)$ be a solution of the problem (2.32)–(2.33). We have already established that $u(x, \lambda)$ is an entire function of genus zero. Hence it is possible to represent it as an infinite product

$$u(x, \lambda) = c \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right),$$

where c is a constant unknown as yet, λ_j are zeros of the function $u(1, \lambda)$. Since the zeros of the function $u(1, \lambda)$ coincide with the eigenvalues of the problem (2.32)–(2.33)

$$-u'' + D_{0x}^\alpha u = \lambda u, \quad (2.32')$$

$$u(0) = 0, \quad u(1) = 0, \quad (2.33')$$

the investigation of the eigenvalues of the problem (2.32')–(2.33') is reduced to the investigation of the zeros of the function $u(1, \lambda)$. First of all we need the following interesting statement.

Lemma 2.3. *All eigenvalues of the problem (2.32')–(2.33') are positive.*

Proof. For proof of the given statement we will consider the following problem

$$-u'' + D_{0x}^\alpha u = \lambda u, \quad u(0) = 0, \quad u(1) = 0.$$

This problem is equivalent to the equation

$$u(x) = x + \int_0^x [(x-t)^{1-\alpha} - \lambda(x-t)]u(t) dt. \quad (2.34)$$

Denote

$$K_1(x, t) = \begin{cases} [(x-t)^{1-\alpha}], & 0 \leq t < x < 1, \\ 0, & x < t \leq 1. \end{cases}$$

We will define the further sequence of kernels $\{K_n(x, t)\}_1^\infty$ by means of the recurrent equalities

$$K_{n+1}(x, t) = \int_t^x K_n(x, t_1)K_1(x, t_1) dt_1.$$

Elementary calculations show that

$$K_2(x, t) = \lambda \frac{\Gamma(1-\alpha)\Gamma(2-\alpha)}{\Gamma(4-2\alpha)} (x-t)^{3-2\alpha} - 2\lambda \frac{\Gamma(2)\Gamma(2-\alpha)}{\Gamma(4-\alpha)} (x-t)^{3-\alpha} + \frac{1}{3!} (x-t)^3.$$

Using induction in n we obtain

$$K_{n+1}(x, t) = K_{n+1}^0(x, t) - \lambda K_{n+1}^1(x, t) + \lambda K_{n+1}^2(x, t) + \dots, \quad x \geq t.$$

Elementary calculations show that $K_{n+1}^i(x, t) \geq 0$ for $x \geq t$ for any n . From here for the resolvent of the equation (2.34) we have the formula

$$R(x, \lambda, t) = \sum_{n=0}^{\infty} K_{n+1}(x, t, \lambda) = a_0(x, t) - \lambda a_1(x, t) + \lambda^2 a_2(x, t) + \dots,$$

where $a_i(x, t) \geq 0$ for $x \geq t$. Since the solution (2.34) is represented as

$$u(x) = x + \int_0^x R(x, \lambda, t)t dt,$$

we have

$$u(1, \lambda)1 + \int_0^1 R(1, \lambda, t)t dt = \tilde{a}_0 - \lambda \tilde{a}_1 + \lambda^2 \tilde{a}_2 - \lambda^3 \tilde{a}_3 + \dots,$$

where $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n, \dots$ are nonnegative numbers. Thus λ is an eigenvalue of the problem (2.32)–(2.33), iff λ is a zero of the function $w(\lambda) = u(1, \lambda) = \tilde{a}_0 - \lambda \tilde{a}_1 + \lambda^2 \tilde{a}_2 - \dots$. But it is obvious that $w(\lambda)$ hasn't negative zeros.

From the estimates obtained in (2.32) it follows that the eigenvalues $\{\lambda_j\}$ of the problem satisfy the equality $\sum_{j=1}^{\infty} \lambda_j^{-1} < \infty$. Therefore for the

Since the solution (2.37) is an entire function of the parameter λ , we have

$$u(x, \lambda) = \sum_{n=0}^{\infty} S_n(x)\lambda_n. \tag{2.38}$$

Substituting (2.38) in (2.37) we have

$$\begin{aligned} & S_0(x) + \lambda S_1(x) + \lambda^2 S_2(x) + \dots = \\ & = x + \int_0^x [(x-t)^{1-\alpha} - \lambda(x-t)] [S_0(t) + \dots + \lambda^n S_n(t) + \dots] dt. \end{aligned} \tag{2.39}$$

From (2.39) it follows

$$S_0(x) = x + \int_0^x (x-t)^{1-\alpha} u(t) dt, \tag{2.40}$$

$$S_1(x) = - \int_0^x (x-t) S_0(t) dt + \int_0^x (x-t)^{1-\alpha} S_1(x) dt, \tag{2.41}$$

$$S_2(x) = \int_0^x (x-t)^{1-\alpha} S_2(x) dt + \int_0^x (x-t)^{1-\alpha} S_1(x) dt, \tag{2.42}$$

.....

Solving the equation (2.40), we obtain

$$S_0(x) = x E_\rho(x^{1/\rho}; 2); \quad S_0(1) = 1.$$

For solution of the equation (2.41) we will calculate

$$\int_0^x (x-t) S_0(t) dt = \int_0^x t E_\rho(t^{1/\rho}; 2) (x-t) dt = x^3 E_\rho(x^{1/\rho}; 4).$$

Then

$$\begin{aligned} a_0(x) &= x^3 E_\rho(x^{1/\rho}; 4) + \int_0^x (x-t)^{1/\rho-1} E_\rho\left((x-t)^{1/\rho}; \frac{1}{\rho}\right) t^3 E_\rho(t^{1/\rho}; 4) dt = \\ &= -c_1 \left(x^3 E_\rho(x^{1/\rho}; 4) + c_0 \rho x^{3+1/\rho} \left[E_\rho\left(x^{1/\rho}; 3 + \frac{1}{\rho}\right) - \right. \right. \\ &\quad \left. \left. - \left(3 + \frac{1}{\rho}\right) E_\rho\left(x^{1/\rho}; 4 + \frac{1}{\rho}\right) \right] \right). \end{aligned}$$

It is likewise possible to show that

$$\begin{aligned} a_2(x) &= -c x^5 E_\rho(x^{1/\rho}; 6) + c \rho x^{5+1/\rho} E_\rho\left(x^{1/\rho}; 5 + \frac{1}{\rho}\right) - \\ &\quad - c \rho \left(5 + \frac{1}{\rho}\right) x^{5+1/\rho} E_\rho\left(x^{1/\rho}; 6 + \frac{1}{\rho}\right) - \end{aligned}$$

$$\begin{aligned}
& -c \left[\rho x^{5+1/\rho} E_\rho \left(x^{1/\rho}; 5 + \frac{1}{\rho} \right) - \rho \left(5 + \frac{1}{\rho} \right) x^{5+1/\rho} E_\rho \left(x^{1/\rho}; 6 + \frac{1}{\rho} \right) \right] + \\
& \quad + c\rho \left\{ \rho x^{2/\rho+5} E_\rho \left(x^{1/\rho}; 5 + \frac{1}{\rho} \right) x^{2/\rho+5} E_\rho \left(x^{1/\rho}; 6 + \frac{1}{\rho} \right) - \right. \\
& - c\rho \left(5 + \frac{1}{\rho} \right) \left[x^{2/\rho+5} E_\rho \left(x^{1/\rho}; 6 + \frac{1}{\rho} \right) - \left(5 + \frac{2}{\rho} \right) E_\rho \left(x^{1/\rho}; 6 + \frac{1}{\rho} \right) x^{2/\rho+5} + \right. \\
& \quad \left. \left. + \frac{1}{2\rho} \sum_{k=0}^{\infty} x^{(k+2)/\rho-5} \frac{k^2}{\Gamma\left(\frac{k}{\rho} + 6 + \frac{2}{\rho}\right)} \right] \right\}.
\end{aligned}$$

Now note that from (2.36) it follows that

$$S_1(1) = -\chi_1, \quad S_2(1) = \frac{1}{2} (\chi_1^2 - \chi_2).$$

As all eigenvalues of the problem (2.27')–(2.28') are positive, obviously

$$\lambda_1 > \frac{1}{\chi_1} = -\frac{1}{S_1(1)}.$$

The estimate from below for λ_1 looks like $\lambda_1 < \chi_1/\chi_2$. \square

Now taking into account that it is possible to calculate S_1 and S_2 to within 10^{-2} , we will obtain

Theorem 2.7. *For the first eigenvalue λ_1 of the problem (2.32')–(2.33') we have the relation*

$$(1.85)^{-1} < \lambda_1 < 3.86.$$

Note that it is likewise possible to find estimates for the eigenvalues of the problem

$$\begin{aligned}
-u'' + D_{0x}^{\alpha_0} u + D_{0x}^{\alpha_1} u &= \lambda u, \\
u(0) = 0, \quad u(1) &= 0.
\end{aligned}$$

6. Mutually Adjoint Problems and Problem of Completeness of Eigenfunctions

For the equation

$$u'' + \sum_{i=1}^n a_i(x) D_{0x}^{\alpha_i} \omega_j(x) u = \lambda u, \quad 0 < \alpha_i < 1, \quad (2.43)$$

consider the problem

$$u(0) = 0, \quad u(1) = 0. \quad (2.44)$$

Along with the problem (2.43)–(2.44), we consider the problem

$$z'' + \sum_{i=1}^n \omega_j(x) (D_{0x}^{\alpha_i})^* a_i z + \lambda z = 0, \quad (2.45)$$

$$z(0) = 0, \quad z(1) = 0. \quad (2.46)$$

Here $(D_{0x}^{\alpha_i})^*$ is the adjoint operator to the operator $D_{0x}^{\alpha_i}$ i.e.

$$(D_{0x}^{\alpha_i})^* u = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^1 \frac{u(t)}{(t-x)^\alpha} dt,$$

in a certain sense associated with the problem (2.43)–(2.44). We will call these problems mutually adjoint.

Let $\{v_n(x)\}_{n=1}^\infty$ be a system of eigenfunctions of the problem (2.43)–(2.44), and $\{z_n(x)\}_{n=1}^\infty$ be a system of eigenfunctions of the problem (2.45)–(2.46).

We will establish that the systems of functions $\{v_n(x)\}_{n=1}^\infty$, $\{z_n(x)\}_{n=1}^\infty$ are biorthogonal on $[0, 1]$.

Theorem 2.8. *The system of eigenfunctions $\{v_n(x)\}_{n=1}^\infty$ is complete in $L_2(0, 1)$.*

Proof. Introduce the operator

$$Mu = \begin{cases} -u'', \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

Then the problem (2.43)–(2.44) is equivalent to the equation

$$u + M^{-1} \left\{ \sum_{i=1}^n a_i(x) D_{0x}^{\alpha_i} \omega_i(x) u \right\} - \lambda M^{-1} u = 0$$

i.e., the problem is reduced to investigation of operators of Keldysh type, where

$$M^{-1} u = \int_0^1 G_T(x, t) u dt, \quad G_T(x, t) = \begin{cases} t(x-1), & t \leq x, \\ x(t-1), & t \geq x. \end{cases}$$

Clearly M^{-1} is a complete self-adjoint operator,

$$M^{-1} \left(\sum_{i=1}^n a_i(x) D_{0x}^{\alpha_i} \omega_i(x) u \right) \in G_1,$$

M^{-1} is a kernel operator. Then, from a theorem of Keldysh [3], it follows the corresponding completeness. Denote by $n(r)$ the exact number of eigenvalues of the problem (2.43)–(2.44) laying in the circle $|\lambda| < r$.

Problems on distribution of eigenvalues consist in studying asymptotic properties of $n(r)$ as $r \rightarrow \infty$. \square

Theorem 2.9. *The following equality holds*

$$\lim_{r \rightarrow \infty} n(r)/r^{1/2} = 1.$$

Proof. Studying the spectrum of the problem is reduced to studying the spectrum of the linear operator bunch

$$L(\lambda) = J + M^{-1} \left\{ \sum_{i=1}^n a_i(x) D_{0x}^{\alpha_i} \omega_i \right\} - \lambda M^{-1}.$$

Clearly,

$$M^{-1} \left(\sum_{i=1}^n a_i(x) D_{0x}^{\alpha_i} \omega_i(x) u \right) \in G_1.$$

M^{-1} is a positive operator. If for the function of distribution $rn(r)$ it is possible choose a nondecreasing function $\varphi(r)$ ($0 \leq r \leq \infty$) possessing the properties

1. $\lim_{r \rightarrow \infty} \varphi(r) = \infty$, $\varphi(r) \uparrow (r \rightarrow \infty)$;
2. $\lim_{r \rightarrow \infty} [\ln(\varphi(r))]' = \lim_{r \rightarrow \infty} \frac{\varphi'(r)}{\varphi(r)} < \infty$;
3. $\lim_{r \rightarrow \infty} \frac{n(r)}{\varphi(r)} = 1$,

then by theorem of Keldysh,

$$\lim_{r \rightarrow \infty} \frac{n(r)}{n(r, M^{-1})} = 1.$$

As $\varphi(r)$, in our case, obviously it is possible to take the function $\varphi(r) = r^{1/2}$. This proves Theorem 2.9. \square

Consider the differential expression

$$Ly = -y'' + D_{0x}^{\alpha} y$$

on the finite interval $[0, 1]$.

Let T be an operator defined in the Hilbert space $H = L_2(0, 1)$ by the operator L with boundary conditions $y(0) = y(1) = 0$. Certainly, the operator T is a weak disturbance of the operator

$$Au = \begin{cases} u'', \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

Formulate a theorem which is quite important in studying the operators of the type T .

Theorem 2.10. *The operator T is dissipative.*

Proof.

$$(Ty, y) = \left(-\frac{d^2}{dx^2} y, y \right) + (D_{0x}^{\alpha} y, y), \quad (D_{0x}^{\alpha} y, y) \geq 0.$$

Thus the operator T is dissipative. \square

7. Construction of a Biorthogonal System

Consider the Sturm–Liouville problem for a fractional differential equation

$$\begin{cases} u'' + a_m(x)u' + a_i(x)D_{0x}^{\alpha_i}\omega_i(x)u + \lambda u = 0, \\ u(0)\cos(\alpha) + u'(0)\sin(\alpha) = 0, \\ u(1)\cos(\beta) + u'(1)\sin(\beta) = 0 \end{cases} \quad (2.47)$$

and the problem

$$\begin{cases} z - [a_m(x)z(x)]' - \omega_i(x)D_{x1}^{\alpha_i}a_i(t)z(t) + \lambda z = 0, \\ z(0)\cos(\alpha) + z'(0)\sin(\alpha) = 0, \\ z(1)\cos(\beta) + z'(1)\sin(\beta) = 0. \end{cases} \quad (2.48)$$

We will call the problems (2.47) and (2.48) mutually adjoint.

Following M. M. Dzhrbashyan [10], we introduce the functions

$$\begin{aligned} \omega(\lambda) &= u(1, \lambda) + u'(1, \lambda)\sin(\beta), \\ \widetilde{\omega(\lambda^*)} &= z(0, \lambda^*) + z'(0, \lambda^*)\sin(\alpha), \end{aligned}$$

where $u(x, \lambda)$ is a solution of the following Cauchy problem

$$u'' + a_m(x)u' + a_i(x)D_{0x}^{\alpha_i}\omega_i(x)u + \lambda u = 0, \quad (2.49)$$

$$u(0) = \sin(\alpha), \quad u'(0) = -\cos(\alpha) \quad (2.50)$$

and $z(x, \lambda^*)$ is a solution of the problem

$$z''(x) + a_m(x)z'(x) + \omega_i(x)D_{x1}^{\alpha_i}a_i(x)z + \lambda^*z = 0, \quad (2.51)$$

$$z(1) = \sin(\beta), \quad z' = -\cos(\beta). \quad (2.52)$$

The existence and uniqueness of solutions of mutually adjoint problems (2.47) and (2.48) are already proved [3].

Clearly if $u(x, \lambda)$ is a solution of the problem (2.49)–(2.50), then a necessary and sufficient condition for it to be also a solution of the problem (2.47) is

$$\omega(\lambda) = u(1, \lambda)\cos(\beta) + u'(1, \lambda)\sin(\beta).$$

There is a similar statement for the solutions of the problems (2.51)–(2.52).

For construction of an orthogonal system of eigenfunctions and associated functions of mutually adjoint problems by method of M. M. Dzhrbashyan we need the following

Theorem 2.11. *Let $a_m(0) = a_m(1) = 0$. Then for any values of the parameters λ, λ^* the identity*

$$(\lambda - \lambda^*) \int_0^1 u(x, \lambda)z(x, \lambda^*) dx = \omega(\lambda) - \widetilde{\omega(\lambda^*)} \quad (2.53)$$

holds.

Proof. Due to the definition of the functions $u(x, \lambda)$ and $z(x, \lambda^*)$ as solutions of problems of Cauchy type, we have (2.49)-(2.50) and

$$\begin{aligned} z''(x, \lambda^*) [a_m z(x, \lambda^*)]' + \omega_i(x) D_{0x}^{\alpha_i} a_i(x) + \lambda^* z &= 0, \\ z(1, \lambda^*) &= \sin \beta, \quad z'(1, \lambda^*) = -\cos \beta. \end{aligned}$$

Multiplying both parts of the first equation of (2.47) by $z(x, \lambda^*)$ and integrating it from 0 to 1, we have

$$\begin{aligned} & \int_0^x z(x, \lambda^*) u''(x, \lambda) dx + \int_0^1 a_m(x) z(x, \lambda^*) u'(x, \lambda) dx + \\ & + \int_0^x a_i(x) z(x, \lambda^*) D_{0x}^{\alpha_i} u(x, \lambda) dx + \int_0^x \lambda u(x, \lambda) z(x, \lambda^*) dx = \\ & = \omega(\lambda) - \tilde{\omega}(\lambda^*) \int_0^x u(x, \lambda) z''(x, \lambda) dx - \int_0^1 [a_m(x) z(x, \lambda^*)]' u(x, \lambda) dx + \\ & + \int_0^1 \omega_i(x) D_{x1}^{\alpha_i} a_i(t) z(x, \lambda^*) u(x, \lambda) dx + \lambda \int_0^1 u(x, \lambda) z(x, \lambda^*) dx. \end{aligned} \quad (2.54)$$

At the same time, we multiply both parts of the first equation of (2.48) by $u(x, \lambda)$ and integrate it from 0 up to 1. We have

$$\begin{aligned} & \int_0^1 z(x, \lambda^*) u(x, \lambda) dx - \int_0^1 [a_m(x) z(x, \lambda^*)]' u(x, \lambda) dx + \\ & + \int_0^1 \omega_i(x) D_{x1}^{\alpha_i} a_i(t) z(x, \lambda^*) u(x, \lambda) dx + \lambda \int_0^1 u(x, \lambda) z(x, \lambda^*) dx = 0. \end{aligned} \quad (2.55)$$

Subtracting (2.55) from (2.56), we obtain

$$\int_0^1 z(x, \lambda^*) dx = \frac{\tilde{\omega}(\lambda^*) - \omega(\lambda)}{\lambda - \lambda^*}.$$

So we obtain an analogue of identity of M. M. Dzhrbashyan. \square

Theorem 2.12. *The formula*

$$\omega(\lambda) = \tilde{\omega}(\lambda) \quad (2.56)$$

holds if $\lambda = \lambda^$.*

The equality (2.56) follows from the identity (2.53).

We needed to construct a biorthogonal system of eigenfunctions and associated functions of mutually adjoint problems (2.47).

Let $\{\lambda_n\}$ be a sequence of all eigenvalues of the problem (2.47) arranged in the non-decreasing order of moduli $0 \leq |\lambda_1| \leq |\lambda_2| \leq \dots \leq |\lambda_n| \leq \dots$. Note that a same eigenvalue may appear repeatedly in the above sequence.

According to M. M. Dzhrbashyan [1], for each natural number $n \geq 1$, we denote by p_n the multiplicity of occurrence of the number λ_n in the sequence $\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|, \dots\}$.

Let $\{u_n(x)\}_{n=1}^{\infty}$, $\{z_n(x)\}_{n=1}^{\infty}$ be sequences of eigenfunctions of the problems (2.47) and (2.48), respectively. As in [7] we prove that the systems of functions $\{u_n(x)\}_{n=1}^{\infty}$, $\{u_n^*(x)\}_{n=1}^{\infty}$ are continuous on $[0, 1)$, and the systems of functions $\{z_n(x)\}_{n=1}^{\infty}$, $\{z_n^*(x)\}_{n=1}^{\infty}$ are continuous on $(0, 1]$.

We will call a system of functions $\{u_n(x)\}_{n=1}^{\infty}$ a normal system of eigenfunctions and associated functions of problem (2.47), and system of functions $\{z_n(x)\}_{n=1}^{\infty}$ a normal system of eigenfunctions and associated functions of problem (2.48).

Then we have the following important theorem of biorthogonality of these constructed systems.

Theorem 2.13. *The system of functions $\{u_n(x), z_n(x)\}$ is biorthogonal on $[0, 1]$ i.e.*

$$\int_0^1 u_n(x) z_m(x) dx = \int_0^1 z_n(x) u_m(x) dx = \begin{cases} \delta_{mn} 1, & \text{for } n \neq m, \\ 0, & \text{for } n = m. \end{cases}$$

Proof of this theorem is bulky and mainly does not differ from the proof of the corresponding theorem of M. M. Dzhrbashyan [1] if there is the identity

$$\int_0^1 u_n(x) z_n(x) dx = \frac{\omega(\lambda) - \tilde{\omega}(\lambda^*)}{\lambda - \lambda^*}.$$

Systems of eigenfunctions and associated functions (3.19)–(3.22) are constructed only on the basis of the identity (3.23) without reference to the problem (2.47). Naturally from the identity it is possible to obtain the sufficient information on the spectrum of the problem (2.47).

Solving Two-Point Boundary Value Problems for Fractional Differential Equations (FBVPs)

1. The Basic Concepts

In [8] and [33], the existence and uniqueness of solutions, and numerical methods for FBVPs with Caputo's derivatives and with Riemann–Liouville derivatives are studied, respectively.

FBVPs with Caputo's derivatives:

$${}^C D_t^\gamma y(t) + f(t, y(t)) = 0, \quad a < t < b, \quad 1 < \gamma \leq 2, \quad (3.1)$$

$$y(a) = \alpha, \quad y(b) = \beta \quad (3.2)$$

and

$${}^C D_t^\gamma y(t) + g(t, y(t), {}^C D_t^\theta y(t)) = 0, \quad a < t < b, \quad (3.3)$$

$$y(a) = \alpha, \quad y(b) = \beta, \quad 1 < \gamma \leq 2, \quad 0 < \theta \leq 1, \quad (3.4)$$

where $y : [a, b] \mapsto \mathbb{R}$, $f : [a, b] \times \mathbb{R} \mapsto \mathbb{R}$, $g : [a, b] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are continuous and satisfy Lipschitz conditions

$$|f(t, x) - f(t, y)| \leq K_f |x - y|, \quad (3.5)$$

$$|g(t, x_2, y_2) - g(t, x_1, y_1)| \leq K_g |x_2 - x_1| + L_g |y_2 - y_1| \quad (3.6)$$

with Lipschitz constants $K_f, K_g, L_g > 0$.

FBVPs with Riemann–Liouville derivatives:

$${}^{RL} D_t^\gamma y(t) + f(t, y(t)) = 0, \quad a < t < b, \quad 1 < \gamma \leq 2, \quad (3.7)$$

$$y(a) = 0, \quad y(b) = \beta, \quad (3.8)$$

and

$${}^{RL} D_t^\gamma y(t) + g(t, y(t), {}^{RL} D_t^\theta y(t)) = 0, \quad a < t < b, \quad (3.9)$$

$$y(a) = 0, \quad y(b) = \beta, \quad 1 < \gamma \leq 2, \quad 0 < \theta \leq \gamma - 1, \quad (3.10)$$

where $y : [a, b] \mapsto \mathbb{R}$, $f : [a, b] \times \mathbb{R} \mapsto \mathbb{R}$, $g : [a, b] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are continuous and satisfy Lipschitz conditions (3.5) and (3.6), respectively.

In order to state the problems in concern, we introduce the definitions and properties of Caputo's derivatives and Riemann–Liouville fractional integrals in [8].

Definition 3.1 (see [29]). Let $\gamma > 0$, $n-1 < \gamma \leq n$ and let $\mathbb{C}^n[a, b] := \{y(t) : [a, b] \rightarrow \mathbb{R}; y(t) \text{ has a continuous } n\text{-th derivative}\}$.

(1) The operator ${}^RLD_t^\gamma$ defined by

$${}^RLD_t^\gamma y(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-\tau)^{n-\gamma-1} y(\tau) d\tau \quad (3.11)$$

for $t \in [a, b]$ and $y(t) \in \mathbb{C}^n[a, b]$, is called the Riemann–Liouville differential operator of order γ .

(2) The operator ${}^CD_t^\gamma$ defined by

$${}^CD_t^\gamma y(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-\tau)^{n-\gamma-1} \left(\frac{d}{d\tau}\right)^n y(\tau) d\tau \quad (3.12)$$

for $t \in [a, b]$ and $y(t) \in \mathbb{C}^n[a, b]$, is called the Caputo differential operator of order γ .

Definition 3.2 (see [29]). Provided $\gamma > 0$, the operator J_a^γ , defined on $\mathbb{L}_1[a, b]$ by

$$J_a^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} y(\tau) d\tau \quad (3.13)$$

for $t \in [a, b]$, is called the Riemann–Liouville fractional integral operator of order γ , where $\mathbb{L}_1[a, b] := \{y(t) : [a, b] \rightarrow \mathbb{R}; y(t) \text{ is measurable on } [a, b] \text{ and } \int_a^b |y(t)| dt < \infty\}$.

Lemma 3.1 (see [31]). (1) Let $\gamma > 0$. Then, for every $f \in L_1[a, b]$,

$${}^RLD_t^\gamma J_a^\gamma f = f \quad (3.14)$$

almost everywhere.

(2) Let $\gamma > 0$ and $n-1 < \gamma \leq n$. Assume that f is such that $J_a^{n-\gamma} f \in \mathbb{A}^n[a, b]$. Then,

$$J_a^{\gamma RL} D_t^\gamma f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\gamma-k-1}}{\Gamma(\gamma-k)} \lim_{z \rightarrow a^+} \frac{d^{n-k-1}}{dz^{n-k-1}} J_a^{n-\gamma} f(z). \quad (3.15)$$

(3) Let $\gamma > \theta > 0$ and f be continuous. Then

$${}^CD_t^\gamma J_a^\gamma f = f, \quad {}^CD_t^\theta J_a^\gamma f = J_a^{\gamma-\theta} f. \quad (3.16)$$

(4) Let $\gamma \geq 0$, $n-1 < \gamma \leq n$ and $f \in \mathbb{A}^n[a, b]$. Then

$$J_a^{\gamma C} D_t^\gamma f(t) = f(t) - \sum_{k=0}^{n-1} \frac{D^k f(a)}{k!} (t-a)^k, \quad (3.17)$$

where $\mathbb{A}^n[a, b]$ is the set of functions with absolutely continuous derivative of order $n-1$.

Let \mathbb{S} be a Banach space, $\mathcal{T} : \mathbb{S} \mapsto \mathbb{S}$ be a mapping, and $\|\cdot\|$ denote the norm of \mathbb{S} .

Definition 3.3 (see [25]). If there exists a constant ρ ($0 \leq \rho < 1$) such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \rho\|x - y\| \quad (3.18)$$

for any $x, y \in \mathbb{S}$, then \mathcal{T} is said to be a *contractive mapping* of \mathbb{S} .

Lemma 3.2 (Contractive Mapping Principle) (see [25]). *If \mathcal{T} is a contractive mapping of a Banach space \mathbb{S} , then there exists a unique fixed point $y \in \mathbb{S}$ satisfying $y = \mathcal{T}y$.*

2. Existence and Uniqueness of the Solutions for FBVPs

In this section, the existence and uniqueness of the solutions for FBVPs (3.7)–(3.8), (3.9)–(3.10), (3.1)–(3.2) and (3.3)–(3.4) are studied. Without loss of generality, only the case of homogeneous boundary conditions $\alpha = \beta = 0$ in the above four kinds of FBVPs are considered.

Lemma 3.3. (1) *FBVP (3.1)–(3.2) is equivalent to*

$$y(t) = \int_a^b G(t, s) f(s, y(s)) ds, \quad (3.19)$$

where $G(t, s)$ is called the fractional Green function defined as follows:

$$G(t, s) = \begin{cases} \frac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} - \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}, & a \leq s \leq t \leq b, \\ \frac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)}, & a \leq t \leq s \leq b. \end{cases} \quad (3.20)$$

(2) *FBVP (3.7)–(3.8) is equivalent to*

$$y(t) = \int_a^b \widehat{G}(t, s) f(s, y(s)) ds, \quad (3.21)$$

where $\widehat{G}(t, s)$ is the fractional Green function defined as follows:

$$\widehat{G}(t, s) = \begin{cases} \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \left(\frac{b-s}{b-a}\right)^{\gamma-1} - \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \left(\frac{b-s}{b-a}\right)^{\gamma-1}, & a \leq t \leq s \leq b. \end{cases} \quad (3.22)$$

(3) *FBVP (3.3)–(3.4) is equivalent to*

$$y(t) = \int_a^b G(t, s) g(s, y(s), {}^C D_s^\rho y(s)) ds, \quad (3.23)$$

where $G(t, s)$ is the fractional Green function defined in (3.20).

(4) FBVP (3.9)–(3.10) is equivalent to

$$y(t) = \int_a^b \widehat{G}(t, s) g(s, y(s), {}_a^{RL}D_s^\theta y(s)) ds, \quad (3.24)$$

where $\widehat{G}(t, s)$ is the fractional Green function defined in (3.22).

Proof. We only give the proof for (1). The case for (3) is similar to that for (1). The proofs of (2) and (4) are referred to [33].

According to Lemma 3.1(4), applying the operator J_a^γ to both sides of the equation in (3.1) yields

$$y(t) - y(a) - y'(a)(t - a) + J_a^\gamma f(t, y(t)) = 0. \quad (3.25)$$

Since $y(a) = y(b) = 0$, from (3.25) one easily obtains

$$y'(a) = \int_a^b \frac{(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} f(s, y(s)) ds, \quad (3.26)$$

and then

$$\begin{aligned} y(t) &= y'(a)(t - a) - J_a^\gamma f(t, y(t)) = \\ &= \int_a^b \frac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} f(s, y(s)) ds - \int_a^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, y(s)) ds = \\ &= \int_a^b G(t, s) f(s, y(s)) ds. \end{aligned}$$

Conversely, applying the operator ${}^C D_t^\gamma$ to both sides of (3.19) yields

$$\begin{aligned} {}^C D_t^\gamma y(t) &= {}^C D_t^\gamma \int_a^b G(t, s) f(s, y(s)) ds = \\ &= \int_a^b \frac{(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} f(s, y(s)) ds {}^C D_t^\gamma (t-a) - {}^C D_t^\gamma J_a^\gamma f(t, y(t)) = \\ &= -f(t, y(t)), \end{aligned}$$

and the homogeneous boundary condition is verified easily. \square

Let

$$\begin{aligned} \mathbb{P} &= \mathbb{C}^0[a, b], \\ \mathbb{P}_1 &= \mathbb{C}^1[a, b] := \{y(t) : y(t), y'(t) \in \mathbb{C}^0[a, b]\}, \\ \mathbb{P}_2 &= \mathbb{C}^\theta[a, b] := \{y(t) : y(t) \in \mathbb{C}^0[a, b], {}_a^{RL}D_t^\theta y(t) \in \mathbb{C}^0[a, b]\}. \end{aligned}$$

where $\mathbb{C}^0[a, b]$ denotes the space of all continuous functions on $[a, b]$. We define the norms $\|\cdot\|_f$, $\|\cdot\|_g$, $\|\cdot\|_{\hat{g}}$ and the operator \mathcal{T}_f , $\hat{\mathcal{T}}_f$, \mathcal{T}_g , $\hat{\mathcal{T}}_g$ as follows:

$$\begin{aligned}\|y\|_f &:= K_f \max_{a \leq t \leq b} |y(t)|, \\ \mathcal{T}_f y(t) &:= \int_a^b G(t, s) f(s, y(s)) ds, \\ \hat{\mathcal{T}}_f y(t) &:= \int_a^b \hat{G}(t, s) f(s, y(s)) ds, \quad \forall y(t) \in \mathbb{P}; \\ \|y\|_g &:= \max_{a \leq t \leq b} [K_g |y(t)| + L_g |{}_a^C D_t^\theta y(t)|], \\ \mathcal{T}_g y(t) &:= \int_a^b G(t, s) g(s, y(s), {}_a^C D_s^\theta y(s)) ds, \quad \forall y(t) \in \mathbb{P}_1, \\ \|y\|_{\hat{g}} &:= \max_{a \leq t \leq b} [K_g |y(t)| + L_g |{}_a^{RL} D_t^\theta y(t)|], \\ \hat{\mathcal{T}}_g y(t) &:= \int_a^b \hat{G}(t, s) g(s, y(s), {}_a^{RL} D_s^\theta y(s)) ds, \quad \forall y(t) \in \mathbb{P}_2.\end{aligned}$$

Obviously, \mathbb{P} , \mathbb{P}_1 , \mathbb{P}_2 are complete normed spaces with respect to $\|\cdot\|_f$, $\|\cdot\|_g$ and $\|\cdot\|_{\hat{g}}$, respectively. The operators \mathcal{T}_f , $\hat{\mathcal{T}}_f$, \mathcal{T}_g , $\hat{\mathcal{T}}_g$ are continuous. Now (3.19) and (3.21) can be rewritten as $y = \mathcal{T}y$ and $y = \hat{\mathcal{T}}y$, for $y \in \mathbb{P}$; (3.23), (3.24) can be rewritten as $y = \mathcal{T}_g y$ and $y = \hat{\mathcal{T}}_g y$, respectively. According to the contractive mapping principle, “finding a sufficient condition for existence and uniqueness of the solution for FBVP (3.1)–(3.2) (or (3.7)–(3.8) or (3.3)–(3.4) or (3.9)–(3.10))” is equivalent to “finding a sufficient condition under which \mathcal{T}_f (or $\hat{\mathcal{T}}_f$ or \mathcal{T} or $\hat{\mathcal{T}}_g$) is a contractive mapping of \mathbb{P} (or \mathbb{P}_1 or \mathbb{P}_2)”. Then, the following theorems are true.

Theorem 3.1. (1) Let f be a continuous function on $[a, b] \times \mathbb{R}$ and satisfy the Lipschitz condition (3.5).

(1.1) If

$$\frac{2K_f(b-a)^\gamma}{\Gamma(\gamma+1)} < 1, \quad (3.27)$$

then there exists a unique solution for FBVP (3.1)–(3.2) in \mathbb{P} .

(1.2) If

$$K_f \frac{(\gamma-1)^{\gamma-1} (b-a)^\gamma}{\gamma \Gamma(\gamma+1)} < 1, \quad (3.28)$$

then there exists a unique solution for FBVP (3.7)–(3.8) in the space \mathbb{P} .

(2) Let g be a continuous function on $[a, b] \times \mathbb{R} \times \mathbb{R}$ and satisfy the Lipschitz condition (3.6).

(2.1) If

$$K_g \frac{2(b-a)^\gamma}{\Gamma(\gamma+1)} + L_g \left[\frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma+1)\Gamma(2-\theta)} + \frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma+1-\theta)} \right] < 1, \quad (3.29)$$

then there exists a unique solution for FBVP (3.3)–(3.4) in \mathbb{P}_1 .

(2.2) If

$$K_g \frac{(\gamma-1)^{\gamma-1}(b-a)^\gamma}{\gamma\Gamma(\gamma+1)} + L_g \frac{(2\gamma-\theta)(b-a)^{\gamma-\theta}}{\gamma\Gamma(\gamma-\theta+1)} < 1, \quad (3.30)$$

then there exists a unique solution for FBVP (3.9)–(3.10) in the space \mathbb{P}_2 .

Proof. The proof for (1.1): For $u(t), v(t) \in \mathbb{P}$, and $(t, s) \in [a, b] \times [a, b]$, according to the definition for the operator “ \mathcal{T}_f ”, we have

$$\begin{aligned} |\mathcal{T}_f u(t) - \mathcal{T}_f v(t)| &\leq \int_a^b |G(t, s)| \cdot |f(s, u(s)) - f(s, v(s))| ds \leq \\ &\leq \|u - v\|_f \left\{ \int_a^b \frac{(t-a)(b-s)^{\gamma-1}}{(b-a)\Gamma(\gamma)} ds + \int_a^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} ds \right\} \leq \\ &\leq \frac{2(b-a)^\gamma}{\Gamma(\gamma+1)} \|u - v\|_f. \end{aligned}$$

Thus

$$\|\mathcal{T}_f u - \mathcal{T}_f v\|_f = \max_{a \leq t \leq b} K_f |\mathcal{T}_f u(t) - \mathcal{T}_f v(t)| \leq \frac{2K_f(b-a)^\gamma}{\Gamma(\gamma+1)} \|u - v\|_f.$$

Considering (3.27), we finish the proof according to Lemma 3.2.

The proof for (2.1): On one hand, we have

$$|\mathcal{T}_g u(t) - \mathcal{T}_g v(t)| \leq \frac{2(b-a)^\gamma}{\Gamma(\gamma+1)} \|u - v\|_g$$

for $u(t), v(t) \in \mathbb{P}_1$, and $(t, s) \in [a, b] \times [a, b]$, which is similar to (1.1). On the other hand, according to Lemma 3.1, we have

$$\begin{aligned} & {}^C D_t^\theta \mathcal{T}_g u(t) = \\ &= \frac{(t-a)^{1-\theta}}{\Gamma(2-\theta)} \int_a^b \frac{(b-s)^{\gamma-1}}{\Gamma(\gamma)(b-a)} g(s, u(s), {}^C D_s^\theta u(s)) ds - J_a^{\gamma-\theta} g(t, u(t), {}^C D_t^\theta u(t)). \end{aligned}$$

Then

$$|{}^C D_t^\theta \mathcal{T}_g u - {}^C D_t^\theta \mathcal{T}_g v| \leq \|u - v\|_g \cdot \left[\frac{(b-a)^{\gamma-\theta}}{\Gamma(2-\theta)\Gamma(\gamma+1)} + \frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma-\theta+1)} \right].$$

Combined with the definition of $\|\cdot\|_g$ and Lemma 3.2, (3.29) holds.

The proof for (1.2) and (2.2) are referred to [33]. \square

3. Shooting Methods for FBVPs

In this section, single shooting methods are applied to solve the FBVPs (3.1)–(3.2), (3.3)–(3.4), (3.7)–(3.8) and (3.9)–(3.10) numerically.

According to the idea of the shooting method, a FBVP is turned into a fractional initial value problem (FIVP) which can be solved by some suitable numerical method (see [26]–[28], [31]). We write down the corresponding procedure for FBVPs.

Denote the corresponding initial value conditions of FBVPs (3.1)–(3.2) and (3.3)–(3.4) as

$$y(a) = a_0, \quad y'(a) = a_1, \quad a_0, a_1 \in \mathbb{R}. \quad (3.31)$$

Then FBVPs (3.1)–(3.2) and (3.3)–(3.4) can turn into FIVPs (3.1), (3.31) and (3.3), (3.31), respectively.

Usually, not all a_k ($k = 0, 1$) are equal to zero, in other words, the initial value conditions (3.31) are inhomogeneous. Setting

$$z(t) = y(t) - a_0 - a_1(t - a), \quad (3.32)$$

(3.1), (3.31) and (3.3)–(3.31) can be transformed into another FIVPs with homogeneous initial value conditions.

For a given equispaced mesh $a = t_0 < t_1 < \dots < t_N = b$ with stepsize $h = (b - a)/N$, we give a fractional backward difference scheme of order one (refer to [31]) to solve FIVPs (3.1) and (3.3) with homogeneous initial value conditions $y(a) = 0, y'(a) = 0$:

$$z_m = -h^\gamma f(t_m, z_m) - \sum_{k=1}^m \omega_k z_{m-k}, \quad (3.33)$$

and

$$z_m = -h^\gamma g\left(t_m, z_m, \frac{1}{h^\theta} \sum_{i=0}^m \tilde{\omega}_i z_{m-i}\right) - \sum_{k=1}^m \omega_k z_{m-k}, \quad (3.34)$$

where

$$\omega_0 = 1, \quad \omega_k = \left(1 - \frac{\gamma + 1}{k}\right) \omega_{k-1}, \quad k = 1, 2, \dots, N, \quad (3.35)$$

$$\tilde{\omega}_0 = 1, \quad \tilde{\omega}_i = \left(1 - \frac{\theta + 1}{i}\right) \tilde{\omega}_{i-1}, \quad i = 1, 2, \dots, N. \quad (3.36)$$

The case for (3.7)–(3.8) and (3.9)–(3.10) is similar. The reader should note that the initial values take the following form

$${}^{RL}D_t^{\gamma-1}y(a) = b_1 \in \mathbb{R}, \quad \lim_{t \rightarrow a^+} J_a^{2-\gamma}y(t) = b_2 \in \mathbb{R}. \quad (3.37)$$

It is easy to check that

$$b_2 = \lim_{t \rightarrow a^+} J_a^{2-\gamma}y(t) = 0. \quad (3.38)$$

Usually, $b_1 \neq 0$ and let

$$z(t) = y(t) - \frac{b_1(t - a)^{\gamma-1}}{\Gamma(\gamma)} \quad (3.39)$$

in (3.7), (3.37) and (3.9), (3.37). Then they are transformed into FIVPs with homogeneous initial conditions. (3.33) or (3.34) is also employed to simulate them.

3.1. Shooting method for linear problems. The linear case of fractional two-point boundary value problem (3.1) with homogeneous boundary value conditions

$${}^C D_t^\gamma y(t) + c(t)y(t) + d(t) = 0, \quad a < t < b, \quad 1 < \gamma \leq 2, \quad (3.40)$$

$$y(a) = 0, \quad y(b) = 0 \quad (3.41)$$

where $c(t), d(t) \in \mathbb{C}^0[a, b]$.

According to Theorem 3.1, if

$$b - a < \left[\frac{\Gamma(\gamma + 1)}{2K_f} \right]^{1/\gamma}$$

with $K_f = \max_{a \leq t \leq b} |c(t)|$, then there exists a unique solution for (3.40)–(3.41).

In order to apply the shooting method, we choose an initial value ξ_1, ξ_2 for y' (denote the initial value of the exact solution $y'(a) = \xi^*$), respectively:

$$y(a) = 0, \quad y'(a) = \xi_1. \quad (3.42)$$

$$y(a) = 0, \quad y'(a) = \xi_2. \quad (3.43)$$

Then FIVP (3.40), (3.42) and FIVP (3.40), (3.43) have unique solution, denoted by $y(t; \xi_1), y(t; \xi_2)$. Usually, $\xi_1 \neq \xi_2 \neq \xi^*, y(b; \xi_1) \neq y(b; \xi_2) \neq 0$, thus $y(t; \xi_1) \neq y$ and $y(t; \xi_2) \neq y$. Suppose that $y(b; \xi_1) \neq y(b; \xi_2)$, and let

$$\lambda := \frac{y(b) - y(b; \xi_2)}{y(b; \xi_1) - y(b; \xi_2)}$$

and

$$y(t) := \lambda y(t; \xi_1) + (1 - \lambda) y(t; \xi_2). \quad (3.44)$$

It is easy to show that (3.44) is a solution of FBVP (3.40)–(3.41).

In [8], the error estimates for the linear case are also considered.

For a given equispaced mesh $a = t_0 < t_1 < \dots < t_N = b$ with stepsize $h = (b - a)/N$, we denote the numerical solution of FIVPs (3.40), (3.42) and FIVPs (3.40), (3.43) by $\{y_n(\xi_1)\}_{n=0}^N$ and $\{y_n(\xi_2)\}_{n=0}^N$, respectively. Then,

$$\hat{\lambda} = \frac{y(b) - y_N(\xi_2)}{y_N(\xi_1) - y_N(\xi_2)}$$

and the numerical solution

$$y_n = \hat{\lambda} y_n(\xi_1) + (1 - \hat{\lambda}) y_n(\xi_2), \quad n = 0, 1, 2, \dots, N. \quad (3.45)$$

Generally, we suppose that the scheme solving FIVPs (3.40), (3.42) have the convergence order $\mathcal{O}(h^r)$, i.e.

$$\begin{cases} \max_{0 \leq n \leq N} |y_n(\xi_1) - y(t_n; \xi_1)| = \mathcal{O}(h^r), \\ \max_{0 \leq n \leq N} |y_n(\xi_2) - y(t_n; \xi_2)| = \mathcal{O}(h^r). \end{cases} \quad (3.46)$$

Due to (3.44)–(3.45), we consider $\{|y(t_n) - y_n|\}_{n=0}^N$. Then we get

$$\max_{0 \leq n \leq N} |y(t_n) - y_n| = \mathcal{O}(h^r). \quad (3.47)$$

As to the linear case of (3.7)–(3.8), we only need to replace (3.42) and (3.43) with the initial value conditions

$${}_a^{RL}D_t^{\gamma-1}y(t)\Big|_{t=a} = \xi_1, \quad \lim_{t \rightarrow a^+} J_a^{2-\gamma}y(t) = 0, \quad (3.48)$$

$${}_a^{RL}D_t^{\gamma-1}y(t)\Big|_{t=a} = \xi_2, \quad ; \quad \lim_{t \rightarrow a^+} J_a^{2-\gamma}y(t) = 0. \quad (3.49)$$

Again, the shooting procedures for linear FBVPs (3.3)–(3.4) and (3.9)–(3.10) are similar to that of (3.1)–(3.2) and (3.7)–(3.8), respectively.

3.2. Shooting method for nonlinear problems. For nonlinear FBVP (3.1) and (3.7) with homogeneous boundary value conditions (3.41), we consider the following FIVP:

$${}_a^C D_t^\gamma y(t) + f(t, y(t)) = 0, \quad a \leq t \leq b, \quad 1 < \gamma \leq 2, \quad (3.50)$$

$$y(a) = 0, \quad y'(a) = \xi, \quad (3.51)$$

The FIVPs (3.50) is corresponding to FBVP (3.1), (3.41) with analytic solutions denoted as $y(t; \xi)$. In general, $y(b; \xi) \neq 0$. Once a zero point ξ^* of $\phi(\xi) := y(b; \xi)$ is found, one obtains $y(t; \xi^*)$, the solution of FBVP (3.1), (3.41). When $y(t; \xi)$, and hence $\phi(\xi)$ are continuously differentiable with respect to ξ , Newton's method can be employed to determine ξ^* . Starting with an initial approximation $\xi^{(0)}$, one gets $\xi^{(k)}$ as follows:

$$\xi^{(k+1)} = \xi^{(k)} - \frac{\phi(\xi^{(k)})}{\phi'(\xi^{(k)})}. \quad (3.52)$$

On one hand, $y(b; \xi)$, hence $\phi(\xi)$ can be determined by solving FIVP (3.50) numerically. On the other hand, it is easy to check that $w(t; \xi) := \frac{\partial y(t; \xi)}{\partial \xi}$ is a solution of the following FIVP (refer to [30]):

$${}_a^C D_t^\gamma w(t; \xi) + \frac{\partial f(t, y(t; \xi))}{\partial y} w(t; \xi) = 0, \quad (3.53)$$

$$w(a; \xi) = 0, \quad w'(a; \xi) = 1, \quad a \leq t \leq b, \quad 1 < \gamma \leq 2. \quad (3.54)$$

So we can compute $w(b; \xi)$, i.e., $\phi'(\xi)$ by solving FIVPs (3.53) numerically. However, considering the computation complexity for $\frac{\partial f(t, y(t; \xi))}{\partial y}$ in practice, we usually calculate the difference quotient

$$\Delta\phi(\xi^{(k)}) := \frac{\phi(\xi^{(k)} + \Delta\xi^{(k)}) - \phi(\xi^{(k)})}{\Delta\xi^{(k)}}$$

instead of the derivative $\phi'(\xi^{(k)})$ itself, where $\Delta\xi^{(k)}$ is a sufficiently small number. And the formula (3.52) is replaced by

$$\xi^{(k+1)} = \xi^{(k)} - \frac{\phi(\xi^{(k)})}{\Delta\phi(\xi^{(k)})}. \quad (3.55)$$

In short, the procedure of solving FBVP (3.1), (3.41) by shooting method is displayed as follows:

- step 1.* choose a starting value $\xi^{(0)}$ and an iterative precision ϵ for Newton's method; then for $k = 0, 1, 2, \dots$;
- step 2.* obtain $y(b; \xi^{(k)})$ by solving FIVP (3.50) with $y'(a) = \xi^{(k)}$ and compute $\phi(\xi^{(k)})$;
- step 3.* choose a sufficiently small number $\Delta\xi^{(k)} \neq 0$ and determine $y(b; \xi^{(k)} + \Delta\xi^{(k)})$ by solving the FIVP (3.50) with $y'(a) = \xi^{(k)} + \Delta\xi^{(k)}$, and then compute $\phi(\xi^{(k)} + \Delta\xi^{(k)})$;
- step 4.* compute $\Delta\phi(\xi^{(k)})$ and determine $\xi^{(k+1)}$ by the formula (3.55); repeat *steps 2-4* until $|\xi^{(k+1)} - \xi^{(k)}| \leq \epsilon$ and denote the final $\xi^{(k+1)}$ by ξ^* ;
- step 5.* finally, we obtain the numerical solution of FBVP (3.1), (3.41) by solving FIVP (3.50) with $y'(a) = \xi^*$.

Next, we want to give the error analysis of shooting method for nonlinear FBVP (3.1), (3.7). The error consists of two parts. One is from the error between $|\xi^* - \tilde{\xi}|$. By shooting method, we can only get the approximation $\tilde{\xi}$ of the exact initial value $\xi^* (= y'(a))$. That is to say, we solve FIVP

$$\begin{cases} {}_a^C D_t^\alpha y(t) + f(t, y(t)) = 0, \\ y(a) = 0, \quad y'(a) = \tilde{\xi} \end{cases} \quad (3.56)$$

instead of FIVP

$$\begin{cases} {}_a^C D_t^\alpha y(t) + f(t, y(t)) = 0, \\ y(a) = 0, \quad y'(a) = \xi^*. \end{cases} \quad (3.57)$$

which is an initial perturbation problem in fact. Denote the exact solutions of (3.56) and (3.57) as $y(t; \tilde{\xi})$ and $y(t; \xi^*)$, respectively.

Lemma 3.4 (see [32]). *Suppose $\alpha > 0$, $A(t)$ is a nonnegative function locally integrable on $[a, b]$ and $B(t)$ is a nonnegative, nondecreasing continuous function defined on $[a, b]$, and suppose $\psi(t)$ is nonnegative and locally integrable on $[a, b]$ with*

$$\psi(t) \leq A(t) + B(t) \int_a^t (t - \tau)^{\alpha-1} \psi(\tau) d\tau, \quad t \in [a, b] \quad (3.58)$$

on this interval. Then

$$\psi(t) \leq A(t) E_\alpha(B(t) \Gamma(\alpha) (t - a)^\alpha), \quad (3.59)$$

where E_α is the Mittag-Leffler function defined by

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \quad (3.60)$$

According to the above lemma, we get the error estimate

$$\begin{aligned} & |y(t; \xi^*) - y(t; \tilde{\xi})| \leq \\ & \leq |\xi^* - \tilde{\xi}|(t-a)E_\alpha(K_f(t-a)^\alpha) \leq C_1|\xi^* - \tilde{\xi}|, \quad t \in [a, b], \end{aligned} \quad (3.61)$$

where $C_1 := \max_{t \in [a, b]} (t-a)E_\alpha(K_f(t-a)^\alpha)$.

Another part of the error is from numerical solving procedure of FIVP (3.50) by scheme (3.33), (3.35). Suppose that the scheme of FIVP converges with order r , and denote the approximate solution of $y(t_n; \tilde{\xi})$ as $y_n(\tilde{\xi})$. We get $\exists C_2 > 0$, s.t.

$$\max_{0 \leq n \leq N} |y(t_n; \tilde{\xi}) - y_n(\tilde{\xi})| \leq C_2 h^r. \quad (3.62)$$

To sum up, we have

$$\begin{aligned} |y(t_n; \xi^*) - y_n(\tilde{\xi})| &= |y(t_n; \xi^*) - y(t_n; \tilde{\xi}) + y(t_n; \tilde{\xi}) - y_n(\tilde{\xi})| \leq \\ &\leq |y(t_n; \xi^*) - y(t_n; \tilde{\xi})| + |y(t_n; \tilde{\xi}) - y_n(\tilde{\xi})| \end{aligned}$$

that is,

$$\max_{0 \leq n \leq N} |y(t_n; \xi^*) - y_n(\tilde{\xi})| \leq C_1|\xi^* - \tilde{\xi}| + C_2 h^r. \quad (3.63)$$

As to (3.7), (3.41), we turn it into its corresponding FIVP

$${}_a^{RL} D_t^\gamma y(t) + f(t, y(t)) = 0, \quad a \leq t \leq b, \quad 1 < \gamma \leq 2, \quad (3.64)$$

$${}_a^{RL} D_t^{\gamma-1} y(t) \Big|_{t=a} = \xi, \quad \lim_{t \rightarrow a^+} J_a^{2-\gamma} y(t) = 0. \quad (3.65)$$

The numerical procedure of simulating (3.7), (3.41) is completely similar to that of (3.1), (3.41).

Again, the procedures of shooting method for (3.2) and (3.8) with homogenous boundary conditions (3.41) are similar to the case of (3.1), (3.41) and (3.7), (3.41), respectively.

4. Numerical Experiments for Solving FBVPs

In this section, two numerical examples are given to show the feasibility and validity of single shooting methods for the FBVPs.

First, let us introduce the parameters. $a = 0 = t_0 < t_1 < \dots < t_n = b$ is a given equispaced mesh with stepsize $h = (b-a)/n$, $n = 100$, $\gamma = 1.5$, $\theta = 0.5$. For the linear cases in Examples 1, the two ‘‘initial speeds’’ $\xi_1 = 0.5$, $\xi_2 = 1.5$. For the nonlinear cases in Examples 2, $\epsilon = 10^{-12}$, $\xi^{(0)} = 0$, $\Delta \xi^{(k)} \equiv 10^{-8}$.

Example 3.1. Consider the following linear FBVP

$$\begin{aligned} {}_0^C D_t^{1.5} y(t) + \frac{1}{3} y(t) + \frac{1}{4} {}_0^C D_t^{0.5} y(t) &= r(t), \quad 0 < t < 1, \\ y(0) = 0, \quad y(1) &= 0, \end{aligned}$$

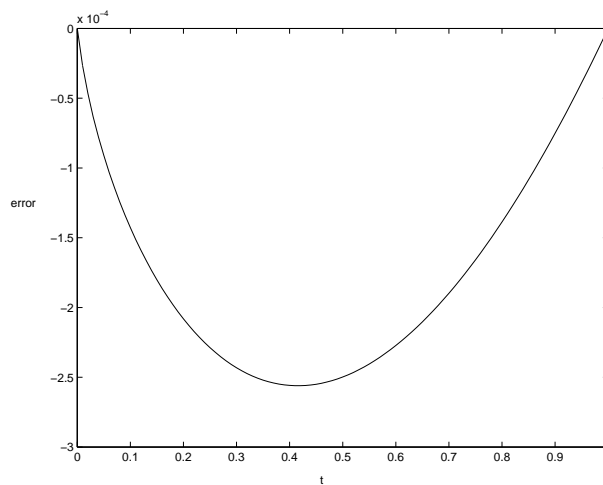


FIGURE 1. The shooting error for Example 3.1

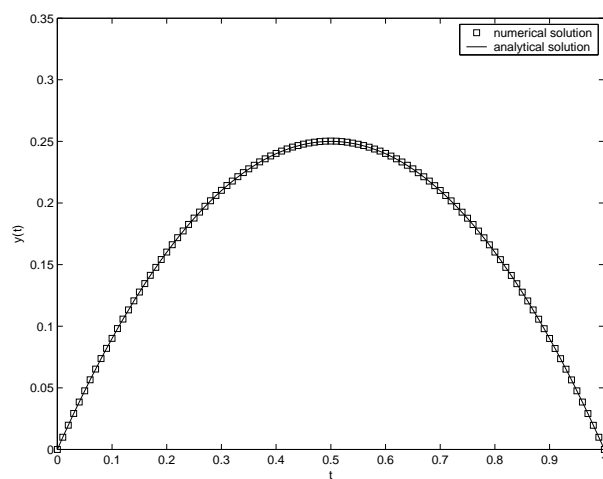


FIGURE 2. The true solution and numerical solution of Example 3.1

where

$$r(t) = -\frac{7t^{0.5}}{2\sqrt{\pi}} + \frac{t}{3} - \frac{2t^{1.5}}{3\sqrt{\pi}} - \frac{t^2}{3}.$$

Since

$$|g(t, u_2, v_2) - g(t, u_1, v_1)| \leq \frac{1}{3} |u_2 - u_1| + \frac{1}{4} |v_2 - v_1|,$$

$$K_g = \frac{1}{3}, \quad L_g = \frac{1}{4}, \quad a = 0, \quad b = 1, \quad \gamma = 1.5, \quad \theta = 0.5,$$

$$K_g \frac{2(b-a)^\gamma}{\Gamma(\gamma+1)} + L_g \left[\frac{(b-a)^{\gamma-\theta}}{\Gamma(2-\theta)\Gamma(\gamma+1)} + \frac{(b-a)^{\gamma-\theta}}{\Gamma(\gamma-\theta+1)} \right] \approx 0.96 < 1,$$

there exists a unique solution for this FBVP according to Theorem 3.1. In fact, one can easily check that $y(t) = t(1-t)$ is the analytical solution. The errors between the numerical solution (obtained by using shooting method mentioned above) and the analytical solution at mesh points are plotted in Figure 1. The true solution (denoted by real line) and numerical solution (denoted by \square) on equispaced mesh are plotted in Figure 2.

Example 3.2. Consider the following nonlinear FBVP

$$\begin{aligned} {}_{-1}^{RL}D_t^{1.5}y(t) + \sin(y) + r(t) &= 0, \quad -1 < t < 1, \\ y(-1) &= 0, \quad y(1) = 0, \end{aligned}$$

where

$$\begin{aligned} r(t) &= \sin(C_{t+1}(2, 2\pi)) + C_{t+1}(0.5, 2\pi), \\ C_{t-a}(\alpha, \omega) &= (t-a)^\alpha \sum_{j=0}^{\infty} \frac{(-1)^j (\omega(t-a))^{2j}}{\Gamma(\alpha+2j+1)}. \end{aligned}$$

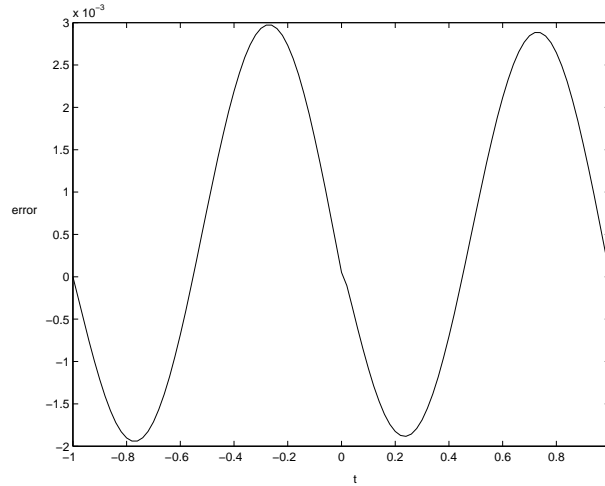


FIGURE 3. the shooting error for Example 3.2

Since

$$\begin{aligned} |f(t, u_2) - f(t, u_1)| &= |\sin(u_2) - \sin(u_1)| \leq |u_2 - u_1|, \\ K_f &= 1, \quad a = -1, \quad b = 1, \quad \gamma = 1.5, \\ K_f \frac{(\gamma-1)^{\gamma-1} (b-a)^\gamma}{\gamma^\gamma \Gamma(\gamma+1)} &\approx 0.82 < 1, \end{aligned}$$

there exists a unique solution for this FBVP according to Theorem 3.1. In fact, one can easily check that $y(t) = C_{t+1}(2, 2\pi)$ is the analytical solution. The errors between the numerical solution (obtained by using shooting method mentioned above) and the analytical solution (approximated by summarizing the first 21 terms of the infinite series $C_{t+1}(2, 2\pi)$ because of $\frac{(2\pi)^{40}}{\Gamma(43)} \approx 6.02 \times 10^{-20}$) at different points are plotted in Figure 3. The analytical solution (denoted by real line) and numerical solution (denoted by \square) on equispaced mesh are plotted in Figure 4.

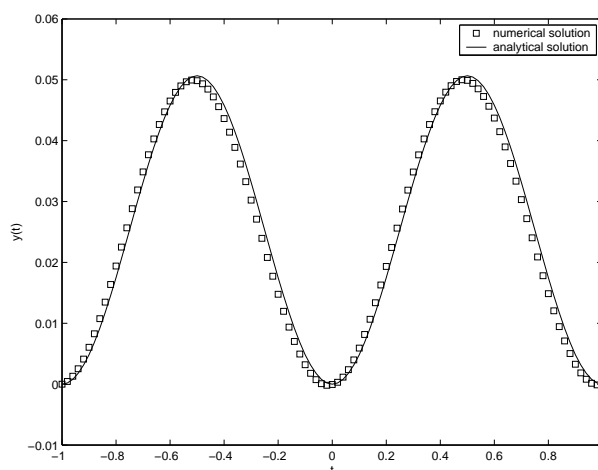


FIGURE 4. the true solution and numerical solution of Example 3.2

TABLE 1

stepsize $h = 1/n$	$\max_{1 \leq i \leq n} y(t_i) - y_n(t_i) $	rates of convergence
1/16	$2.46e - 2$	
1/32	$1.06e - 2$	1.2198
1/64	$4.9e - 3$	1.1013
1/128	$2.34e - 3$	1.0740
1/256	$1.31e - 3$	1.0471

The rates of convergence and maximum errors for Example 3.2 between the numerical solution and the analytical solution are given in Table 1. Table 1 shows that the shooting method is of order one, which is in good

agreement with the fact that (3.33), (3.35) or (3.34), (3.36) is a method of order one.

The above numerical results indicate that the single shooting method is a successful tool to solve fractional boundary value problems (3.1), (3.2), (3.3), (3.4). The numerical experiments for FBVPs (3.7), (3.8) and (3.9), (3.10) can be referred to [33]. The results are very similar.

Bibliography

1. T. S. ALEROEV, On a class of operators associated with differential equations of fractional order. (Russian) *Sibirsk. Mat. Zh.* **46** (2005), No. 6, 1201–1207; English transl.: *Siberian Math. J.* **46** (2005), No. 6, 963–968.
2. G. M. GUBREEV, Regular Mittag–Leffler kernels and spectral decomposition of a class of nonselfadjoint operators. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **69** (2005), No. 1, 17–60; English transl.: *Izv. Math.* **69** (2005), No. 1, 15–57.
3. T. S. ALEROEV, The boundary problems for differential equations with fractional derivatives. *Dissertation, doctor of Physical and Mathematical Sciences, Moscow State University*, 2000.
4. M. S. Brodskii, Triangular and Jordan representations of linear operators. Translated from the Russian by J. M. Danskin. Translations of Mathematical Monographs, Vol. 32. *American Mathematical Society, Providence, R.I.*, 1971; Russian original: Nauka, Moscow, 1969.
5. T. S. ALEROEV, The Sturm–Liouville problem for a second-order differential equation with fractional derivatives in the lower terms. (Russian) *Differentsial'nye Uravneniya* **18** (1982), No. 2, 341–342.
6. T. S. ALEROEV, On a boundary value problem for a fractional-order differential operator. (Russian) *Differ. Uravn.* **34** (1998), No. 1, 123, 144; English transl.: *Differential Equations* **34** (1998), No. 1, 126.
7. T. S. ALEROEV, Aleroev, T. S. Some problems in the theory of linear differential operators of fractional order. (Russian) *Dokl. Akad. Nauk* **341** (1995), No. 1, 5–6.
8. M. LI, N. M. NIE, S. JIMÉNEZ, Y. F. TANG, AND L. VÁZQUEZ, Solving two-point boundary value problems of fractional differential equations, http://www.cc.ac.cn/2009research_report/0902.pdf.
9. M. M. DZHRBASHYAN AND A. B. NERSESYAN, Fractional derivative and Cauchy's problem for differential equations of fractional order. (Russian) *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* **3** (1968), No. 1, 3–29.
10. M. M. DZHRBASHYAN, A boundary value problem for a Sturm–Liouville type differential operator of fractional order. (Russian) *Izv. Akad. Nauk Armyan. SSR, Ser. Mat.* **5** (1970), No. 2, 71–96.
11. A. M. GACHAEV, The boundary problems for differential equations of fractional order, *Dissertation, Nalchik, Russia*, 2005.
12. T. S. ALEROEV, The boundary problem for differential operator of fractional order, *The reports of Circassian International Academy of Sciences (Doklady Adygskoy (Cherkesskoy) Mezhdunarodnoy Akademii Nauk* **1** (1994), No. 1.
13. M. M. MALAMUD AND L. L. ORIDOROGA, On some questions of the spectral theory of ordinary differential equations of fractional order. *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki* **1998**, No. 9, 39–47.

14. M. M. MALAMUD AND L. L. ORIDOROGA, Analog of the Birkhoff theorem and the completeness results for fractional order differential equations. *Russ. J. Math. Phys.* **8** (2001), No. 3, 287–308.
15. T. S. ALEROEV AND H. T. ALEROEVA, Some applications of the theory of disturbance in fractional calculation. *Mem. Differ. Equations Math. Phys.* (to appear).
16. T. KATO, Perturbation theory for linear operators. *Die Grundlehren der mathematischen Wissenschaften*, Band 132. Springer-Verlag New York, Inc., New York, 1966; Russian transl.: Mir, Moscow, 1972.
17. I. C. GOHBERG AND M. G. KREIN, Theory of Volterra operators in Hilbert space and its applications. (Russian) *Nauka, Moscow*, 1967.
18. M. M. DZHRBASHJAN, Integral transformations and presentations of functions in complex space. *Nauka, Moscow*, 1966.
19. V. I. MATSAEV AND Y. A. POLANT, Über die Potenzen eines beschränkten dissipativen Operators. (Russian) *Ukr. Mat. Zh.* **14** (1962), 329–337 (1962).
20. F. R. GANTMAHER AND M. G. KREIN, Oscillation matrices and kernels and small oscillations of mechanical systems. (Russian) 2d ed. *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad*, 1950.
21. E. HILLE AND R. FILLIPS, Functional analysis and semi-groups. (Russian) Translated from the English by D. A. Vasil'kov. Edited by V. M. Alekseev and S. V. Fomin *Izdat. Inostran. Lit., Moscow*, 1962.
22. I. C. GOHBERG AND M. G. KREIN, Introduction to the theory of linear non-selfadjoint operators in Hilbert space. (Russian) *Izdat. Nauka, Moscow*, 1965.
23. T. S. ALEROEV AND H. T. ALEROEVA, A problem on the zeros of the Mittag-Leffler function and the spectrum of a fractional-order differential operator. *Electron. J. Qual. Theory Differ. Equ.* **2009**, No. 25, 18 pp.
24. V. B. LIDSKIIĬ, Conditions for completeness of a system of root subspaces for non-selfadjoint operators with discrete spectrum. (Russian) *Trudy Moskov. Mat. Obsc.* **8** (1959), 83–120.
25. P. B. BAILEY, L. F. SHAMPINE, AND P. E. WALTMAN, Nonlinear two point boundary value problems. Mathematics in Science and Engineering, Vol. 44 *Academic Press, New York-London*, 1968.
26. K. DIETHELM AND N. J. FORD, Numerical solution of the Bagley–Torvik equation. *BIT* **42** (2002), No. 3, 490–507.
27. K. Diethelm, N. J. Ford, and A. D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations. Fractional order calculus and its applications. *Nonlinear Dynam.* **29** (2002), No. 1-4, 3–22.
28. CH. LUBICH, Discretized fractional calculus. *SIAM J. Math. Anal.* **17** (1986), No. 3, 704–719.
29. I. PODLUBNY, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering, 198. *Academic Press, Inc., San Diego, CA*, 1999.
30. J. STOER AND R. BULIRSCH, Introduction to numerical analysis. Translated from the German by R. Bartels, W. Gautschi and C. Witzgall. *Springer-Verlag, New York-Heidelberg*, 1980.
31. M. Weilbeer, Efficient numerical methods for fractional differential equations and their analytical background. *Univ. Braunschweig, Fakultät für Mathematik und Informatik (Diss.)*, Braunschweig, 2005, 218 p.
32. H. YE, J. GAO, AND Y. DING, A generalized Gronwall inequality and its application to a fractional differential equation. *J. Math. Anal. Appl.* **328** (2007), No. 2, 1075–1081.
33. N. M. NIE, Y. M. ZHAO, S. JIMÉNEZ, M. LI, Y. F. TANG AND L. VÁZQUEZ, Solving two-point boundary value problems of fractional differential equations with Riemann–Liouville derivatives, *J. Syst. Simul.* **22** (2010), No. 1.

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