Memoirs on Differential Equations and Mathematical Physics
Volume 52, 2011, 1-16

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THE SUPPORT THEOREM FOR THE SINGLE RADIUS SPHERICAL MEAN TRANSFORM

This article is dedicated to the 120-th anniversary of N. Muskhelishvili

Abstract. Let $f \in L^p(\mathbb{R}^n)$ and $R > 0$. The transform is considered that integrates the function f over (almost) all spheres of radius R in $\mathbb{R}^n$. This operator is known to be non-injective (as one can see by taking Fourier transform). However, the counterexamples that can be easily constructed using Bessel functions of the 1st kind, only belong to L^p if $p > 2n/(n-1)$. It has been shown previously by S. Thangavelu that for p not exceeding the critical number $2n/(n-1)$, the transform is indeed injective.

A support theorem that strengthens this injectivity result can be deduced from the results of [12], [13]. Namely, if K is a convex bounded domain in $\mathbb{R}^n$, the index p is not above $2n/(n-1)$, and (almost) all the integrals of f over spheres of radius R not intersecting K are equal to zero, then f is supported in the closure of the domain K.

In fact, convexity in this case is too strong a condition, and the result holds for any what we call R-convex domain.

We provide a simplified and self-contained proof of this statement.

2010 Mathematics Subject Classification. 35L05, 92C55, 65R32, 44A12.

Key words and phrases. Spherical mean, Radon transform, support.

რეზიუმე. განვიხილოთ $f \in L^p(\mathbb{R}^n)$ და $R > 0$. განიხილება ინტეგრალური გარდაქმნა, რომელიც გარდასაქმნელ f ფუნქციას აინტეგრებს (თითქმის) ყველა R-რადიუსიან სფეროზე $\mathbb{R}^n$ სივრცეში. ცნობილია, რომ ეს ოპერატორი არაინექციურია (ამაში შეიძლება დავრწმუნდეთ ფურიეს გარდაქმნის მაგალითზე). მიუხედავად ამისა, კონტრმაგალითები, რომელთა აგებაც ადვილია ბესელის I გვარის ფუნქციების გამოყენებით, გარდაქმნა მიეკუთვნება L^p სივრცეს, თუ $p > 2n/(n-1)$. ს. ტანგაველუს მიერ ადრე ნაჩვენები იყო, რომ თუ p პარამეტრი არ აღემატება $2n/(n-1)$ მნიშვნელობას, ინტეგრალური გარდაქმნა მართლაც ინექციურია.

საყრდენი თეორემა, რომელიც აძლიერებს შედეგს ინექციურობის შესახებ, შეიძლება მიღებულ იქნას [12], [13] ნაშრომთა შედეგებზე დაყრდნობით. კერძოდ, თუ K წარმოადგენს ამოზნექილ შემოსაზღვრულ არეს $\mathbb{R}^n$ სივრცეში, ხოლო p ინდექსი არ აღემატება $2n/(n-1)$ მნიშვნელობას, და (თითქმის) ყველა ინტეგრალი f ფუნქციიდან სფეროებზე R რადიუსით, რომლებიც არ იკვეთებიან K სიმრავლესთან უტოლდებიან ნულს, მაშინ f ფუნქციის სუპორტი განლაგებულია K სიმრავლის ჩაკეტვაში.

სინამდვილეში, სიმრავლის ამოზნექილობა ამ შემთხვევაში წარმოადგენს გადამეტებულ მოთხოვნას და შედეგი სამართლიანია არეებისათვის, რომელთაც ჩვენ ვუწოდებთ R-ამოზნექილს.

ჩვენ მოგვყავს აღნიშნული დებულების გამარტივებული და თვითკმარი დამტკიცება.

1. Introduction

We consider the transform acting on functions defined on $\mathbb{R}^n$ by integrating them over all spheres of a fixed radius $R > 0$. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then such integrals exist for almost every center. One can easily construct examples of non-injectivity of this transform at least for some values of p (see the proof of Theorem 1 below for details). However, such constructions, which use Bessel functions of the 1st kind, work only when $p > 2n/(n-1)$. And indeed, it was shown by S. Thangavelu [10] that for $p \leq 2n/(n-1)$, the transform is injective. In this text, we prove a stronger statement (comparable to S. Helgason's "hole" support theorem [4, Theorem 2.6 and Corollary 2.8] for the Radon transform):

Theorem 1. *Let K be the closure of a bounded convex domain in $\mathbb{R}^n$ with $n > 1$, a function $f(x)$ belong to $L^p(\mathbb{R}^n)$ with $p \leq 2n/(n-1)$, and $R > 0$. If the integrals of f over almost all spheres of radius R contained in $\mathbb{R}^n \setminus K$ are equal to zero, then f is compactly supported and its support is contained in K.*

This conclusion does not hold for $p > 2n/(n-1)$.

It is interesting to notice the appearance of the same critical power $2n/(n-1)$ in a similar situation, where however the set of spheres of integration is defined differently: one allows arbitrary radii of the spheres, but restricts the set of their centers to the points of a closed hypersurface $S \subset \mathbb{R}^n$ only. It is shown in [1] that this transform is injective when $p \leq 2n/(n-1)$ and injectivity fails otherwise, for instance when S itself is a sphere.

Convexity is too strong condition in this case. The statement holds for a larger class of domains that is natural for the problem under the consideration.

Definition 2. Let R be a positive number. A bounded closed domain $K \subset \mathbb{R}^n$ is said to be R-convex, if

(1) Its complement $CK := \mathbb{R}^n \setminus K$ is the union of all closed balls $B \in CK$ of radius R.
(2) The set of centers of all such balls is connected.

Theorem 3. *The statement of Theorem 1 holds for R-convex bounded domains K.*

Theorem 1 is proven in the next section. In the following section, an auxiliary local result is established in Theorem 10. In the next section, Theorem 3 is derived from Theorems 1 and 10. The paper ends with the remarks and acknowledgments sections.

A few month after the paper was posted on May 8th 2009, the book [13] appeared, which apparently contains results implying the theorems of this article. Moreover, it has been pointed out to us that these results can be derived from [12, Chapter 3.3, Section 3.2, Corollary 3.3]. It is, however, difficult to reconstruct the proof, which is distributed among various parts of the very technical books [12], [13]. The authors thus think that a streamlined and self-contained proof would be useful to researchers in the area of integral geometry and harmonic analysis.

2. Proof of Theorem 1

We start with the hardest part of the proof, when $p \leq 2n/(n-1)$.

Let $f \in L^p(\mathbb{R}^n)$, $p \in [1, 2n/(n-1)]$ be such that

$$\int\limits_{|\omega|=1} f(y + R\omega)\, d\sigma(\omega) = 0 \tag{1}$$

for almost all $y \in \mathbb{R}^n$ such that dist $(y, K) > R$, where $d\sigma(\omega)$ is the standard surface area measure on the unit sphere in $\mathbb{R}^n$. We need to show that then $f(x) = 0$ for almost all $x \notin K$.

Since K, being a closed bounded convex domain, is the intersection of all balls it is contained within, it is sufficient to prove the statement when K is a ball. Rescaling and shifting, we can assume without loss of generality that K is the unit ball $B(0,1)$ centered at the origin.

Convolving with small support smooth radial functions, one reduces the problem to the case when f is infinitely differentiable and, moreover, all its derivatives belong to the same space L^p as f itself.

Consider for each $m \in \mathbb{Z}^+$ an orthonormal basis Y_l^m, $1 \leq l \leq d(m)$ of the space of all spherical harmonics of degree m in $\mathbb{R}^n$ (the natural representation of the group $O(n)$ in this space is irreducible). Then function f can be expanded into the Fourier series with respect to spherical harmonics as follows:

$$f(x) = \sum_{m,l} f_{m,l}(|x|) Y_l^m(\theta), \tag{2}$$

where $\theta = \frac{x}{|x|}$ and

$$f_{m,l}(|x|) = \int\limits_{\theta \in S} f(|x|\theta) Y_l^m(\theta)\, d\theta. \tag{3}$$

Due to the obvious rotational invariance of the problem, each term $f_{m,l}(|x|)Y_l^m(\theta)$ of the series also has the corresponding spherical integrals (1) vanishing (see a more detailed consideration in Lemma 6 below). Since clearly f_m belongs to the same L^p-space that f does, it is sufficient to prove the statement of the theorem for the functions of the form

$$f(|x|) Y_l^m(\theta) \tag{4}$$

only, where, as before, $\theta = \frac{x}{|x|}$. Hence, we will assume from now on that f is in the form (4).

Let $\delta_R(x)$ be the delta function supported on the sphere of radius R centered at the origin. Then condition (1) can be rewritten as follows:

$$h(x) := (f * \delta_R)(x) = 0 \text{ for } |x| > R + 1, \tag{5}$$

where the star $*$ denotes the n-dimensional convolution. Considering $f(x)$ as a tempered distribution, one can pass to Fourier images in the left hand side of (5) to get

$$\widehat{h}(\xi) = \widehat{f}(\xi)\widehat{\delta_R}(\xi), \quad \xi \in \mathbb{R}^n_\xi. \tag{6}$$

Notice that due to (5), function $h := f * \delta_R$ is compactly supported (with the support in the ball of radius $R+1$) and smooth, and thus the standard Paley–Wiener theorem applies [9]. Therefore, the Fourier transform $\widehat{h}(\xi)$ of h is an entire function satisfying for any $N > 0$ the estimate:

$$|\widehat{h}(\xi)| \leq C_N(1+|\xi|)^{-N} e^{(R+1)|\Im \xi|}. \tag{7}$$

We also recall that $\widehat{\delta_R}(\xi)$ coincides, up to a constant factor, with $j_{(n-2)/2}(R|\xi|)$, where j_p is the so called *normalized* or *spherical* Bessel function [8]:

$$j_p(\lambda) = \frac{2^p \Gamma(p+1) J_p(\lambda)}{\lambda^p}. \tag{8}$$

Here we use the standard notation $J_p(\lambda)$ for Bessel functions of the first kind.

Due to (6), we have

$$\widehat{h}(\xi) = \operatorname{const} j_{(n-2)/2}(R|\xi|)\widehat{f}(\xi). \tag{9}$$

We can now explain the strategy of the proof. The claim we are proving is equivalent to $\widehat{f}(\xi)$ being an entire function of the following Paley–Wiener class:

$$|\widehat{f}(\xi)| \leq C_N(1+|\xi|)^{-N} e^{R|\Im \xi|} \tag{10}$$

(notice the exponent R in (10) instead of $R+1$ present in (7)). Taking into account (9), this task will be achieved, if we could show that:

(1) The distribution $\widehat{f}$ does not have any delta-type terms supported at zeros of $j_{(n-2)/2}(R|\xi|)$, and thus $\widehat{f}$ can be obtained by dividing $\widehat{h}$ by $j_{(n-2)/2}(R|\xi|)$.
(2) This ratio is entire, i.e. $\widehat{h}$ in fact vanishes at zeros of $j_{(n-2)/2}(R|\xi|)$.
(3) The estimate (10) holds, which due to (7) requires one to get an estimate from below for $j_{(n-2)/2}(R|\xi|)$ that would eliminate the unnecessary $+1$ in $R+1$ in (7).

We will deal with these steps in the reverse order. The last one is achieved by the following simple statement:

Lemma 4 (e.g., [3, Lemma 6] or [2, Lemma 4]). *On the entire complex plane, except for a disk S_0 centered at the origin and a countable number of disks S_k of radii $\pi/6$ centered at points $\pi(k + \frac{2\nu+3}{4})$, one has*

$$|J_\nu(z)| \geq \frac{C e^{|Im\, z|}}{\sqrt{|z|}}, \quad C > 0. \tag{11}$$

In order to handle the other two issues, we need to do some preparations.

The following lemma allows one to represent spherical means as volume integrals.

Lemma 5. *Let* $\lambda_0 > 0$ *satisfy*

$$j_{(n-2)/2}(R\lambda_0) = 0.$$

Then the spherical mean $h = \delta_R * f$ *can be represented as*

$$h = \text{const}\,(\Delta + \lambda_0^2)(f * \Psi_R), \tag{12}$$

where

$$\Psi_R(x) = j_{(n-2)/2}(\lambda_0|x|)\chi_R(x)$$

and χ_R *is the characteristic function of the ball of radius* R *centered at the origin.*

Proof. Indeed, this follows easily from Stokes formula. Denoting by $B(x,R)$ and $S(x,R)$ the ball and sphere centered at x and of radius R, one gets

$$\int\limits_{B(x,R)} \Big\{ \big[(\Delta+\lambda_0^2)f\big](v) j_{(n-2)/2}(\lambda_0|x-v|) - f(v)\big[(\Delta+\lambda_0^2)j_{(n-2)/2}\big](\lambda_0|x-v|)\Big\}\, dv =$$

$$= \int\limits_{S(x,R)} \Big\{ f(v)\,\frac{dj_{(n-2)/2}}{dr}\,(R\lambda_0) - \frac{df}{dr}\,(v) j_{(n-2)/2}(R\lambda_0)\Big\}\, dA(v). \tag{13}$$

Here $r = |x|$ and $\partial/\partial r$ is the external normal derivative on the sphere $|x| = t$.

We now take into account that, according to our choice of λ_0, the Bessel function $j_{(n-2)/2}(\lambda_0 u)$ satisfies the following two equalities:

$$j_{(n-2)/2}(R\lambda_0) = 0$$

and

$$(\Delta + \lambda_0^2) j_{(n-2)/2}(\lambda_0|y|) = 0.$$

Also, due to the simplicity of zeros of $j_{(n-2)/2}$,

$$j'_{(n-2)/2}(R\lambda_0) \neq 0.$$

These features, combined with (13), prove the statement of the lemma. □

Lemma 6. *Let*

$$f(x) = \sum_{l=1}^{d(m)} f_l(r) Y_l^m(\theta), \quad x = r\theta, \quad |\theta| = 1.$$

Then for any radial compactly supported continuous function ψ *the convolution* $F = \psi * f$ *has the similar representation*

$$F(x) = \sum_{l=1}^{d(m)} F_l(r) Y_l^m(\theta).$$

Proof. The convolution operator $f \to \psi * f$ is rotationally invariant. Indeed:

$$(\psi * f)(x) = \int f(y)\psi(|x-y|)\, dy,$$

and thus for any rotation T and the rotated function $f_T(x) = f(Tx)$ one has:

$$(\psi * f_T)(x) = \int f(Ty)\psi(|x-y|)\, dy = \int f(Ty)\psi(|Tx-Ty|)\, dy = \\ = \int f(y)\psi(|Tx-y|)\, dy = (\psi * f)_T(x).$$

This implies that the convolution preserves the subspaces of harmonics of a fixed degree, which proves the lemma. □

According to our strategy, the next step is to prove that $\widehat{f}$ is an entire function.

Due to (9), outside of zeros of $j_{(n-2)/2}(R|\xi|)$, one has

$$\widehat{f}(\xi) = \mathrm{const}\, \frac{\widehat{h}(\xi)}{j_{(n-2)/2}(R|\xi|)}\,. \tag{14}$$

Notice that the denominator is an entire function of the variable $\xi \in \mathbb{C}^n$, since $j_\nu(u)$ is an even entire function of the real argument u and hence is an entire function of u^2. The next lemma shows that the numerator in (14) vanishes at the (simple) zeros of the denominator. Therefore, the zeros cancel, and the ratio in the right hand side of (14) is an entire function, as needed.

Lemma 7. *For any λ_0 such that $j_{(n-2)/2}(R\lambda_0) = 0$, function $\widehat{h}(\xi)$ vanishes on the complex quadric*

$$Q = \left\{\xi \in \mathbb{C}^n : \ \xi_1^2 + \cdots + \xi_n^2 = \lambda_0^2\right\}.$$

Proof. Since $\lambda_0 \neq 0$, the quadric Q is irreducible and has a maximal dimension intersection with the real subspace. Thus, due to analytic continuation, it suffices to check vanishing of the entire function $\widehat{h}(\xi)$ on the intersection $Q \bigcap \mathbb{R}^n$, i.e. on the sphere $|\xi| = \lambda_0$ in $\mathbb{R}^n$.

Since, by assumption, h vanishes outside of the unit ball, we can write

$$\widehat{h}(\xi) = \int\limits_{|x|\le t} h(x) e^{-i\xi\cdot x}\, dx,$$

for arbitrary $t > 1$.

Let us substitute for h the representation (12). Then, by Stokes' formula,

$$\widehat{h}(\xi) = \text{const} \int\limits_{|x|\le t} (\Delta+\lambda_0^2)(f*\Psi_R)e^{-i\xi\cdot x}\,dx =$$
$$= \text{const} \int\limits_{|x|\le t} (f*\Psi_R)(\Delta+\lambda_0^2)e^{-i\xi\cdot x}\,dx +$$
$$+ \text{const} \int\limits_{|x|=t} \Big(\frac{\partial}{\partial r}(f*\Psi_R)e^{-i\xi\cdot x} - (f*\Psi_R)\frac{\partial}{\partial r}e^{-i\xi\cdot x}\Big)\,dA(x). \quad (15)$$

Since $|\xi| = \lambda_0$, the exponential function $e^{-i\xi\cdot x}$ is annihilated by the operator $\Delta+\lambda_0^2$. Therefore, $\widehat{h}(\xi)$ is expressed by the surface term alone:

$$\widehat{h}(\xi) = \text{const} \int\limits_{|x|=t} \Big(\frac{\partial}{\partial r}(f*\Psi_R)e^{-i\xi\cdot x} - (f*\Psi_R)\frac{\partial}{\partial r}e^{-i\xi\cdot x}\Big)\,dA(x).$$

Here, as before, $r = |x|$ and $\frac{\partial}{\partial r}$ is the external normal derivative on the sphere $|x| = t$.

The function Ψ_R is radial and thus, due to Lemma 6, the convolution $F := f*\Psi_R$ has the form $F(x) = \sum\limits_{l=1}^{d(m)} F_l(r)Y_l(\theta)$. Projection of the exponential function $e^{-i\xi\cdot x}$ on the space of spherical harmonics of degree m can be given in terms of Bessel functions (see [9, Theorem 3.10]), which leads to the following formula:

$$\widehat{h}(\xi) = c_m\lambda_0^m t^{n+m-1}\sum_{l=1}^{d(m)}\Big(F_l'(t)j_{n/2+m-1}(\lambda_0 t) - F_l(t)j'_{n/2+m-1}(\lambda_0 t)\Big). \quad (16)$$

In what follows, the estimate is done the same way for any l between 1 and $d(m)$, so we will drop the sum over l and work with a single term.

In order to prove that $\widehat{h}(\xi) = 0$, it suffices to check that the expression in the right hand side tends to 0 as $t\to\infty$. This can now be easily shown using the L^p condition on F and the known estimate for Bessel functions:

$$j_{n/2+m-1}(t), j'_{n/2+m-1}(t) = O(t^{-\frac{n+2m-1}{2}}), \quad t\to\infty. \quad (17)$$

Indeed, let us pick $t_0 > t$ and average both sides of (16) for t from t_0 to $2t_0$:

$$\widehat{h}(\xi) = c_m\frac{1}{t_0}\int\limits_{t_0}^{2t_0}\Big[F_l'(t)j_{n/2+l-1}(\lambda_0 t) - F_l(t)j'_{n/2+l-1}(\lambda_0 t)\Big]t^{n+m-1}\,dt. \quad (18)$$

Let $A(t) := |F_l'(t)| + |F_l(t)|$. From (17) and (18) one obtains:

$$|\widehat{h}(\xi)| \le \frac{c_m}{t_0}\int\limits_{t_0}^{2t_0} A(t)t^{\frac{n-1}{2}}\,dt = \frac{c_m}{t_0}\int\limits_{t_0}^{2t_0} A(t)t^{\frac{n-1}{p}}t^{(n-1)\frac{p-2}{2p}}\,dt. \quad (19)$$

Functions $F_l(r)$ and $F_l'(r)$ are the radial parts of functions in $L^p(\mathbb{R}^n)$ and therefore belong to $L^p((0,\infty), r^{n-1}dr)$. So is the function $A(r)$. We now apply Hölder inequality to (19) to get

$$|\widehat{h}(\xi)| \leq \frac{c_m}{t_0} \left(\int_{t_0}^{2t_0} A^p(t) t^{n-1}\, dt \right)^{\frac{1}{p}} \left(\int_{t_0}^{2t_0} t^{(\frac{n-1}{2} - \frac{n-1}{p})q}\, dt \right)^{\frac{1}{q}}, \tag{20}$$

where the index q dual to p is introduced in the standard manner: $p^{-1} + q^{-1} = 1$, or $q = p/(p-1)$. The second factor in (20) can be easily computed:

$$\left(\int_{t_0}^{2t_0} t^{(\frac{n-1}{2} - \frac{n-1}{p})q}\, dt \right)^{\frac{1}{q}} = C t_0^{\frac{n-1}{2} - \frac{n}{p} + 1},$$

and hence (20) leads to the estimate:

$$|\widehat{h}(\xi)| \leq c_m \|A\|_{L^p((t_0, 2t_0), t^{n-1}dt)} t_0^{\frac{n-1}{2} - \frac{n}{p}}. \tag{21}$$

The condition $p \leq 2n/(n-1)$ shows that $(n-1)/2 - n/p \leq 0$, and hence the last factor in (21) is bounded. Since the condition that $F \in L^p(\mathbb{R}^n)$ implies

$$\|A\|_{L^p((t_0, 2t_0), t^{n-1}dt)} \to 0 \text{ when } t_0 \to \infty,$$

this shows the required equality $\widehat{h}(\xi) = 0$. □

Corollary 8. *The function*

$$\Phi(\xi) := \frac{\widehat{h}(\xi)}{j_{(n-2)/2}(R|\xi|)}$$

is entire of the Paley–Wiener class (10).

The only remaining step is to show that the same statement as in Corollary 8 applies to the function $\widehat{f}(\xi)$:

Lemma 9. *The Fourier transform $\widehat{f}(\xi)$ is an entire function of the Paley–Wiener class* (10)*.*

Proof. Corollary 8 says that the right hand side in (14) is an entire function of the Paley–Wiener class (10). The Lemma (and thus the Theorem 1) will be proven if we show that in fact $\widehat{f} = \Phi$.

The tempered distribution $\widehat{f}(\xi)$, $\xi \in \mathbb{R}^n$ coincides with $\Phi(\xi)$ outside of the union of the discrete set of spheres S_k defined by (simple) zeros of Bessel function:

$$S_k = \left\{ \xi \in \mathbb{R}^n : \ \xi_1^2 + \cdots + \xi_n^2 = \lambda_k^2 \right\},$$

where

$$j_{(n-2)/2}(\lambda_k R) = 0.$$

This means that $\widehat{f}$ can differ from Φ only by terms supported on these spheres:

$$\widehat{f}(\xi) = \Phi(\xi) + \sum_k c_k(\xi) \delta(|\xi| - |\lambda_k|).$$

Although in principle higher order distibituions concentrated on the spheres could have arised, the equality (9), together with the simplicity of zeros of the Bessel function involved, shows that these higher order terms are not present.

We now observe that since

$$f(x) = \sum_{l=1}^{d(m)} f_l(r) Y_l^m(\theta),$$

the coefficients $c_k(\xi)$ must have the similar form

$$c_k(\xi) = \sum_{l=1}^{d(m)} a_{k,l} Y_l^m(\eta), \quad a_k = \text{const}\,, \quad \xi = |\xi|\eta, \quad |\eta| = 1.$$

Our aim is to show that there are no such distributional terms in $\widehat{f}$, i.e. all coefficients $a_{k,l}$ must vanish.

Fix k and choose a positive number ε so small that the spherical layer

$$L := \left\{\lambda_k - \varepsilon \le |\xi| \le \lambda_k + \varepsilon\right\}$$

containing S_k, does not contain other spheres S_m with $m \neq k$.

Let now ψ be a radial function from the Schwartz class, whose Fourier transform vanishes outside the spherical layer L and such that $\widehat{\psi}(\xi) = 1$, $\xi \in S_k$. We can now localize the sphere S_k in the spectrum of f by considering the convolution $g = \psi * f$. By construction,

$$\widehat{g}(\xi) = \Phi(\xi)\widehat{\varphi}(\xi) + c_k Y_l(\eta)(\delta(|\xi| - \lambda_k).$$

The first term is in the Schwartz class, while the second one is, up to a constant factor, Fourier transform of Bessel function $j_{n/2+l-1}(|x|)$ and therefore after convolving with ψ we have

$$g(x) = \psi * \varphi + \text{const}\, a_k j_{n/2+l-1}(|x|),$$

where ψ is inverse Fourier transform of Ψ and hence is also a Schwartz function. By the condition for f and by the construction, the functions g and $\psi * \varphi$ belong to $L^p(\mathbb{R}^n)$ with $p < 2n/n - 1$, while Bessel function $j_{n/2+l-1}$ is not in this class. Therefore the coefficient a_k must be equal to zero.

Thus, there is no δ-function terms in $\widehat{f}$ and $\widehat{f} = \Psi$ is an entire function in $\mathbb{C}^n$ satisfying, as it was explained above, the Paley–Wiener estimate that implies that $supp f \subset \overline{B}(0, R)$.

Let now $p > 2n/(n-1)$. Then one can find a counterexample, where even compactness of support of f cannot be guaranteed, using Bessel functions. The function

$$f(x) = |x|^{1-n/2} J_{n/2-1}(\lambda|x|) \tag{22}$$

provides such a counterexample (due to L. Zalcman). Indeed, consider the following spherical mean mapping M:

$$Mg(x,t) = \frac{1}{\omega_n} \int\limits_{S(0,1)} g(x + t\theta)\, d\theta,$$

which averages any continuous function g over spheres. It is well known that f defined in 22 satisfies the following functional identity:

$$Mf(x,t) = \text{const}\, f(x) f(t). \tag{23}$$

Thus, if λ is chosen as a zero of $J_{n/2-1}$, the relation (23) implies that the spherical means of $f(x)$ over all spheres of radius 1 are equal to zero. Also, the known asymptotic behavior of Bessel functions shows that $f \in L^q(\mathbb{R}^n)$ for any $q > 2n/(n-1)$. This completes the proof of Theorem 1. $\square$

3. A Local Result

In order to extend the statement of Theorem 1 to all R-convex domains, we need to establish first the following local theorem, which in some particular cases as well as in different related versions has been established previously [6], [12].

Theorem 10. *Let $f(x)$ be an infinitely differentiable function in the ball $B(0, R+\varepsilon) \subset \mathbb{R}^n$ and its spherical averages over all spheres of radius R contained in this ball are equal to zero. If f vanishes in the ball $B(0,R)$, then it vanishes in the whole ball $B(0, R+\varepsilon)$.*

Proof. Without loss of generality, we can assume that $R = 1$. As in [6], [12], we will exploit relations between spherical and plane waves [6, Ch. 1 and 4].

For a function $u(x)$ on $\mathbb{R}^n$ we will denote by $u^\#(x)$ its *radialization*

$$u^\#(x) := \int\limits_{k \in O(n)} u(kx)\, dk,$$

where dk is the normalized Haar measure on $O(n)$. Function $u^\#(x)$ is clearly radial and thus is a function of a single variable $|x|$. Abusing notations, we will write $u^\#(x) = u^\#(|x|)$.

The following simple statement (which we will prove for completeness) will be useful:

Lemma 11. *Let $u(x)$, $v(x)$ be continuous functions on $\mathbb{R}^n$ and $v(x)$ be radial and compactly supported. Then*

$$(u * v)^\# = u^\# * v.$$

Proof. Indeed,

$$(u * v)^\#(x) = \int\limits_{O(n)} \int\limits_{\mathbb{R}^n} u(kx - y) v(y)\, dy\, dk.$$

Changing the variables in the y-integral from y to ky, using the rotational invariance of v, and changing the order of integration, one gets

$$(u*v)^{\#}(x) = \int\limits_{\mathbb{R}^n} \Bigg(\int\limits_{O(n)} u(kx-ky)\,dk \Bigg) v(y)\,dy = (u^{\#} * v)(x).$$

This proves the lemma. □

In particular, the convolution of two radial functions is radial.

The relation between plane waves and radial functions that we need is contained in the following result of [6, Ch.4, formulas (4.13) and (4.16)]:

Lemma 12 ([6]). *Let $e \in \mathbb{R}^n$ and $g(p)$ be a function of a scalar variable $p \in \mathbb{R}$. We consider the ridge function $g(\langle x, e\rangle)$ and its radialization $g(\langle \cdot, e\rangle)^{\#}$, which we will identify with a function $f(r)$ of scalar variable r. Then the relations between the functions $f(r)$ and $g(p)$ are provided by the following Abel type transforms:*

$$f(r) = (\mathcal{A}g)(r) := 2\frac{\omega_{n-1}}{\omega_n} r^{2-n} \int\limits_0^r (r^2 - s^2)^{\frac{n-3}{2}} g(p)\,dp \tag{24}$$

and

$$g(p) = (\mathcal{A}^{-1}f)(p) := \frac{2^{n-1}p}{(n-2)!} \Big(\frac{d}{dp^2}\Big)^{n-1} \int\limits_0^p r^{n-1}(p^2 - r^2)^{\frac{n-3}{2}} f(r)\,dr. \tag{25}$$

We can now derive the following useful relation:

Lemma 13. *Let δ_S denote the normalized measure supported by the unit sphere. Let also $g(p)$ be a continuous function on $\mathbb{R}$. Then*

$$(\mathcal{A}g * \delta_S)(p) = \mathrm{const}\, \mathcal{A}\big(g *_1 (1-|p|^2)_+^{\frac{n-3}{2}}\big), \tag{26}$$

where $*_1$ denotes one-dimensional convolution and $\mathcal{A}g$ in the left hand side is considered as a radial function on $\mathbb{R}^n$, i.e. $\mathcal{A}g(|x|)$ for $x \in \mathbb{R}^n$.

Proof. Since $\mathcal{A}g = g(\langle\cdot, e\rangle)^{\#}$, the left hand side, according to Lemma 11 can be rewritten as

$$\big(g(\langle\cdot, e\rangle) * \delta_S\big)^{\#}.$$

It is straightforward to check that

$$\big(g(\langle\cdot, e\rangle) * \delta_S\big)(x)$$

is equal to the ridge function

$$\big(g *_1 (1-|p|^2)_+^{\frac{n-3}{2}}\big)\Big|_{p=\langle x, e\rangle}.$$

Now radialization of this ridge function gives the right hand side expression in (26). □

We can complete now the proof of our theorem. We start with the case of a radial function, which we write as $f(|x|)$ for some function $f(r)$ of a single variable. By the assumption, $(f * \delta_S)(x) = 0$ for $|x| < \varepsilon$. Then (26) implies that

$$\big(g *_{\mathbb{R}^1} (1 - |p|^2)^{\frac{n-3}{2}}\big)(s) = 0 \tag{27}$$

for $s \leq \varepsilon$, where $g(p) := (\mathcal{A}^{-1} f)(p)$. It follows from (25) that the condition $f(x) = 0$ for $|x| \leq 1$ implies $g(p) = 0$ for $|p| \leq 1$, and therefore (27) can be rewritten as

$$\int\limits_{1}^{1+\varepsilon} g(p)(1 - |p - s|^2)_+^{\frac{n-3}{2}}\, dp = 0, \quad s \leq \varepsilon.$$

Thus the Titchmarsh theorem [11] (see also [5, Theorem 4.3.3], [7, Lecture 16], or [14, Ch. VI]) implies that $g(p) = 0$ for $1 \leq p \leq 1 + \varepsilon$. Since $f = \mathcal{A}g$, the relation (24) leads to the conclusion that $f(x) = 0$ for $|x| \leq 1 + \varepsilon$. This proves the statement of the theorem in the radial case.

It remains now to pass from radial to non-radial functions. To this end, we observe that the C^∞ function f has zero integrals over all spheres of radius 1 centered in the open ball $B(0, \varepsilon)$. Thus, all its partial derivatives $D^\alpha f$ have the same property. Since this vanishing condition is invariant under rotations, according to Lemma 11, it also holds for radializations $(D^\alpha f)^\#$. Since the theorem is already proven for radial functions, all these radializations vanish, i.e.

$$\int\limits_{|x|=r} D^\alpha f(x)\, dA(x) = 0 \tag{28}$$

for all $0 < r < 1 + \varepsilon$.

Let us prove now that on each sphere $|x| = t$ for $t \in [0, 1+\varepsilon)$ the function f, along with all its derivatives, is orthogonal to all monomials. This, due to the Weierstrass Theorem will imply the needed property that $f = 0$ in $B(0, 1 + \varepsilon)$.

We prove this claim by induction with respect to the degree of the monomial. For a zero degree monomial, the claim is true, due to (28). Suppose that

$$\int\limits_{|x|=t} p(x) D^\alpha f(x)\, dA(x) = 0, \tag{29}$$

for all monomials $p(x)$ of degree not exceeding N and all multiindices α. Integrating both sides of this identity with respect to t from 0 to any $r < 1+\varepsilon$ yields

$$\int\limits_{|x|\leq r} p(x) D^\alpha f(x)\, dx = 0.$$

We now replace the multiindex α with $\beta = \alpha + \delta_j$, where the multiindex δ_j has 1 in jth place and 0s otherwise and write

$$p(x)D^{\beta} f(x) = \frac{\partial}{\partial x_j}\left(p(x)D^{\alpha} f(x)\right) - \frac{\partial p}{\partial x_j} D^{\alpha} f(x). \tag{30}$$

The second term on the right does not contribute to the integral over the ball $|x| \le r$, due to the induction assumption, and thus identity (29), where α is replaced by β, reduces to

$$\int\limits_{|x|\le r} \frac{\partial}{\partial x_j}\left(p(x)D^{\alpha} f(x)\right) dx = 0.$$

Using Stokes' formula, we obtain

$$\int\limits_{|x|=r} x_j p(x) D^{\alpha} f(x)\, dA(x) = 0.$$

Since $j = 1, \ldots, n$ is arbitrary, we conclude that identity (29) holds for all monomials of degree $N + 1$. This completes the proof of theorem. $\square$

4. Proof of Theorem 3

We can now prove Theorem 3 that extends Theorem 1 to the case of R-convex domains. So, we assume that $K \subset \mathbb{R}^n$ is a closed R-convex domain and a function $f \in L^p(\mathbb{R}^n)$ with $p \le 2n/(n-1)$ is such that its spherical means over almost every sphere of radius R not intersecting K is zero. As it has been shown before, one can assume, without restriction of generality, that the function is smooth. Consider the set C of centers of all balls of radius K not intersecting K. Due to R-convexity of K, this set is connected, and the union of the corresponding balls covers the whole complement of K. Consider also the subset $C_f \subset C$ of such centers x that f vanishes in the ball $B(x, R)$. If we establish that in fact $C_f = C$, this will prove the theorem.

Theorem 1 implies that $f = 0$ outside the convex hull of K. Thus, in particular, the set C_f is non-empty, since it contains all points x with a sufficiently large norm. It is also obvious that, due to continuity of f, the set C_f is relatively closed in C. Let us now prove that it is also relatively open. Due to connectedness of C, this will imply that $C_f = C$ and thus $f = 0$ in the whole complement of K, which is the statement of the theorem.

Indeed, let $x \in C_f$. This means that $f = 0$ in $B(x, R)$. There exists a positive ε such that the ball $B(x, R+\varepsilon)$ is inside the complement of K. Then the function f satisfies the conditions of Theorem 10 in $B(x, R+\varepsilon)$, and thus $f = 0$ in $B(x, R+\varepsilon)$. In particular, f vanishes in the ball $B(y, R)$ for any $y \in \mathbb{R}^n$ such that $|y - x| < \varepsilon$. This means that all such points y belong to C_f, and hence C_f is open. This finishes the proof of the theorem. $\square$

5. Remarks

(1) The statement of Theorem 10 holds also for functions of finite smoothness, if one knows that spherical averages of f vanish for all spheres of radius $r < R$ (rather than $r = R$ as in Theorem 10).

(2) The local Theorem 10, has been established previously in some particular cases, as well as in different related versions in [6], [12]. For instance, one can check that the consideration in the second section of [6, Ch. VI] provides such a result in $3D$, although the local formulation is not stated there. In [12], a theorem similar to Theorem 10 is proven for the case of integrals over balls.

Acknowledgments

The work of the first author was performed when he was visiting Texas A&M University and was partially supported by the ISF (Israel Science Foundation) grant 688/08. The second author was partially supported by the NSF grants DMS 0604778 and 0908208 and by the IAMCS. The authors express their gratitude to ISF, NSF, Texas A&M University, and IAMCS for the support. We are also grateful to the reviewer of the first version of the text for pointing to us the reference to [13] and to Y. Lyubarskii for useful comments and references.

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(Received 12.10.2010)

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