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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER. APPROXIMATION OF INVERSE OPERATORS BY MATRICES

Abstract. We investigate a Cauchy problem and a boundary value problem for a fractional order differential operator, where the order of the operator is within the range between 2 and 3. Relationship is established between the eigenvalues of such operators and zeroes of functions of Mittag–Leffler type. Approximation matrices are also investigated.

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რეზიუმე. ნაშრომში შესწავლილია კოშის და სასაზღვრო ამოცანები წილადი რიგის დიფერენციალური ოპერატორისათვის, სადაც ოპერატორის რიგი მოთავსებულია 2-სა და 3-ს შორის. დადგენილია კავშირი ამ ოპერატორის საკუთრივ მნიშვნელობებსა და მიტტაგ–ლეფლერის ტიპის ერთი ფუნქციის ნულებს შორის. გარდა ამისა, შესწავლილია მატრიცებით აპროქსიმირების საკითხები.

1. Boundary Value Problems for Differential Equations of Fractional Order

Let $\{\gamma_k\}_0^n \equiv \left\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_n\right\}$ be a random set of real numbers meeting the following requirement: $0 < \gamma_k < 1$ $(k = 0, 1, 2, \ldots, n)$. Denote

$$\sigma_n = \sum_{j=0}^{n} \gamma_j - 1, \quad n - 1 \le \sigma_n \le n.$$

Let function $f(x)$ be defined on the interval $[0; 1]$. Under the term "fractional derivative of the order $\alpha \in [0; 1]$ of the function $f(x)$" it is understood the following expression:

$$\frac{d^\alpha u}{dx^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{u(t)}{(x-t)^\alpha}\, dt.$$

Consider the following differential operators

$$D^{(\sigma_0)} f(x) \equiv \frac{d^{-(1-\gamma_0)}}{dx^{-(1-\gamma_0)}} f(x),$$

$$D^{(\sigma_1)} f(x) \equiv \frac{d^{-(1-\gamma_1)}}{dx^{-(1-\gamma_1)}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D^{(\sigma_n)} f(x) \equiv \frac{d^{-(1-\gamma_n)}}{dx^{-(1-\gamma_n)}} \frac{d^{\gamma_{n-1}}}{dx^{\gamma_{n-1}}} \cdots \frac{d^{\gamma_1}}{dx^{\gamma_1}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} f(x).$$

Assume that all these operators are defined almost everywhere on $[0; 1]$.

Consider the operator

$$D^{(\sigma_3)}(y) = \frac{d^{-(1-\gamma_3)}}{dx^{-(1-\gamma_3)}} \frac{d^{\gamma_2}}{dx^{\gamma_2}} \frac{d^{\gamma_1}}{dx^{\gamma_1}} \frac{d^{\gamma_0}}{dx^{\gamma_0}} y$$

with the parameters $\gamma_0 = 1$, $\gamma_1 = \alpha$, $\gamma_2 = 1$, $\gamma_3 = 1$ the operator takes the form

$$D^{(\sigma_3)} y(x) = \frac{d^2}{dx^2} \int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt,$$

its order being $\sigma_3 = 2 + \alpha$.

Pose the following boundary value problem for the selected operator:

$$\frac{d^2}{dx^2} \int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt - \{\lambda + q(x)\}\, y = 0, \tag{1}$$

$$y(0) = 0, \quad y^\alpha(0) = 0, \quad y(1) = 0. \tag{2}$$

Lemma 1.1. *Let* $(y(x; \lambda)$ *be the solution of the following Cauchy problem for Equation* (1)

$$y(0) = 0, \quad y^\alpha(0) = 0, \quad y^{1+\alpha}(0) = c. \tag{3}$$

Then the following identity holds:

$$y(x;\lambda) = \frac{c}{\Gamma(2+\alpha)}\, x^{1+\alpha} + \frac{c}{\Gamma(2+\alpha)} \int\limits_0^x (x-t)^{1+\alpha} \left\{\lambda + q\,(t)\right\} y\,(t;\lambda)\; dt. \qquad (4)$$

The proof of the lemma is provided by word-for-word repetition of similar arguments from [1].

Corollary. *In case* $q(x) \equiv 0$*, the solution of Problem* (1)–(3) *satisfies the following identity*

$$y(x;\lambda) = \frac{c}{\Gamma\,(2+\alpha)}\, x^{1+\alpha} + \frac{c\lambda}{\Gamma\,(2+\alpha)} \int\limits_0^x (x-t)^{1+\alpha}\, y\,(t;\lambda)\; dt. \qquad (5)$$

For further arguments, introduce specific notation for the order of the operator: $\sigma_3 = 2 + \alpha = \frac{1}{\rho}$.

Theorem 1.1. a) λ_j *is an eigenvalue of Problem* (1)–(2) *if and only if* λ_j *is a zero of the following Mittag–Leffler function*

$$E_\rho(\lambda; \frac{1}{\rho}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\rho^{-1} + k\rho^{-1})},$$

that is, the eigenvalues of Problem (1)–(2) *coincide with the roots of the equation* $E_\rho\big(\lambda; \frac{1}{\rho}\big) = 0$.

b) *The eigenfunctions of Problem* (1)–(2) *have the form:*

$$y_j(x) = x^{\frac{1}{\rho}-1} E_\rho\Big(\lambda_j x^{\frac{1}{\rho}}; \frac{1}{\rho}\Big) \quad (j = 1, 2, 3, \ldots),$$

where λ_j *are the zeroes of the function* $E_\rho(\lambda; \frac{1}{\rho})$.

Proof. Rewrite (5) as

$$\left\{ y(x,\lambda) - \frac{c\lambda}{\Gamma(2+\alpha)} \int\limits_0^x (x-t)^{1+\alpha} y(t,\lambda)\, dt \right\} = \frac{c x^{1+\alpha}}{\Gamma(2+\alpha)}.$$

It is not difficult to see that

$$\frac{1}{\Gamma\,(2+\alpha)} \int\limits_0^x (x-t)^{1+\alpha}\, y\,(t;\lambda)\; dt = \frac{d^{\frac{1}{\rho}}}{dx^{\frac{1}{\rho}}}\, y(x,\lambda).$$

Therefore, the latter equality can be rewritten as

$$\left\{ y(x,\lambda) - c\lambda \frac{d^{\frac{1}{\rho}}}{dx^{\frac{1}{\rho}}} y(x,\lambda) \right\} = \frac{c x^{\frac{1}{\rho}-1}}{\Gamma(\rho^{-1})}\, .$$

To solve that integral equation, we will use the Dzhrbashyan theorem.

M. M. Dzhrbashyan theorem. Suppose the function $f(x)$ belongs to $L_1(0;1)$. Then the equation

$$u(x) = f\,(x) + \lambda \frac{d^{\frac{1}{\rho}}}{dx^{\frac{1}{\rho}}}\, u(t)$$

will have a unique solution, namely:

$$u(x) = f\,(x) + \lambda \int\limits_0^x (x-t)^{\frac{1}{\rho}-1}\, E_\rho\Big(\lambda\,(x-t)^{\frac{1}{\rho}}\,;\frac{1}{\rho}\Big) f\,(t)\; dt.$$

It follows from this theorem that the solution of Problem (1)–(3) can be represented as:

$$y(x;\lambda) = c_1\Big[x^{\frac{1}{\rho}-1} + \lambda \int\limits_0^x (x-t)^{\frac{1}{\rho}-1} E_\rho\Big(\lambda(x-t)^{\frac{1}{\rho}}, \frac{1}{\rho}\Big) t^{\frac{1}{\rho}-1}\, dt\Big].$$

The integral on the right-hand side can be calculated with the use of the known M. M. Dzhrbashyan's formula:

$$\int\limits_0^l x^{\alpha-1} E_\rho\left(\lambda x^{\frac{1}{\rho}};\alpha\right)(l-x)^{\beta-1}\, E_\rho\left(\lambda^*\,(l-x)^{\frac{1}{\rho}}\,;\beta\right)\, dx =$$

$$= \frac{\lambda E_\rho\left(\lambda l^{\frac{1}{\rho}};\alpha+\beta\right) - \lambda^* E_\rho\left(\lambda^* l^{\frac{1}{\rho}};\alpha+\beta\right)}{\lambda-\lambda^*}\, l^{\alpha+\beta-1}. \tag{6}$$

Taking $\alpha = \frac{1}{\rho}$, $\lambda^* = 0$ in the formula, we will express the integral as follows:

$$\lambda \int\limits_0^x (x-t)^{\frac{1}{\rho}-1}\, E_\rho\left(\lambda\,(x-t)^{\frac{1}{\rho}}\,;\frac{1}{\rho}\right) t^{\frac{1}{\rho}-1}\, dt = \lambda E_\rho\left(\lambda x^{\frac{1}{\rho}}, \frac{2}{\rho}\right),$$

wherefrom the following general solution for the Cauchy problem (1)-(3) can be derived:

$$y(x,\lambda) = c x^{\frac{1}{\rho}-1} E_\rho\left(\lambda x^{\frac{1}{\rho}}, \frac{1}{\rho}\right). \tag{7}$$

It follows from (7) that λ is an eigenvalue of Problem (1)–(2) if and only if λ is a zero of the function $E_\rho(\lambda;\frac{1}{\rho})$, with the eigenfunctions having the form

$$y_j(x) = x^{\frac{1}{\rho}-1} E_\rho\left(\lambda_j x^{\frac{1}{\rho}};\frac{1}{\rho}\right) \quad (j=1,2,3,\ldots).$$

Thus the theorem is proved. □

Now let us discuss inverse operators. The object of our discussion is the operator generated by the following differential expression and appropriate boundary conditions:

$$Au = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d^2}{dx^2} \displaystyle\int_0^x \dfrac{u'(t)}{(x-t)^{1-\alpha}}\, dt \\ u(0) = 0; \quad u^{(\alpha)}(0) = 0; \quad u(1) = 0. \end{cases}$$

It should not be forgotten that the order of the operator is within the range between 2 and 3.

Theorem 1.2. *The operator $\widetilde{A}$ inverse to the operator* A *looks as follows:*

$$\widetilde{A}u = \frac{1}{\Gamma(3-\alpha)} \left[x^{2-\alpha} \int_0^1 (1-t)^{2-\alpha} u(t)\, dt - \int_0^x (x-t)^{2-\alpha} u(t)\, dt \right].$$

Proof. We have to prove that $\widetilde{A}Au = A\widetilde{A}u = u$, which can be proved by direct integration, with the boundary conditions taken into account. □

Theorem 1.3. *The system of eigenfunctions of the operator* A *is complete in $L_2(0;1)$.*

Proof. To prove this, let us use the Lidskii theorem [3]. What should be proved here is that the operator A is dissipative.

Consider the following expression:

$$(D^{(\sigma_3)}y, y) = \int_0^1 \left\{ \frac{d^2}{dx^2} \int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt \right\} \overline{y}(x)\, dx = \int_0^x \overline{y}(x)\, d\left(\int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt \right)' =$$

$$= \overline{y}(x) \left\{ \int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt \right\} \Bigg|_0^1 - \int_0^1 \left\{ \int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt \right\} \overline{y}'(x)\, dx.$$

In the latter expression, the first of the summands is equal to zero due to the boundary conditions, while the second summand equals

$$\int_0^1 \left\{ \int_0^x \frac{y'(t)}{(x-t)^\alpha}\, dt \right\} \overline{y}'(x)\, dx = (D^{(\sigma_2)}y', y').$$

The dissipativity of the operator $D^{(\sigma_2)}$ is proved in [4].

The operator $\widetilde{A}$ is nuclear, because $\frac{1}{\rho} - 1 = 2 + \alpha > 2$. Therefore, it follows from the Lidskii theorem that the system of the eigenfunctions of the operator A is complete in $L_2(0;1)$. The theorem is proved. □

Note. It follows from the proof that the assertion of the theorem is true if the function $q(x)$ is semi-bounded.

Theorem 1.4. *The operator* A *has at least one positive eigenvalue. Such eigenvalue has the largest module.*

Proof. The assertion of the theorem follows from the non-negative nature of the nucleus of the operator A [4]. □

2. Approximation of Inverse Operators by Matrices

Thus the operator

$$A^{-1}u = \Gamma \frac{1}{(3-\alpha)} \left[\int_0^E (x-t)^{2-\alpha} u(t)\,dx - x^{2-\alpha} \int_0^1 (1-t)^{2-\alpha} u(t)\,dt \right]$$

is inverse to the basic operator generated by the boundary value problem. For convenience we will denote its order by $\mu = 2-\alpha$. Then, to within a factor, it is possible to represent it as follows:

$$A_1 u = \int_0^x (x-t)^\mu u(t)\,dt - \int_0^1 x^\mu (1-t)^\mu u(t)\,dt.$$

Then its kernel equals

$$K(x,t) = \theta(x,t)(x-t)^\mu - x^\mu (1-t)^\mu,$$

where $\theta(x,t) = \begin{cases} 0, & t \geq x, \\ 1, & t < x. \end{cases}$

For detection of some remarkable properties of the operator A, we approximate the continuous kernel by a matrix using the elementary partition of segment the $[0,1]$: $x_0 = 0$, $x_i = \frac{i}{n}$, $x_n = 1$; $t_0 = 0$, $t_j = \frac{j}{n}$, $t_n = 1$ $(i = 0,\dots,n;\ j = 0,\dots,n)$.

Then the elements of the matrix $K = \|K_{ij}\|$ are defined by the formula

$$K_{ij} = K(x_i,t_j) = \theta(i,j)\Big(\frac{i-j}{n}\Big)^\mu - \frac{i^\mu}{n^\mu} \cdot \Big(1 - \frac{j}{n}\Big)^\mu.$$

For simplicity we will multiply all elements of the matrix K_{ij} by $n^{2\mu}$ which will not change its basic properties:

$$K^*_{ij} = n^{2\mu} K_{ij} = \theta(i,j) n^\mu (i-j)^\mu - i^\mu (n-j)^\mu.$$

Thus we obtain a matrix of the order $n+1$.

Let us consider the structure of the matrix K^*. The first and the last rows, as well as the first and the last columns of the matrix are zero, since $K(0,j) = K(n,j) = K(i,0) = K(i,n) = 0$; all other elements are negative.

The elements to the right of the principal diagonal (including the diagonal itself) are calculated by the formula $K_{ij} = -i^\mu (n-j)^\mu$, because here $\theta(i,j) = 0$, and this means that their modules increase across the rows and columns from edges to the principal diagonal.

The elements below the principal diagonal $(\theta(i,j) = 1)$ are calculated by the formula $K_{ij} = n^\mu (i-j)^\mu - i^\mu (n-j)^\mu$.

Let us construct the matrix K^* for some values of n:

$n = 4$, $\mu = 2$:

$$K_4^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & -4 & -1 & 0 \\ 0 & -20 & -16 & -4 & 0 \\ 0 & -17 & -20 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$n = 5$, $\mu = 2$:

$$K_5^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16 & -9 & -4 & -1 & 0 \\ 0 & -39 & -36 & -16 & -4 & 0 \\ 0 & -44 & -56 & -36 & -9 & 0 \\ 0 & -31 & -44 & -39 & -16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 2.1. *The matrix K^* is symmetric with respect to the secondary diagonal: $K_{ij} = K_{n-j,n-i}$.*

Proof. The statement of the theorem is verified by elementary substitution. □

Note. In fact, a more general statement about symmetry of the kernel is also correct: $K(x,t) = K(1-t, 1-x)$. Indeed:

$$\begin{gathered} K(1-t, 1-x) = \\ = \theta(1-t, 1-x) \cdot (1-t-(1-x))^\mu - (1-t)^\mu (1-(1-x))^\mu = \\ = \big\{\theta(1-t, 1-x) = \theta(x,t)\big\} = \theta(x,t) \cdot (x-t)^\mu - (1-t)^\mu x^\mu = K(x,t). \end{gathered}$$

For further study we will slightly simplify the matrix K^* by removing bordering zeros (the matrix becomes of the order $n-1$), and multiplying by -1 (the matrix elements become positive). We denote the new matrix of the order $n-1$ by L_n:

$$L_{ij} = i^\mu (n-j)^\mu - \theta(i,j) n^\mu (i-j)^\mu \quad (i = 1, \ldots, n-1; \;\; j = 1, \ldots, n-1),$$

$$L_n =$$

$$= \begin{pmatrix} 1^\mu (n-1)^\mu & 1^\mu (n-2)^\mu & \ldots & 1^\mu \\ 2^\mu (n-1)^\mu - n^\mu 1^\mu & 2^\mu (n-2)^\mu & \ldots & 2^\mu \\ 3^\mu (n-1)^\mu - n^\mu 2^\mu & 3^\mu (n-2)^\mu - n^\mu & \ldots & 3^\mu \\ \ldots & \ldots & \ldots & \ldots \\ (n-1)^\mu (n-1)^\mu - n^\mu (n-2)^\mu & (n-1)^\mu (n-2)^\mu - n^\mu (n-3)^\mu & \ldots & (n-1)^\mu \end{pmatrix}.$$

Let us divide all the matrix rows by i^μ, and the columns by $(n-j)^\mu$; we will obtain the matrix M: $M_{ij} = \frac{L_{ij}}{i^\mu \, (n-j)^\mu}$. The matrix elements are calculated by the following formula:

$$M_{ij} = \begin{cases} 1, & i \le j, \\ 1 - \Big[\dfrac{n(i-j)}{i(n-j)}\Big]^{\mu}, & i > j, \end{cases}$$

$$M =$$

$$= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1-\Big[\dfrac{n}{2(n-1)}\Big]^{\mu} & 1 & 1 & \dots & 1 \\ 1-\Big[\dfrac{2n}{3(n-1)}\Big]^{\mu} & 1-\Big[\dfrac{n}{3(n-2)}\Big]^{\mu} & 1 & \dots & 1 \\ 1-\Big[\dfrac{3n}{4(n-1)}\Big]^{\mu} & 1-\Big[\dfrac{2n}{4(n-2)}\Big]^{\mu} & 1-\Big[\dfrac{n}{4(n-3)}\Big]^{\mu} & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1-\Big[\dfrac{n(n-2)}{(n-1)^2}\Big]^{\mu} & 1-\Big[\dfrac{n(n-3)}{(n-1)(n-2)}\Big]^{\mu} & 1-\Big[\dfrac{n(n-4)}{(n-1)(n-3)}\Big]^{\mu} & \dots & 1 \end{bmatrix}.$$

This is a matrix of the order $n-1$.

Theorem 2.2. *The elements of the matrix M below the principal diagonal increase as they approach the principal diagonal, i.e. $M_{i,j+1} \ge M_{i,j}$, $M_{i-1,j} \ge M_{i,j}$, $\forall\, i, j$.*

Proof. For the elements of the principal diagonal and the ones above it the statement is obvious: all of them equal 1. The elements below the principal diagonal are calculated by the formula

$$M_{ij} = 1 - \Big[\frac{n(i-j)}{i(n-j)}\Big]^{\mu}.$$

We will prove that they increase across rows (the increasing nature across columns is proved similarly). We will consider n and i as constants, denote the variable j by x, $x \in [0, i-1]$ and consider the elements of a row as a function of x:

$$M(x) = 1 - \Big(\frac{n}{i}\Big)^{\mu}\Big(\frac{i-x}{n-x}\Big)^{\mu}, \quad x \in [0, i-1]\,.$$

Let us calculate the derivative of this function:

$$M'(x) = -\Big(\frac{n}{i}\Big)^{\mu}\mu\Big(\frac{i-x}{n-x}\Big)^{\mu-1}\frac{-(n-x)+(i-x)}{(n-x)^2} =$$

$$= \mu\Big(\frac{n}{i}\Big)^{\mu}\Big(\frac{i-x}{n-x}\Big)^{\mu-1}\frac{n-i}{(n-x)^2}.$$

$M'(x) > 0$ because $n > i > x$, hence $M(x)$ is an increasing function, i.e. the matrix elements increase. The theorem is proved. □

It is known that the operator A is an oscillatory operator for $\mu \ge 1$ and it has a real spectrum. For $\mu \in (0;1)$ the spectrum becomes mixed, and for $\mu \in (0;1)$ it becomes complex. It is logical to assume that the same statements are true for the matrices as well.

A matrix is oscillatory if all its minors are positive [5]. Presently the proof of the oscillatory nature seems to be a rather complicated task. If at least one minor of a matrix is negative, the matrix loses its oscillatory nature.

Theorem 2.3. *The matrix M is not oscillatory if $\mu < 1$.*

Proof. It turns out that in the matrix M there is a minor which changes its sign when the parameter μ passes 1. Note that the transformation from the initial matrix K to the matrix M does not change the signs of minors, therefore they do not influence the oscillatory nature.

Let us take a matrix of an odd order, i.e. $n - 1 = 2k + 1$.

Let us consider in it a central minor of the 2nd order symmetric with respect to the secondary diagonal

$$m = \begin{vmatrix} M_{k+1,k} & 1 \\ M_{k+2,k} & M_{k+2,k+1} \end{vmatrix}.$$

Its elements are:

$$M_{k+1,k} = M_{k+2,k+1} = 1 - \Big(\frac{n \cdot 1}{(k+1)(n-k)}\Big)^{\mu} = 1 - \Big(\frac{2}{k+2}\Big)^{\mu},$$
$$M_{k+2,k} = 1 - \Big(\frac{2n}{(k+2)(n-k)}\Big)^{\mu} = 1 - \Big(\frac{4(k+1)}{(k+2)^2}\Big)^{\mu}.$$

This minor equals

$$m = M_{k+1,k}^2 - M_{k+2,k} = \Big[1 - \Big(\frac{2}{k+2}\Big)^{\mu}\Big]^2 - \Big[1 - \Big(\frac{4(k+1)}{(k+2)^2}\Big)^{\mu}\Big] =$$
$$= \Big[\frac{2}{k+2}\Big]^{\mu}\Big[\Big(\frac{2}{k+2}\Big)^{\mu} + \Big(\frac{2(k+1)}{k+2}\Big)^{\mu} - 2\Big].$$

The second multiplier equals zero for $\mu = 1$. It is necessary to prove that it will be negative for $\mu < 1 \; \forall \, k$.

Let us introduce the function

$$f(\mu) = \Big[\Big(\frac{2}{k+2}\Big)^{\mu} + \Big(\frac{2(k+1)}{k+2}\Big)^{\mu} - 2\Big]$$

and calculate its derivative:

$$f'(\mu) = \Big(\frac{2}{k+2}\Big)^{\mu} \ln\Big(\frac{2}{k+2}\Big) + \Big(\frac{2(k+1)}{k+2}\Big)^{\mu} \ln\Big(\frac{2(k+1)}{k+2}\Big).$$

This derivative is positive at the point $\mu = 1$ for any value of k: $f'(\mu) > 0$. Hence, the function $f(\mu)$ (and the considered minor together with it) will be negative for $\mu < 1$ and positive for $\mu > 1$. This proves the statement of the theorem. □

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