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SOLUTION OF THE KIRSCH PROBLEM IN VIEW OF SURFACE STRESSES

The work is dedicated to the 120th birthday anniversary of N. I. Muskhelishvili, one of the greatest mechanicians and mathematicians of the 20th century

Abstract. The Kirsch problem on the tension of an elastic plane with a circular hole free from external traction is considered. It is assumed that complementary surface stresses are applied at the boundary. Based on Kolosov–Muskhelishvili's method, the solution of the problem is reduced to the solution of a singular integro-differential equation for an unknown surface stress. A solution to the obtained equation is derived in an explicit form and shows that stress concentration at the boundary depends on the elastic properties of a surface and bulk material, and the radius of a hole as well if surface stresses are taken into account.

The paper is an example of the modern applications of Muskhelisvili's outstanding achievements to the problems of the nanomechanics.

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Key words and phrases. Kirsch problem, surface stress, singular integro-differential equation, stress concentration.

რეზიუმე. განხილულია კირშის ამოცანა დრეკადი ფირფიტისათვის მრგვალი ხვრელით, რომელიც თავისუფალია გარეშე ძალების ზემოქმედებისაგან. დაშვებულია, რომ საზღვარზე მოდებულია დამატებითი ზედაპირული ძაბვები. კოლოსოვ–მუსხელიშვილის მეთოდის გამოყენებით დასმული ამოცანა მიყვანილია სინგულარული ინტეგრო-დიფერენციალური განტოლების ამოხსნამდე უცნობი ზედაპირულ დაძაბულობების მიმართ. მიღებული განტოლების ამოხსნა ჩაწერილია ცხადი სახით, რაც საშუალებას გვაძლევს დავინახოთ, რომ ძაბვების კონცენტრაცია საზღვარზე დამოკიდებულია ფირფიტის ელასტიურ თვისებებზე და მასალაზე, აგრეთვე ხვრელის რადიუსზე, თუ გავითვალისწინებთ ზედაპირულ ძაბვებს.

სტატია წარმოადგენს ნ. მუსხელიშვილის გამორჩეული მეცნიერული მიღწევების გამოყენების ნიმუშს ნანომექანიკის თანამედროვე პრობლემებში.

It is known, that taking into account surface stresses in rigid bodies [1]–[4] might be most important for nanoobjects. Here unexpected effects, not correspond to our traditional representations, may turn out [5], [6].

From these positions the classical Kirsch problem concerning the tension of an elastic plane weakened by a circular hole will be considered. Assume that complementary surface stresses occur along the boundary of the circular hole [1]–[4]. The problem will be treated with the help of the Kolosov–Muskhelishvili method [7].

According to the Laplace–Young law [1], [4], the boundary conditions in the absence of external stresses on the circular boundary are given as follows

$$\sigma_{rr} + \frac{\sigma^s_{\theta\theta}}{r} = 0, \quad \sigma_{r\theta} - \frac{1}{r}\frac{\partial \sigma^s_{\theta\theta}}{\partial\theta} = 0. \tag{1}$$

Here $\sigma^s_{\theta\theta}$ is the surface stress, r, θ are the polar coordinates with the center coinciding with that of the circular hole.

First, we construct a solution for the hole of unit radius and, therefore, introduce $r = 1$ in equation (1).

Suppose that the conditions of uniaxial tension along the x_1-axis at infinity are imposed, i.e.,

$$\sigma^\infty_{11} = \sigma, \quad \sigma^\infty_{22} = \sigma^\infty_{12} = \omega^\infty = 0, \tag{2}$$

where ω is a turning angle of the material particle.

In the complex writing the conditions (1) for $r = 1$ take the form

$$\sigma_{rr} + i\sigma_{r\theta} = -\sigma^s_{\theta\theta} + i\,\frac{\partial \sigma^s_{\theta\theta}}{\partial\theta} \equiv t^s, \tag{3}$$

where i is the imaginary unit.

To solve the problem, we will apply the Kolosov–Muskhelishvili formulas [7] which express stresses in the plane σ_{jk} and displacements u_j $(j, k = 1, 2)$ in the Cartesian coordinates x_1, x_2 in terms of complex functions Φ, Ψ holomorphic for $r = \sqrt{x_1^2 + x_2^2} > 1$:

$$\begin{gathered}\sigma_{11} + \sigma_{22} = 4Re\Phi(z),\\ \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left(\bar{z}\Phi'(z) + \Psi(z)\right),\end{gathered} \tag{4}$$

$$2\mu(u_1 + iu_2) = \varkappa\int \Phi(z)dz - z\overline{\Phi(z)} - \int \overline{\Psi(z)}\,d\bar{z}. \tag{5}$$

Here $z = x_1 + ix_2$, $\varkappa = 3 - 4\nu$ for the plane strain, $\varkappa = (3 - \nu)/(1 + \nu)$ for the plane stress, ν and μ are, respectively, the Poisson ratio and the shear modulus of the elastic medium. A quantity with the bar denotes complex conjugation and the prime denotes the derivative with respect to the argument.

We will introduce a local orthogonal system of coordinates n, t, rotated with respect to the system x_1, x_2 by the angle $\alpha - \pi/2$. Then from formulas (4), (5) we derive the joint expression for the traction $\sigma_n = \sigma_{nn} + i\sigma_{nt}$ on an element of area with the normal vector $\mathbf{n}$ and the displacement vector

$u = u_1 + iu_2$ [8]

$$G(z,\bar z) = \eta\Phi(z) + \overline{\Phi(z)} + \frac{d\bar z}{dz}\big(z\overline{\Phi'(z)} + \overline{\Psi(z)}\big), \tag{6}$$

where $G = \sigma_n$ for $\eta = 1$ and $G = -2\mu du/dz$ for $\eta = -\varkappa$. The increment dz is taken in the direction of the axis t. i.e., along the chosen area element. Thus in (6), $dz = |dz|e^{i\alpha}$, $d\bar z = \overline{dz}$.

Following N. I. Muskhelishvili's method [7], we introduce the function $\Upsilon(z)$, holomorphic in the circle $|z| < 1$, except the point $z = 0$, where it might have a pole up to the second order, inclusive:

$$\Upsilon(z) = -\overline{\Phi(\bar z^{-1})} + z^{-1}\overline{\Phi'(\bar z^{-1})} + z^{-2}\overline{\Psi(\bar z^{-1})}. \tag{7}$$

Using equality (7) from (6) we derive the following

$$G(z,\bar z) = \eta\Phi(z) + \overline{\Phi(z)} + \frac{d\bar z}{dz}\left[\frac{1}{\bar z^2}\left(\overline{\Phi(z)} + \Upsilon\Big(\frac{1}{\bar z}\Big)\right) + \Big(z - \frac{1}{\bar z}\Big)\overline{\Phi'(z)}\right], \tag{8}$$
$$|z| > 1.$$

We take the limit $z \to \zeta = e^{i\theta}$ in equation (8) and direct the vector $\mathbf{n}$ towards the center $z = 0$. Since in this case $\alpha = \theta + 3\pi/2$ and $dz = -i|dz|e^{i\theta}$, by virtue of conditions (3) from (8) we derive that

$$\Phi(\zeta) - \Upsilon(\zeta) = t^s(\zeta). \tag{9}$$

Here $\Phi(\zeta)$, $\Upsilon(\zeta)$ are the limiting values of the corresponding functions on the circumference of unit radius γ.

Introducing the function $W(z)$, holomorphic in the complex plane except the circumference γ,

$$W(z) = \begin{cases} \Phi(z), & |z| > 1 \\ \Upsilon(z), & |z| < 1 \end{cases}. \tag{10}$$

we reduce equation (9) to the following Hilbert problem

$$W^+(\zeta) - W^-(\zeta) = -t^s(\zeta), \quad |\zeta| = 1. \tag{11}$$

Taking into account the existence of the pole, a solution to the problem (11) is written in the form (cf. [7])

$$W(z) = -I(z) + S(z) + D_1, \tag{12}$$

where

$$I(z) = \frac{1}{2\pi i}\int\limits_\gamma \frac{t^s(\eta)}{\eta - z}\,d\eta, \quad S(z) = \frac{c_1}{z} + \frac{c_2}{z^2} \tag{13}$$

and

$$D_1 = \lim_{z\to\infty}\Phi(z) = \sigma/4.$$

Since the principal vector of forces applied to the boundary of the hole equals zero we have $c_1 = 0$, $c_2 = -\sigma/2$.

For the problem under consideration the constitutive relation, connecting the surface stress and the corresponding strain, takes the form (cf. [4])

$$\sigma^s_{\theta\theta} = (2\mu_s + \lambda_s)\varepsilon^s_{\theta\theta}, \quad z = \zeta, \tag{14}$$

where λ_s, μ_s are the modules of the surface elasticity, similar to the Lamé constants of the bulk material.

We impose the continuity constraint on the displacements vector under passing from the volume to the boundary

$$\lim_{\substack{|z|>1 \\ z\to\zeta}} u(z) = u^s(\zeta), \qquad \zeta \in \gamma, \tag{15}$$

where $u^s(\zeta)$ is the displacement vector of the boundary point $\zeta \in \gamma$. From (15) follows the same for the volume deformations $\varepsilon_{\theta\theta}$ and the deformation on the boundary $\varepsilon^s_{\theta\theta}$, i.e.,

$$\lim_{\substack{|z|>1 \\ z\to\zeta}} \varepsilon_{\theta\theta}(z) = \varepsilon^s_{\theta\theta}(\zeta), \qquad \zeta \in \gamma. \tag{16}$$

The relations (14)–(16) result in the equation for the surface stress

$$\sigma^s_{\theta\theta} = (2\mu_s + \lambda_s)\varepsilon_{\theta\theta}, \quad z = \zeta. \tag{17}$$

The expression for the deformation $\varepsilon_{\theta\theta}$ is derived by using the relation (8). Putting in (8) successively $dz = dx_1$ and $dz = idx_2$ for $\eta = -\varkappa$, and $z = \zeta$, after some transformations we find expressions for the deformations ε_{jk} in (x_1, x_2)–system of coordinates. After passing to the polar coordinates r, θ, we obtain

$$2\mu\varepsilon_{\theta\theta} = \operatorname{Re}\big[\varkappa\Phi(\zeta) + \Upsilon(\zeta)\big]. \tag{18}$$

Introducing (18) into (17) and taking into account (10) and (11), we arrive at the following equation:

$$\sigma^s_{\theta\theta} = -M\operatorname{Re}\big[\varkappa I^-(\zeta) + I^+(\zeta)\big] + \frac{M(\varkappa+1)\sigma}{4}\big(1 - \zeta^2 - \zeta^{-2}\big), \tag{19}$$

where $M = \frac{2\mu_s+\lambda_s}{2\mu}$.

Let $\tau = \sigma^s_{\theta\theta}$. Since $\partial\tau/\partial\theta = i\zeta\partial\tau/\partial\zeta = i\zeta\tau'(\zeta)$, the Sokhotskii–Plemelj formulas for the Cauchy type integral $I(z)$ acquire the form

$$I^{\pm}(\zeta) = \mp\frac{\tau(\zeta)}{2} \mp \frac{\zeta\tau'(\zeta)}{2} - \frac{1}{2\pi i}\int_\gamma \frac{\tau(\eta) + \eta\tau'(\eta)}{\eta - \zeta}\,\eta, \tag{20}$$

where the integral is understood in the sense of the Cauchy principal value.

Introducing $I^{\pm}(\zeta)$ from (20) into (19), we obtain the following integral equation

$$\tau(\zeta) - \frac{M(\varkappa+1)}{M(\varkappa-1)+2}\times \\ \times\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\tau(\eta)+\eta\tau'(\eta)}{\eta-\zeta}\,d\eta - \frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\tau(\eta)}+\overline{\eta\tau'(\eta)}}{\bar\eta-\bar\zeta}\,d\bar\eta\right] = \\ = \frac{M(\varkappa+1)\sigma}{2M(\varkappa-1)+4}\left(1-\zeta^2-\zeta^{-2}\right). \quad (21)$$

Taking into account the relations $\bar\eta = \eta^{-1}$, $\bar\zeta = \zeta^{-1}$, $\overline{\tau(\eta)} = \tau(\eta)$, $\overline{\eta\tau'(\eta)} = -\eta\tau'(\eta)$, $d\bar\eta = -\eta^{-2}d\eta$, equation (21) for the hole of radius r transforms into the following singular integro-differential equation

$$\tau(\zeta) - \frac{M(\varkappa+1)}{M(\varkappa-1)+2r}\times \\ \times\left[\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\tau(\eta)+\eta\tau'(\eta)}{\eta-\zeta}\,d\eta - \frac{\zeta}{2\pi i}\int\limits_{\gamma}\frac{\eta^{-1}\tau(\eta)-\tau'(\eta)}{\eta-\zeta}\,d\eta\right] = \\ = \frac{Mr(\varkappa+1)\sigma}{2M(\varkappa-1)+4r}\left(1-\zeta^2-\zeta^{-2}\right). \quad (22)$$

In (22) we denote $\eta = \eta_1/r$, $\zeta = \zeta_1/r$, where η_1, ζ_1 are points on the circumference of radius r.

From physical considerations for $\sigma = 0$ the surface stress $\sigma^s_{\theta\theta}$ is absent. This implies that the homogeneous equation corresponding to the integral equation (21), or (22), has only the trivial solution $\tau = 0$.

A particular solution to equation (22) is sought in the form of infinite sum

$$\tau = \sum_{k=-\infty}^{+\infty} d_k\zeta^k. \quad (23)$$

Introducing (23) into (22), after integration and reduction of similar terms, we get

$$d_0 = \frac{Mr(\varkappa+1)}{4(r-M)}\sigma, \quad d_2 = d_{-2} = -\frac{Mr(\varkappa+1)}{2[2r-M(\varkappa+3)]}\,\sigma, \quad d_k = 0, \quad k \neq 0,-2,2. \quad (24)$$

Find now the hoop stresses $\sigma_{\theta\theta}$ on the boundary. From (8), when $z \to \zeta = e^{i\theta}$ and $dz = dre^{i\theta}$, we obtain

$$\sigma_{\theta\theta}(\zeta_1) + i\sigma_{r\theta}(\zeta_1) = \Phi(\zeta) + 2\overline{\Phi(\zeta)} + \Upsilon(\zeta), \quad |\zeta_1| = r. \quad (25)$$

Using (10), (11)–(13), in view of (23) and (24), the equality (25) yields

$$\sigma_{\theta\theta} = d_0 + 6d_2\cos 2\theta + (1-2\cos 2\theta)\sigma. \quad (26)$$

The first two summands in the right-hand side of (26) show the influence of surface stresses on the hoop stress $\sigma_{\theta\theta}$ on the boundary of the hole. The tensile stress $\sigma_{\theta\theta}$ in the absence of surface stresses attains its maximum at the points $\theta = \pm\pi/2$ on the boundary of the hole. In the presence of surface stresses for the value $\sigma_{\theta\theta}$ we get a different formula, namely,

$$\sigma_{\theta\theta}\big|_{\theta=\pi/2} = \frac{M(\varkappa+1)[14r - M(15+\varkappa)]}{4(r-M)[2r - M(\varkappa+3)]}\,\sigma + 3\sigma. \qquad (27)$$

It is rather evident from (27) that if $M > 0$ then for $r < M$ and $M(15+\varkappa) < 14r < 7M(3+\varkappa)$ the stress concentration diminishes when the surface stresses are present, while for $14M < 14r < M(15+\varkappa)$ and $2r > M(3+\varkappa)$ the stress concentration increases.

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