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EXISTENCE RESULTS FOR MULTI-POINT NONLINEAR BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. In this paper, we obtain some existence results in a Banach space for a multi-point boundary value problem involving a nonlinear fractional differential equation given by

$$ {}^{c}D^{q}x(t) = f(t, x(t)), \quad 0 < t < 1, \ \ 1 < q \leq 2, $$

$$ \alpha_1 x(0) - \beta_1 x'(0) = \gamma_1 x(\eta_1), \quad \alpha_2 x(1) + \beta_2 x'(1) = \gamma_2 x(\eta_2), \ \ 0 < \eta_1, \eta_2 < 1. $$

Our results are based on contraction mapping principle and Krasnoselskii's fixed point theorem.

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$$ {}^{c}D^{q}x(t) = f(t, x(t)), \quad 0 < t < 1, \ \ 1 < q \leq 2, $$

$$ \alpha_1 x(0) - \beta_1 x'(0) = \gamma_1 x(\eta_1), \quad \alpha_2 x(1) + \beta_2 x'(1) = \gamma_2 x(\eta_2), \ \ 0 < \eta_1, \eta_2 < 1. $$

მიღებული შედეგები ეფუძნება კუმშვითი ასახვის პრინციპს და კრასნოსელსკის უძრავი წერტილის თეორემას.

1. Introduction

In some real world applications, fractional-order models are found to be more adequate than integer-order models as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. In fact, fractional differential equations arise in many engineering and scientific disciplines as mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see [1]–[7], [10], [12], [17], [18], [21]–[23] and the references therein.

Multi-point nonlocal boundary value problems, initiated by Il'in and Moiseev [15], [16], have been addressed by many authors, for instance, [8], [9], [11], [13], [14], [19]. The multi-point boundary conditions appear in certain problems of thermodynamics, elasticity and wave propagation, see [20] and the references therein. The multi-point boundary conditions may be understood in the sense that the controllers at the end points dissipate or add energy according to censors located at intermediate positions.

In this paper, we consider the following nonlinear fractional differential equation with multi-point boundary conditions

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \;\; 1 < q \le 2, \\ \alpha_1 x(0) - \beta_1 x'(0) = \gamma_1 x(\eta_1), \quad \alpha_2 x(1) + \beta_2 x'(1) = \gamma_2 x(\eta_2), \end{cases} \tag{1.1}$$

where ${}^cD$ is Caputo's fractional derivative, $f : [0,1] \times X \to X$, $0 < \eta_1$, $\eta_2 < 1$, and α_1, α_2, β_1, β_2, γ_1, γ_2 are real numbers. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathrm{C} = \mathrm{C}([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1]$ into X endowed with the topology of uniform convergence with the norm denoted by $\|\cdot\|$.

2. Preliminaries

Let us recall some basic definitions [17], [21], [23] from fractional calculus.

Definition 2.1. For a function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \quad n-1 < q \le n, \quad q > 0,$$

where Γ denotes the gamma function.

Definition 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Definition 2.3. The Riemann–Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}}\, ds, \quad n-1 < q \leq n, \quad q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

We remark that the Caputo derivative becomes the conventional nth derivative of the function as $q \to n$, and the initial conditions for fractional differential equations retain the same form as that for ordinary differential equations with integer order derivatives. On the other hand, the Riemann–Liouville fractional derivative could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations (the same applies to the boundary value problems for fractional differential equations). Moreover, the Caputo derivative of a constant is zero while the Riemann–Liouville fractional derivative of a constant is nonzero. For more details, see [23].

Lemma 2.1 ([18])**.** *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$. Here, $[q]$ denotes the integer part of the real number q.

In view of Lemma 2.1, it follows that

$$I^q\, {}^cD^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [q]+1)$.

Now we state a known result due to Krasnoselskii [24] which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.1. *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be operators such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*
(ii) *A is compact and continuous;*
(iii) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Lemma 2.2. *For a given $\sigma \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \sigma(t), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ \alpha_1 x(0) - \beta_1 x'(0) = \gamma_1 x(\eta_1), \quad \alpha_2 x(1) + \beta_2 x'(1) = \gamma_2 x(\eta_2), \end{cases} \tag{2.2}$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds +$$

$$+ \frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds +$$

$$+ \frac{1}{\Delta}\Big((\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)\Big)\Big[-\alpha_2\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds -$$

$$- \beta_2\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,\sigma(s)\,ds + \gamma_2\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds\Big],$$

where

$$\Delta = \Big[(\beta_1+\gamma_1\eta_1)(\alpha_2-\gamma_2)+(\alpha_2+\beta_2-\gamma_2\eta_2)(\alpha_1-\gamma_1)\Big] \neq 0. \qquad (2.3)$$

Proof. Using (2.1), for arbitrary constants c_0, $c_1 \in \mathbb{R}$ we have

$$x(t) = I^q\sigma(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds - c_0 - c_1 t. \qquad (2.4)$$

In view of the relations ${}^cD^q\ I^q x(t) = x(t)$ and $I^q\ I^p x(t) = I^{q+p}x(t)$ for $q,p>0$, $x\in L(0,1)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\,\sigma(s)\,ds - c_1.$$

Applying the boundary conditions from (2.2), we find that

$$c_0 = -\frac{\gamma_1(\alpha_2+\beta_2-\gamma_2\eta_2)}{\Delta}\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds +$$

$$+ \frac{(\beta_1+\gamma_1\eta_1)}{\Delta}\Big[\alpha_2\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds +$$

$$+ \beta_2\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,\sigma(s)\,ds - \gamma_2\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds\Big],$$

$$c_1 = \frac{\gamma_1(\alpha_2-\gamma_2)}{\Delta}\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\,\sigma(s)\,ds +$$

$$+\frac{(\alpha_1-\gamma_1)}{\Delta}\Bigg[\alpha_2\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds+$$

$$+\beta_2\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds-\gamma_2\int\limits_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds\Bigg],$$

where Δ is given by (2.3). Substituting the values of c_0 and c_1 in (2.4), we obtain

$$x(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds+$$

$$+\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\int\limits_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds+$$

$$+\frac{1}{\Delta}\Big((\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)\Big)\Bigg[-\alpha_2\int\limits_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds-$$

$$-\beta_2\int\limits_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)\,ds+\gamma_2\int\limits_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\sigma(s)\,ds\Bigg].$$

This completes the proof. □

3. Main results

To prove the main results, we need the following assumptions:

$(\mathbf{A_1})$ $\|f(t,x)-f(t,y)\|\le L\|x-y\|$, $\forall\, t\in[0,1]$, $x,\,y\in X$;
$(\mathbf{A_2})$ $\|f(t,x)\|\le\mu(t)$, $\forall\,(t,x)\in[0,1]\times X$, and $\mu\in L^1([0,1],R^+)$.

Theorem 3.1. *Assume that $f:[0,1]\times X\to X$ is a continuous function satisfying the assumption* $(\mathbf{A_1})$*. Then the boundary value problem* (1.1) *has a unique solution provided*

$$L\le\frac{1}{2}\Bigg[\frac{1}{\Gamma(q+1)}\Big(1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big)\Bigg]^{-1}.$$

Proof. Define $\mathcal{F}:\mathrm{C}\to\mathrm{C}$ by

$$(\mathcal{F}x)(t)=\int\limits_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds+$$

$$+\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\int\limits_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\,f(s,x(s))\,ds+$$

$$+\frac{1}{\Delta}\Big((\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)\Big)\Big[-\alpha_2\int\limits_0^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\,f(s,x(s))\,ds-$$

$$-\beta_2\int\limits_0^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,f(s,x(s))\,ds+\gamma_2\int\limits_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\,f(s,x(s))\,ds\Big],\quad t\in[0,1].$$

Setting $\sup_{t\in[0,1]}\|f(t,0)\|=M$ and choosing

$$r\geq 2\frac{M}{\Gamma(q+1)}\Big[1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+ \\ +\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big],$$

we show that $\mathcal{F}B_r\subset B_r$, where $B_r=\{x\in \mathrm{C}:\|x\|\leq r\}$. For $x\in B_r$, we have

$$\|(\mathcal{F}x)(t)\|\leq\int\limits_0^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,\|f(s,x(s))\|\,ds+$$

$$+\Big|\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\Big|\int\limits_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\,\|f(s,x(s))\|\,ds+$$

$$+\Big|\frac{(\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)}{\Delta}\Big|\Big[|\alpha_2|\int\limits_0^{1}\frac{(1-s)^{q-1}}{\Gamma(q)}\,\|f(s,x(s))\|\,ds+$$

$$+|\beta_2|\int\limits_0^{1}\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,\|f(s,x(s))\|\,ds+|\gamma_2|\int\limits_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\,\|f(s,x(s))\|\,ds\Big]\leq$$

$$\leq\int\limits_0^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\big(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|\big)\,ds+$$

$$+\Big|\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\Big|\times$$

$$\times\int\limits_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\big(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|\big)\,ds+$$

$$+\Big|\frac{(\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)}{\Delta}\Big|\times$$

$$\times\Big[|\alpha_2|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\big(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|\big)\,ds+$$

$$+|\beta_2|\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\big(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|\big)\,ds+$$

$$+|\gamma_2|\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\big(\|f(s,x(s))-f(s,0)\|+\|f(s,0)\|\big)\,ds\Big]\le$$

$$\le(Lr+M)\Big[\frac{t^q}{\Gamma(q+1)}+\Big|\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\Big|\frac{\eta_1^q}{\Gamma(q+1)}+$$

$$+\Big|\frac{(\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)}{\Delta}\Big|\Big(\frac{|\alpha_2|}{\Gamma(q+1)}+\frac{|\beta_2|}{\Gamma(q)}+\frac{|\gamma_2|\eta_2^q}{\Gamma(q+1)}\Big)\Big]\le$$

$$\le L\Big[\frac{1}{\Gamma(q+1)}\Big(1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big)\Big]r+$$

$$+\frac{M}{\Gamma(q+1)}\Big[1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big]\le r.$$

Now, for $x,y\in\mathrm{C}$ and each $t\in[0,1]$, we obtain

$$\big\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\big\|\le\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\big\|f(s,x(s))-f(s,y(s))\big\|\,ds+$$

$$+\Big|\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\Big|\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\big\|f(s,x(s))-f(s,y(s))\big\|\,ds+$$

$$+\Big|\frac{(\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)}{\Delta}\Big|\Big[|\alpha_2|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\big(\|f(s,x(s))-f(s,y(s))\|\big)\,ds+$$

$$+|\beta_2|\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\big\|f(s,x(s))-f(s,y(s))\big\|\,ds+$$

$$+|\gamma_2|\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\big\|f(s,x(s))-f(s,y(s))\big\|\,ds\Big]\le$$

$$\le L\|x-y\|\Big[\frac{t^q}{\Gamma(q+1)}+\Big|\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\Big|\frac{\eta_1^q}{\Gamma(q+1)}+$$

$$+\Big|\frac{(\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)}{\Delta}\Big|\Big(\frac{|\alpha_2|}{\Gamma(q+1)}+\frac{|\beta_2|}{\Gamma(q)}+\frac{|\gamma_2|\eta_2^q}{\Gamma(q+1)}\Big)\Big]\le$$

$$\le L\|x-y\|\Big[\frac{1}{\Gamma(q+1)}\bigg(1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\bigg)\Big]=$$

$$=\Lambda_{\alpha_i,\beta_i,\gamma_i,\eta_i,q,L}\|x-y\|,\ \ i=1,2,$$

where

$$\Lambda_{\alpha_i,\beta_i,\gamma_i,\eta_i,q,L}=L\Big[\frac{1}{\Gamma(q+1)}\Big\{1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big\}\Big],$$

which depends only on the parameters involved in the problem. As $\Lambda_{\alpha_i,\beta_i,\gamma_i,\eta_i,q,L}<1$, therefore $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. □

Theorem 3.2. *Let $f:[0,1]\times X\to X$ be a continuous function mapping bounded subsets of $[0,1]\times X$ into relatively compact subsets of X, and the assumptions $(A_1)-(A_2)$ hold with*

$$L\Big[\frac{1}{\Gamma(q+1)|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big]<1.$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Let us fix

$$r\ge\|\mu\|_{L_1}\Big[\frac{1}{\Gamma(q+1)}\Big\{1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$

$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big\}\Big],$$

and consider $B_r=\big\{x\in \mathrm{C}:\ \|x\|\le r\big\}$. We define the operators Φ and Ψ on B_r as

$$(\Phi x)(t)=\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))\,ds,$$

$$(\Psi x)(t)=\frac{\gamma_1}{\Delta}\Big(\alpha_2(1-t)+\beta_2+\gamma_2(t-\eta_2)\Big)\int_0^{\eta_1}\frac{(\eta_1-s)^{q-1}}{\Gamma(q)}\,f(s,x(s))\,ds+$$

$$+\frac{1}{\Delta}\Big((\beta_1+\gamma_1\eta_1)+t(\alpha_1-\gamma_1)\Big)\Big[-\alpha_2\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\,f(s,x(s))\,ds-$$
$$-\beta_2\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\,f(s,x(s))\,ds+\gamma_2\int_0^{\eta_2}\frac{(\eta_2-s)^{q-1}}{\Gamma(q)}\,f(s,x(s))\,ds\Big].$$

For $x,\,y\in B_r$, we find that

$$\|\Phi x+\Psi y\|\le\|\mu\|_{L_1}\Big[\frac{1}{\Gamma(q+1)}\Big(1+\frac{1}{|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$
$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big)\Big]\le r.$$

Thus, $\Phi x+\Psi y\in B_r$. It follows from the assumption (A_1) that Ψ is a contraction mapping for

$$L\Big[\frac{1}{\Gamma(q+1)|\Delta|}\Big\{\big(|\alpha_2|+|\beta_2|+|\gamma_2|\big)|\gamma_1|\eta_1^q+$$
$$+\big(|\beta_1+\gamma_1\eta_1|+|\alpha_1-\gamma_1|\big)\big(|\alpha_2|+|\beta_2|q+|\gamma_2|\eta_2^q\big)\Big\}\Big]<1.$$

The continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\|\le\frac{\|\mu\|_{L^1}}{\Gamma(q+1)}\,.$$

Now we prove the compactness of the operator Φ. In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_r}\|f(t,x)\|=f_{\max}$, and consequently we have

$$\big\|(\Phi x)(t_1)-(\Phi x)(t_2)\big\|=\Big\|\frac{1}{\Gamma(q)}\int_0^{t_1}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]f(s,x(s))\,ds+$$
$$+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))\,ds\Big\|\le\frac{f_{\max}}{\Gamma(q+1)}\big|2(t_2-t_1)^q+t_1^q-t_2^q\big|,$$

which is independent of x. So Φ is relatively compact on B_r. Hence, by Arzela–Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 2.1 are satisfied and the conclusion of Theorem 2.1 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

Example. Consider the following boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t)=\dfrac{1}{(t+5)^2}\,\dfrac{\|x\|}{1+\|x\|}\,,\quad t\in[0,1],\\ x(0)-x'(0)=x\Big(\dfrac{1}{3}\Big),\quad x(1)+x'(1)=\dfrac{1}{2}\,x\Big(\dfrac{1}{2}\Big). \end{cases}\tag{3.1}$$

Here,

$$f(t, x(t)) = \frac{1}{(t+5)^2} \frac{\|x\|}{1+\|x\|},$$
$$\alpha_1 = 1, \ \ \beta_1 = 1, \ \ \alpha_2 = 1, \ \ \beta_2 = 1, \quad \gamma_1 = 1, \ \ \gamma_2 = 1/2.$$

As

$$\|f(t,x) - f(t,y)\| \le \frac{1}{25} \|x-y\|,$$

therefore (A_1) is satisfied with $L = \frac{1}{25}$. Further,

$$2L\Bigg[\frac{1}{\Gamma(q+1)} \Big\{1 + \frac{1}{|\Delta|} \Big\{\big(|\alpha_2| + |\beta_2| + |\gamma_2\eta_2|\big)|\gamma_1|\eta_1^q + \\ + \big(|\beta_1 + \gamma_1\eta_1| + |\alpha_1 - \gamma_1|\big)\big(|\alpha_2| + |\beta_2|q + |\gamma_2|\eta_2^q\big)\Big\}\Big\}\Bigg] \\ = \frac{144 + 9\sqrt{3} + 6\sqrt{2}}{225\sqrt{\pi}} < 1.$$

Thus, by Theorem 3.1, the boundary value problem (3.1) has a unique solution on $[0, 1]$.

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