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INVESTIGATION OF INTERIOR AND EXTERIOR NEUMANN-TYPE STATIC BOUNDARY-VALUE PROBLEMS OF THERMO-ELECTRO-MAGNETO ELASTICITY THEORY

Abstract. We investigate the three-dimensional interior and exterior Neumann-type boundary-value problems of statics of the thermo-electromagneto-elasticity theory. We construct explicitly the fundamental matrix of the corresponding strongly elliptic non-self-adjoint 6×6 matrix differential operator and study their properties near the origin and at infinity. We apply the potential method and reduce the corresponding boundary-value problems to the equivalent system of boundary integral equations. We have found efficient asymptotic conditions at infinity which ensure the uniqueness of solutions in the space of bounded vector functions. We analyze the solvability of the resulting boundary integral equations in the Hölder and Sobolev-Slobodetski spaces and prove the corresponding existence theorems. The necessary and sufficient conditions of solvability of the interior Neumann-type boundary-value problem are written explicitly.

რეზიუმე. სტატიაში გამოკვლეულია ნეიმანის შიგა და გარე სამგანზომილებიანი ამოცანები თერმო-ელექტრო-მაგნეტო დრეკადობის თეორიის სტატიკის განტოლებებისათვის. შესაბამისი ოპერატორისათვის, რომელიც წარმოადგენს 6×6 განზომილების მატრიცულ არათვითშეუღლებულ, ძლიერად ელიფსურ დიფერენციალურ ოპერატორს, ცხადი სახითაა აგებული ფუნდამენტურ ამონახსნთა მატრიცა და დადგენილია მისი ასიმპტოტური თვისებები სათავისა და უსასრულობის მიდამოში. პოტენციალთა მეთოდის გამოყენებით სასაზღვრო ამოცანები დაყვანილია ეკვივალენტურ სასაზღვრო ინტეგრალურ (ფსევდოდიფერენციალურ) განტოლებათა სისტემაზე. გამოკვლეულია ამ ინტეგრალური განტოლების ამოხსნადობის საკითხი და დამტკიცებულია შესაბამისი სასაზღვრო ამოცანების ამონახსნების არსებობის თეორემები პელდერისა და სობოლევ–სლობოდეცკის ფუნქციურ სივრცეებში. აღსანიშნავია, რომ ცხადი სახითაა ამოწერილი ნეიმანის შიგა ამოცანის არსებობის აუცილებელი და საკმარისი პირობები.

1. Introduction

Modern industrial and technological processes apply widely, on the one hand, composite materials with complex microstructure and, on the other hand, complex composed structures consisting of materials having essentially different physical properties (for example, piezoelectric, piezomagnetic, hemitropic materials, two- and multi-component mixtures, nano-materials, bio-materials, and solid structures constructed by composition of these materials, such as, e.g., Smart Materials and other meta-materials). Therefore the investigation and analysis of mathematical models describing the mechanical, thermal, electric, magnetic and other physical properties of such materials have a crucial importance for both fundamental research and practical applications. In particular, the investigation of correctness of corresponding mathematical models (namely, existence, uniqueness, smoothness, asymptotic properties and stability of solutions) and construction of appropriate adequate numerical algorithms have a crucial role for fundamental research.

In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. The mathematical model of statics of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with appropriate boundary conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, and heat flux vector) can be then determined by the gradient and constitutive equations (for details see [2], [3], [4], [5], [6], [16], [21], [24], [27]).

For the equations of dynamics the uniqueness theorems of solutions for some initial-boundary-value problems are well studied. In particular, in the reference [16] the uniqueness theorem is proved without making restrictions on the positive definiteness on the elastic moduli, while the uniqueness theorems for the basic boundary-value problems (BVP) of statics of the thermo-electro-magneto-elasticity theory are proved in [20]. Existence theorems for the Dirichlet-type boundary-value problems are established in [19]. To the best of our knowledge, the existence of solutions to the Neumann-type BVPs of statics are not treated in the scientific literature.

In this paper, with the help of the potential method we reduce the three-dimensional interior and exterior Neumann-type boundary-value problems of the thermo-electro-magneto-elasticity theory to the equivalent 6×6 systems of integral equations and analyze their solvability in the Hölder and Sobolev-Slobodetski spaces and prove the corresponding uniqueness and existence theorems.

Essential difficulties arise in the study of exterior BVPs for unbounded domains. The case is that one has to consider the problem in a class of

vector functions which are bounded at infinity. This complicates the proof of uniqueness and existence theorems since Green's formulas do not hold for such vector functions and analysis of null spaces of the corresponding integral operators needs special consideration. We have found efficient and natural asymptotic conditions at infinity which ensure the uniqueness of solutions in the space of bounded vector functions. Moreover, for the interior Neumann-type boundary-value problem, the complete system of linearly independent solutions of the corresponding homogeneous adjoint integral equation is constructed in polynomials and the necessary and sufficient conditions of solvability of the problem are written explicitly.

2. FORMULATION OF PROBLEMS

Here we collect the basic field equations of the thermo-electro-magneto-elasticity theory and formulate the interior and exterior Neumann-type boundary-value problems of statics.

2.1. **Field equations.** Throughout the paper $u = (u_1, u_2, u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ is the strain tensor, the vectors $E = (E_1, E_2, E_3)^\top$ and $H = (H_1, H_2, H_3)^\top$ are electric and magnetic fields respectively, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials and $E = -\operatorname{grad} \varphi$, $H = -\operatorname{grad} \psi$, ϑ is the temperature increment, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density.

We employ also the notation $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation. In what follows the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators. To this end, we recall here the basic relations of the theory:

Constitutive relations:

$$\sigma_{rj} = \sigma_{jr} = c_{rjkl}\varepsilon_{kl} - e_{lrj}E_l - q_{lrj}H_l - \lambda_{rj}\vartheta, \ \ r,j = 1,2,3, \tag{2.1}$$

$$D_j = e_{jkl}\varepsilon_{kl} + \varkappa_{jl}E_l + a_{jl}H_l + p_j\vartheta, \ \ j = 1,2,3, \tag{2.2}$$

$$B_j = q_{jkl}\varepsilon_{kl} + a_{jl}E_l + \mu_{jl}H_l + m_j\vartheta, \ \ j = 1,2,3, \tag{2.3}$$

$$\mathcal{S} = \lambda_{kl}\varepsilon_{kl} + p_k E_k + m_k H_k + \gamma\vartheta. \tag{2.4}$$

Fourier Law:

$$q_j = -\eta_{jl}\partial_l\vartheta, \ \ j = 1,2,3. \tag{2.5}$$

Equations of motion:

$$\partial_j\sigma_{rj} + X_r = \varrho\partial_t^2 u_r, \ \ r = 1,2,3. \tag{2.6}$$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free):

$$\partial_j D_j = \varrho_e, \quad \partial_j B_j = 0. \tag{2.7}$$

Linearized equation of the entropy balance:

$$T_0 \partial_t \mathcal{S} - Q = -\partial_j q_j. \tag{2.8}$$

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma = \varrho c T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, c is the specific heat per unit mass, $X = (X_1, X_2, X_3)^\top$ is a mass force density, Q is a heat source intensity.

The constants involved in these equations satisfy the symmetry conditions:

$$c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl},$$
$$q_{klj} = q_{kjl}, \quad \varkappa_{kj} = \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad r, j, k, l = 1, 2, 3. \tag{2.9}$$
$$\mu_{kj} = \mu_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad a_{kj} = a_{jk},$$

From physical considerations it follows that (see, e.g., [16], [27]):

$$c_{rjkl}\xi_{rj}\xi_{kl} \geq c_0 \xi_{kl}\xi_{kl}, \quad \varkappa_{kj}\xi_k\xi_j \geq c_1 |\xi|^2,$$
$$\mu_{kj}\xi_k\xi_j \geq c_2 |\xi|^2, \quad \eta_{kj}\xi_k\xi_j \geq c_3 |\xi|^2, \tag{2.10}$$
$$\text{for all } \xi_{kj} = \xi_{jk} \in \mathbb{R} \text{ and for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,$$

where c_0, c_1, c_2, and c_3 are positive constants.

It is easy to see that due to the symmetry conditions (2.9)

$$c_{rjkl}\xi_{rj}\overline{\xi_{kl}} \geq c_0 \xi_{kl}\overline{\xi_{kl}}, \quad \varkappa_{kj}\xi_k\overline{\xi_j} \geq c_1 |\xi|^2,$$
$$\mu_{kj}\xi_k\overline{\xi_j} \geq c_2 |\xi|^2, \quad \eta_{kj}\xi_k\overline{\xi_j} \geq c_3 |\xi|^2,$$
$$\text{for all } \xi_{kj} = \xi_{jk} \in \mathbb{C} \text{ and for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3.$$

More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure that for arbitrary ζ', $\zeta'' \in \mathbb{C}^3$ and $\theta \in \mathbb{C}$ there is a positive constant δ_0 depending on the material constants such that (cf. [27])

$$\varkappa_{kj}\zeta_k'\overline{\zeta_j'} + a_{kj}\left(\zeta_k'\overline{\zeta_j''} + \overline{\zeta_k'}\zeta_j''\right) + \mu_{kj}\zeta_k''\overline{\zeta_j''} \pm 2\Re\left[\overline{\theta}(p_j\zeta_j' + m_j\zeta_j'')\right] + \gamma|\theta|^2 \geq$$
$$\geq \delta_0\left(|\zeta'|^2 + |\zeta''|^2 + |\theta|^2\right). \tag{2.11}$$

This condition is equivalent to positive definiteness of the matrix

$$
\Xi := \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} & [p_j]_{3\times1} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} & [m_j]_{3\times1} \\ [p_j]_{1\times3} & [m_j]_{1\times3} & \gamma \end{bmatrix}_{7\times7} .
$$

In particular, it follows that the matrix

$$
\Lambda := \begin{bmatrix} [\varkappa_{kj}]_{3\times3} & [a_{kj}]_{3\times3} \\ [a_{kj}]_{3\times3} & [\mu_{kj}]_{3\times3} \end{bmatrix}_{6\times6} \tag{2.12}
$$

is positive definite, i.e.,

$$
\varkappa_{kj}\zeta_k'\overline{\zeta_j'} + a_{kj}\big(\zeta_k'\overline{\zeta_j''} + \overline{\zeta_k'}\zeta_j''\big) + \mu_{kj}\zeta_k''\overline{\zeta_j''} \geq \kappa(|\zeta'|^2 + |\zeta''|^2)
$$

with some positive constant κ depending on the material parameters involved in (2.12). A sufficient condition for the quadratic form in the left hand side of (2.11) to be positive definite then reads as $\nu^2 < \frac{\kappa\gamma}{6}$ with $\nu = \max\big\{|p_1|, |p_2|, |p_3|, |m_1|, |m_2|, |m_3|\big\}$.

With the help of the symmetry conditions (2.9) we can rewrite the constitutive relations (2.1)–(2.4) as follows

$$
\begin{aligned}
\sigma_{rj} &= c_{rjkl}\partial_l u_k + e_{lrj}\partial_l\varphi + q_{lrj}\partial_l\psi - \lambda_{rj}\vartheta, \quad r,j = 1,2,3, \\
D_j &= e_{jkl}\partial_l u_k - \varkappa_{jl}\partial_l\varphi - a_{jl}\partial_l\psi + p_j\vartheta, \quad j = 1,2,3, \\
B_j &= q_{jkl}\partial_l u_k - a_{jl}\partial_l\varphi - \mu_{jl}\partial_l\psi + m_j\vartheta, \quad j = 1,2,3, \\
\mathcal{S} &= \lambda_{kl}\partial_l u_k - p_l\partial_l\varphi - m_l\partial_l\psi + \gamma\vartheta.
\end{aligned}
$$

In the theory of thermo-electro-magneto-elasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a unit normal vector $n = (n_1, n_2, n_3)$ have the form

$$
\sigma_{rj}n_j = c_{rjkl}n_j\partial_l u_k + e_{lrj}n_j\partial_l\varphi + q_{lrj}n_j\partial_l\psi - \lambda_{rj}n_j\vartheta, \quad r = 1,2,3,
$$

while the normal components of the electric displacement vector, magnetic induction vector and heat flux vector read as

$$
\begin{aligned}
D_jn_j &= e_{jkl}n_j\partial_l u_k - \varkappa_{jl}n_j\partial_l\varphi - a_{jl}n_j\partial_l\psi + p_jn_j\vartheta, \\
B_jn_j &= q_{jkl}n_j\partial_l u_k - a_{jl}n_j\partial_l\varphi - \mu_{jl}n_j\partial_l\psi + m_jn_j\vartheta, \\
q_jn_j &= -\eta_{jl}n_j\partial_l\vartheta.
\end{aligned}
$$

For convenience we introduce the following matrix differential operator

$$
\mathcal{T}(\partial, n) = \big[\mathcal{T}_{pq}(\partial, n)\big]_{6\times6} :=
$$

$$
:= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times3} & [e_{lrj}n_j\partial_l]_{3\times1} & [q_{lrj}n_j\partial_l]_{3\times1} & [-\lambda_{rj}n_j]_{3\times1} \\ [-e_{jkl}n_j\partial_l]_{1\times3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & -p_jn_j \\ [-q_{jkl}n_j\partial_l]_{1\times3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & -m_jn_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}_{6\times6} . \tag{2.13}
$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)U = \big(\sigma_{1j}n_j, \sigma_{2j}n_j, \sigma_{3j}n_j, -D_j n_j, -B_j n_j, -q_j n_j\big)^\top. \qquad (2.14)$$

The components of the vector $\mathcal{T}U$ given by (2.14) have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

As we see, all the thermo-mechanical and electro-magnetic characteristics can be determined by the six functions: the three displacement components u_j, $j = 1, 2, 3$, temperature distribution ϑ, and the electric and magnetic potentials φ and ψ. Therefore, all the above field relations and the corresponding boundary-value problems we reformulate in terms of these six functions.

First of all from the equations (2.1)–(2.8) we derive the basic linear system of dynamics of the theory of thermo-electro-magneto-elasticity:

$$c_{rjkl}\partial_j\partial_l u_k(x,t) + e_{lrj}\partial_j\partial_l \varphi(x,t) + q_{lrj}\partial_j\partial_l \psi(x,t) - \lambda_{rj}\partial_j \vartheta(x,t) -$$
$$-\varrho\partial_t^2 u_r(x,t) = -X_r(x,t), \quad r = 1, 2, 3,$$
$$-e_{jkl}\partial_j\partial_l u_k(x,t) + \varkappa_{jl}\partial_j\partial_l \varphi(x,t) + a_{jl}\partial_j\partial_l \psi(x,t) - p_j\partial_j \vartheta(x,t) = -\varrho_e(x,t),$$
$$-q_{jkl}\partial_j\partial_l u_k(x,t) + a_{jl}\partial_j\partial_l \varphi(x,t) + \mu_{jl}\partial_j\partial_l \psi(x,t) - m_j\partial_j \vartheta(x,t) = 0,$$
$$-T_0\lambda_{kl}\partial_t\partial_l u_k(x,t) + T_0 p_l\partial_t\partial_l \varphi(x,t) + T_0 m_l\partial_t\partial_l \psi(x,t) + \eta_{jl}\partial_j\partial_l \vartheta(x,t) -$$
$$-T_0\gamma\partial_t\vartheta(x,t) = -Q(x,t).$$

If all the functions involved in these equations are harmonic time dependent, that is they can be represented as the product of a function of the spatial variables (x_1, x_2, x_3) and the multiplier $\exp\{\tau t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have then the *pseudo-oscillation equations* of the theory of thermo-electro-magneto-elasticity. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If τ is a pure imaginary number, $\tau = i\omega$ with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$ we get the *equations of statics*:

$$c_{rjkl}\partial_j\partial_l u_k(x) + e_{lrj}\partial_j\partial_l \varphi(x) + q_{lrj}\partial_j\partial_l \psi(x) - \lambda_{rj}\partial_j \vartheta(x) =$$
$$= -X_r(x), \quad r = 1, 2, 3,$$
$$-e_{jkl}\partial_j\partial_l u_k(x) + \varkappa_{jl}\partial_j\partial_l \varphi(x) + a_{jl}\partial_j\partial_l \psi(x) - p_j\partial_j \vartheta(x) = -\varrho_e(x), \quad (2.15)$$
$$-q_{jkl}\partial_j\partial_l u_k(x) + a_{jl}\partial_j\partial_l \varphi(x) + \mu_{jl}\partial_j\partial_l \psi(x) - m_j\partial_j \vartheta(x) = 0,$$
$$\eta_{jl}\partial_j\partial_l \vartheta(x) = -Q(x).$$

In matrix form these equations can be written as

$$A(\partial)U(x) = \Phi(x),$$

where

$$U = (u_1, u_2, u_3, u_4, u_5, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top,$$

$$\Phi = (\Phi_1, \ldots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top,$$

and $A(\partial)$ is the matrix differential operator generated by equations (2.15),

$$A(\partial) = [A_{pq}(\partial)]_{6\times 6} :=$$

$$:= \begin{bmatrix} [c_{rjkl}\partial_j\partial_l]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -m_j\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times 6} . \quad (2.16)$$

2.2. Formulation of the boundary-value problems.

Let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $S = \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\overline{\Omega^\pm}$ are filled by an anisotropic homogeneous material with thermo-electro-magneto-elastic properties.

Throughout the paper $n = (n_1, n_2, n_3)$ stands for the outward unit normal vector with respect to Ω^+ at the point $x \in \partial\Omega^+$.

Neumann-type problems $(N)^\pm$: Find a regular solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ (resp. $U \in [C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$), to the system of equations

$$A(\partial)U = \Phi \ \text{ in } \ \Omega^\pm,$$

satisfying the Neumann-type boundary conditions

$$\{\mathcal{T}U\}^\pm = f \ \text{ on } \ S,$$

where $A(\partial)$ is a nonselfadjoint strongly elliptic matrix partial differential operator generated by the equations of statics of the theory of thermo-electro-magneto-elasticity defined in (2.16), while $\mathcal{T}(\partial, n)$ is the matrix boundary operator defined in (2.13). The symbols $\{\cdot\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$.

In our analysis we need special asymptotic conditions at infinity in the case of unbounded domains [20].

Definition 2.1. We say that a continuous vector $U = (u, \varphi, \psi, \vartheta)^\top \equiv (U_1, \cdots, U_6)^\top$ in the domain Ω^- has the property $Z(\Omega^-)$ if the following conditions are satisfied

$$\widetilde{U}(x) := (u(x), \varphi(x), \psi(x))^\top = \mathcal{O}(1),$$
$$U_6(x) = \vartheta(x) = \mathcal{O}(|x|^{-1}), \qquad \text{as } |x| \to \infty,$$
$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} U_k(x)\,d\Sigma_R = 0, \ \ k = \overline{1,5},$$

where Σ_R is a sphere centered at the origin and radius R.

In what follows we always assume that in the case of exterior boundary-value problem a solution possesses $Z(\Omega^-)$ property.

2.3. **Potentials and their properties.** Denote by $\Gamma(x) = [\Gamma_{kj}(x)]_{6\times 6}$ the matrix of fundamental solutions of the operator $A(\partial)$, $A(\partial)\Gamma(x) = I_6\,\delta(x)$, where $\delta(\cdot)$ is the Dirac's delta distribution and I_6 stands for the unit 6×6 matrix. Applying the generalized Fourier transform technique, the fundamental matrix can be constructed explicitly,

$$\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}[A^{-1}(-i\,\xi)]\,, \qquad (2.17)$$

where $\mathcal{F}^{-1}$ is the generalized inverse Fourier transform and $A^{-1}(-i\,\xi)$ is the matrix inverse to $A(-i\,\xi)$. The properties of the fundamental matrix near the origin and at infinity are established in [23]. The entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in x and at the origin and at infinity the following asymptotic relations hold

$$\Gamma(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}.$$

Moreover, the columns of the matrix $\Gamma(x)$ possess the property $Z(\mathbb{R}^3\setminus\{0\})$. With the help of the fundamental matrix we construct the generalized single and double layer potentials, and the Newton-type volume potentials,

$$V(h)(x) = \int\limits_S \Gamma(x-y)\,h(y)\,dS_y, \quad x\in\mathbb{R}^3\setminus S,$$

$$W(h)(x) = \int\limits_S [\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)]^\top\, h(y)\,dS_y, \quad x\in\mathbb{R}^3\setminus S,$$

$$N_{\Omega^\pm}(g)(x) = \int\limits_{\Omega^\pm} \Gamma(x-y)\,g(y)\,dy, \quad x\in\mathbb{R}^3,$$

where $S = \partial\Omega^\pm \in C^{m,\kappa}$ with integer $m \geq 1$ and $0 < \kappa \leq 1$; $h = (h_1,\dots,h_6)^\top$ and $g = (g_1,\cdots,g_6)^\top$ are density vector-functions defined respectively on S and in $\Omega^\pm$; the so called *generalized stress operator* $\mathcal{P}(\partial, n)$, associated with the adjoint differential operator $A^*(\partial) = A^\top(-\partial)$, reads as

$$\mathcal{P}(\partial, n) = \big[\mathcal{P}_{pq}(\partial, n)\big]_{6\times 6} =$$

$$= \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l & 0 \\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}n_j\partial_l \end{bmatrix}. \qquad (2.18)$$

The following properties of layer potentials immediately follow from their definition.

Theorem 2.2. The generalized single and double layer potentials solve the homogeneous differential equation $A(\partial)U = 0$ in $\mathbb{R}^3\setminus S$ and possess the property $Z(\Omega^-)$.

In what follows by L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev–Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [29]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

With the help of Green's formulas, one can derive general integral representations of solutions to the homogeneous equation $A(\partial)U = 0$ in $\Omega^{\pm}$. In particular, the following theorems hold.

Theorem 2.3. Let $S = \partial\Omega^+ \in C^{1,\kappa}$ with $0 < \kappa \leq 1$ and U be a regular solution to the homogeneous equation $A(\partial)U = 0$ in Ω^+ of the class $[C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$. Then there holds the integral representation formula

$$W(\{U\}^+)(x) - V(\{\mathcal{T}U\}^+)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases}$$

Theorem 2.4. Let $S = \partial\Omega^-$ be $C^{1,\kappa}$-smooth with $0 < \kappa \leq 1$ and let U be a regular solution to the homogeneous equation $A(\partial)U = 0$ in Ω^- of the class $[C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$ having the property $Z(\Omega^-)$. Then there holds the integral representation formula

$$-W(\{U\}^-)(x) + V(\{\mathcal{T}U\}^-)(x) = \begin{cases} 0 & \text{for } x \in \Omega^+, \\ U(x) & \text{for } x \in \Omega^-. \end{cases}$$

By standard limiting procedure, these formulas can be extended to Lipschitz domains and to solution vectors from the spaces $[W_p^1(\Omega^+)]^6$ and $[W_{p,loc}^1(\Omega^-)]^6 \cap Z(\Omega^-)$ with $1 < p < \infty$ (cf., [12], [17], [25]).

The qualitative and mapping properties of the layer potentials are described by the following theorems (cf. [7], [9], [15], [17], [23]).

Theorem 2.5. Let $S = \partial\Omega^{\pm} \in C^{m,\kappa}$ with integers $m \geq 1$ and $k \leq m - 1$, and $0 < \kappa' < \kappa \leq 1$. Then the operators

$$V : [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(\overline{\Omega^{\pm}})]^6, \quad W : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(\overline{\Omega^{\pm}})]^6 \quad (2.19)$$

are continuous.

For any $g \in [C^{0,\kappa'}(S)]^6$, $h \in [C^{1,\kappa'}(S)]^6$, and any $x \in S$ we have the following jump relations:

$$\{V(g)(x)\}^{\pm} = V(g)(x) = \mathcal{H}g(x), \qquad (2.20)$$

$$\{\mathcal{T}(\partial_x, n(x))V(g)(x)\}^{\pm} = [\mp 2^{-1}I_6 + \mathcal{K}]g(x), \qquad (2.21)$$

$$\{W(g)(x)\}^{\pm} = [\pm 2^{-1}I_6 + \mathcal{N}]g(x), \qquad (2.22)$$

$$\{\mathcal{T}(\partial_x, n(x))W(h)(x)\}^+ =$$
$$= \{\mathcal{T}(\partial_x, n(x))W(h)(x)\}^- = \mathcal{L}h(x), \quad m \geq 2, \qquad (2.23)$$

where $\mathcal{H}$ is a weakly singular integral operator, $\mathcal{K}$ and $\mathcal{N}$ are singular integral operators, and $\mathcal{L}$ is a singular integro-differential operator,

$$
\begin{aligned}
\mathcal{H}g(x) &:= \int\limits_{S} \Gamma(x-y)g(y)\,dS_y, \\[2mm]
\mathcal{K}g(x) &:= \int\limits_{S} \mathcal{T}(\partial_x, n(x))\Gamma(x-y)\,g(y)\,dS_y, \\[2mm]
\mathcal{N}g(x) &:= \int\limits_{S} \left[\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)\right]^\top g(y)\,dS_y, \\[2mm]
\mathcal{L}h(x) &:= \lim_{\Omega^\pm \ni z \to x \in S} \mathcal{T}(\partial_z, n(x)) \int\limits_{S} \left[\mathcal{P}(\partial_y, n(y))\Gamma^\top(z-y)\right]^\top h(y)\,dS_y.
\end{aligned}
\tag{2.24}
$$

Theorem 2.6. Let S be a Lipschitz surface. The operators V and W can be extended to the continuous mappings

$$
V : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6, \quad V : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_{2,loc}^1(\Omega^-)]^6 \cap Z(\Omega^-),
$$
$$
W : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6, \quad W : [H_2^{\frac{1}{2}}(S)]^6 \to [H_{2,loc}^1(\Omega^-)]^6 \cap Z(\Omega^-).
$$

The jump relations (2.20)–(2.23) on S remain valid for the extended operators in the corresponding function spaces.

Theorem 2.7. Let S, m, κ, κ' and k be as in Theorem 2.5. Then the operators

$$\mathcal{H} : [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6, \quad m \geq 1, \tag{2.25}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6, \quad m \geq 1, \tag{2.26}$$

$$\mathcal{K} : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \quad m \geq 1, \tag{2.27}$$

$$: [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6, \quad m \geq 1, \tag{2.28}$$

$$\mathcal{N} : [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6, \quad m \geq 1, \tag{2.29}$$

$$: [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6, \quad m \geq 1, \tag{2.30}$$

$$\mathcal{L} : [C^{k,\kappa'}(S)]^6 \to [C^{k-1,\kappa'}(S)]^6, \quad m \geq 2, \; k \geq 1, \tag{2.31}$$

$$: [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6, \quad m \geq 2, \tag{2.32}$$

are continuous. The operators (2.26), (2.28), (2.30), and (2.32) are bounded if S is a Lipschitz surface.

Proofs of the above formulated theorems are word for word proofs of the similar theorems in [8], [10], [11], [13], [14], [15], [22], [26].

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [1], [5], [9], [12], [28], and the references therein).

Theorem 2.8. Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$ and $\mathcal{L}$ be as in Theorems 2.5 and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The layer potential operators (2.19) and the boundary integral (pseudodifferential) operators (2.25)–(2.32) can be extended to the following continuous operators

$$V : [B_{p,p}^s(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6, \quad W : [B_{p,p}^s(S)]^6 \to [H_p^{s+\frac{1}{p}}(\Omega^+)]^6,$$

$$V : [B_{p,p}^s(S)]^6 \to [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6, \quad W : [B_{p,p}^s(S)]^6 \to [H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)]^6,$$

$$\mathcal{H} : [H_p^s(S)]^6 \to [H_p^{s+1}(S)]^6, \quad \mathcal{K} : [H_p^s(S)]^6 \to [H_p^s(S)]^6,$$

$$\mathcal{N} : [H_p^s(S)]^6 \to [H_p^s(S)]^6, \quad \mathcal{L} : [H_p^{s+1}(S)]^6 \to [H_p^s(S)]^6.$$

The jump relations (2.20)–(2.23) remain valid for arbitrary $g \in [B_{p,q}^s(S)]^6$ with $s \in \mathbb{R}$ if the limiting values (traces) on S are understood in the sense described in [28].

Remark 2.9. Let either $\Phi \in [L_p(\Omega^+)]^6$ or $\Phi \in [L_{p,comp}(\Omega^-)]^6$, $p > 1$. Then the Newtonian volume potentials $N_{\Omega^\pm}(\Phi)$ possess the following properties (see, e.g., [18]):

$$N_{\Omega^+}(\Phi) \in [W_p^2(\Omega^+)]^6, \quad N_{\Omega^-}(\Phi) \in [W_{p,loc}^2(\Omega^-)]^6,$$

$$A(\partial) N_{\Omega^\pm}(\Phi) = \Phi \text{ almost everywhere in } \Omega^\pm.$$

Therefore, without loss of generality, we can assume that in the formulation of the Neumann-type problems the right hand side function in the differential equations vanishes, $\Phi(x) = 0$ in $\Omega^\pm$.

3. Investigation of the Exterior Neumann BVP

Let us consider the exterior Neumann-type BVP for the domain Ω^-:

$$A(\partial)U(x) = 0, \quad x \in \Omega^-, \tag{3.1}$$

$$\left\{ \mathcal{T}(\partial, n)U(x) \right\}^- = F(x), \quad x \in S. \tag{3.2}$$

We assume that $S \in C^{1,\kappa}$ and $F \in C^{0,\kappa'}(S)$ with $0 < \kappa' < \kappa \le 1$. We investigate this problem in the space of regular vector functions $[C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$. In [20] it is shown that the homogeneous version of the exterior Neumann-type problem possesses only the trivial solution.

To prove the existence result, we look for a solution of the problem (3.1)–(3.2) as the single layer potential

$$U(x) \equiv V(h)(x) = \int_S \Gamma(x - y)h(y) \, dS_y, \tag{3.3}$$

where Γ is defined by (2.17) and $h = (h_1, \ldots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is unknown density. By Theorem 2.5 and in view of the boundary condition (3.2), we get the following integral equation for the density vector h

$$[2^{-1}I_6 + \mathcal{K}]h = F \text{ on } S, \tag{3.4}$$

where $\mathcal{K}$ is a singular integral operator defined by (2.24). Note that the operator $2^{-1}I_6 + \mathcal{K}$ has the following mapping properties

$$2^{-1}I_6 + \mathcal{K} : [C^{0,\kappa'}(S)]^6 \to [C^{0,\kappa'}(S)]^6, \qquad (3.5)$$

$$: [L_2(S)]^6 \to [L_2(S)]^6. \qquad (3.6)$$

These operators are compact perturbations of their counterpart operators associated with the pseudo-oscillation equations which are studied in [23]. Applying the results obtained in [23] one can show that $2^{-1}I_6 + \mathcal{K}$ is a singular integral operator of normal type (i.e., its principal homogeneous symbol matrix is non-degenerate) and its index equals to zero.

Let us show that the operators (3.5) and (3.6) have trivial null spaces. To this end, it suffices to prove that the corresponding homogeneous integral equation

$$[2^{-1}I_6 + \mathcal{K}]h = 0 \text{ on } S, \qquad (3.7)$$

has only the trivial solution in the appropriate space. Let $h^{(0)} \in [L_2(S)]^6$ be a solution to equation (3.7). By the embedding theorems (see, e.g., [15], Ch.4), we actually have that $h^{(0)} \in [C^{0,\kappa'}(S)]^6$. Now we construct the single layer potential $U_0(x) = V(h^{(0)})(x)$. Evidently, $U_0 \in [C^{1,\kappa'}(\overline{\Omega^{\pm}})]^6 \cap [C^2(\Omega^{\pm})]^6 \cap Z(\Omega^-)$ and the equation $A(\partial)U_0 = 0$ in $\Omega^{\pm}$ is automatically satisfied. Since $h^{(0)}$ solves equation (3.7), we have $\{\mathcal{T}(\partial,n)U_0\}^- = [2^{-1}I_6 + \mathcal{K}]h^{(0)} = 0$ on S. Therefore U_0 is a solution to the homogeneous exterior Neumann problem satisfying the property $Z(\Omega^-)$. Consequently, due to the uniqueness theorem [20], $U_0 = 0$ in Ω^-. Applying the continuity property of the single layer potential we find: $0 = \{U_0\}^- = \{U_0\}^+$ on S, yielding that the vector $U_0 = V(h^{(0)})$ represents a solution to the homogeneous interior Dirichlet problem. Now by the uniqueness theorem for the Dirichlet problem [20], we deduce that $U_0 = 0$ in Ω^+. Thus $U_0 = 0$ in $\Omega^{\pm}$. By virtue of the jump formula

$$\left\{\mathcal{T}(\partial,n)U_0\right\}^+ - \left\{\mathcal{T}(\partial,n)U_0\right\}^- = -h^{(0)} = 0 \text{ on } S,$$

whence it follows that the null space of the operator $2^{-1}I_6 + \mathcal{K}$ is trivial and the operators (3.5) and (3.6) are invertible. As a ready consequence, we finally conclude that the non-homogeneous integral equation (3.4) is solvable for arbitrary right hand side vector $F \in [C^{0,\kappa'}(S)]^6$, which implies the following existence result.

Theorem 3.1. Let $m \geq 0$ be a nonnegative integer and $0 < \kappa' < \kappa \leq 1$. Further, let $S \in C^{m+1,\kappa}$ and $F \in [C^{m,\kappa'}(S)]^6$. Then the exterior Neumann-type BVP (3.1)–(3.2) is uniquely solvable in the space of regular vector functions, $[C^{m+1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$, and the solution is representable by the single layer potential $U(x) = V(h)(x)$ with the density $h = (h_1, \ldots, h_6)^{\top} \in [C^{m,\kappa'}(S)]^6$ being a unique solution of the integral equation (3.4).

Remark 3.2. Let S be Lipschitz and $F \in \left[H^{-1/2}(S)\right]^6$. Then by the same approach as in the reference [17], the following propositions can be established:

(i) the integral equation (3.4) is uniquely solvable in the space $[H^{-1/2}(S)]^6$;

(ii) the exterior Neumann-type BVP (3.1)–(3.2) is uniquely solvable in the space $[H^1_{2,loc}(\Omega^-)]^6 \cap Z(\Omega^-)$ and the solution is representable by the single layer potential (3.3), where the density vector $h \in [H^{-1/2}(S)]^6$ solves the integral equation (3.4).

4. Investigation of the Interior Neumann BVP

Before we go over to the interior Neumann problem we prove some preliminary assertions needed in our analysis.

4.1. **Some auxiliary results.** Let us consider the adjoint operator $A^*(\partial)$ to the operator $A(\partial)$

$$A^*(\partial) :=$$

$$:= \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times3} & [-e_{jkl}\partial_j\partial_l]_{3\times1} & [-q_{jkl}\partial_j\partial_l]_{3\times1} & [0]_{3\times1} \\ [e_{lrj}\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & 0 \\ [q_{lrj}\partial_j\partial_l]_{1\times3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & 0 \\ [\lambda_{rj}\partial_j]_{1\times3} & p_j\partial_j & m_j\partial_j & \eta_{jl}\partial_j\partial_l \end{bmatrix}_{6\times6} . \quad (4.1)$$

The corresponding matrix of fundamental solutions $\Gamma^*(x-y) = [\Gamma(y-x)]^\top$ has the following property at infinity

$$\Gamma^*(x-y) = \Gamma^\top(y-x) := \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times5} & [0]_{5\times1} \\ [\mathcal{O}(1)]_{1\times5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times6}$$

as $|x| \to \infty$. With the help of the fundamental matrix $\Gamma^*(x-y)$ we construct the single and double layer potentials, and the Newtonian volume potentials

$$V^*(h^*)(x) \equiv \int_S \Gamma^*(x-y)h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \quad (4.2)$$

$$W^*(h^*)(x) \equiv \int_S \left[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^\top\right]^\top h^*(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \quad (4.3)$$

$$N^*_{\Omega^\pm}(g^*)(x) \equiv \int_{\Omega^\pm} \Gamma^*(x-y)g^*(y)\,dy, \quad x \in \mathbb{R}^3,$$

where the density vector $h^* = (h_1^*, \ldots, h_6^*)^\top$ is defined on S, while $g^* = (g_1^*, \ldots, g_6^*)^\top$ is defined in $\Omega^\pm$. We assume that in the case of the domain Ω^- the vector g^* has a compact support.

It can be shown that the layer potentials V^* and W^* possess exactly the same mapping properties and jump relations as the potentials V and W (see Theorems 2.5–2.8). In particular,

$$\{V^*(h^*)\}^+ = \{V^*(h^*)\}^- = \mathcal{H}^* h^*,$$

$$\{W^*(h^*)\}^{\pm} = \pm\, 2^{-1} h^* + \mathcal{K}^* h^*, \tag{4.4}$$

$$\left\{\mathcal{P}V^*(h^*)\right\}^{\pm} = \mp\, 2^{-1} h^* + \mathcal{N}^* h^*, \tag{4.5}$$

where $\mathcal{H}^*$ is a weakly singular integral operator, while $\mathcal{K}^*$ and $\mathcal{N}^*$ are singular integral operators,

$$\mathcal{H}^* h^*(x) := \int_S \Gamma^*(x-y) h^*(y)\, dS_y,$$

$$\mathcal{K}^* h^*(x) := \int_S \left[\mathcal{T}(\partial_y, n(y))[\Gamma^*(x-y)]^{\top}\right]^{\top} h^*(y)\, dS_y, \tag{4.6}$$

$$\mathcal{N}^* h^*(x) := \int_S [\mathcal{P}(\partial_x, n(x))\Gamma^*(x-y)] h^*(y)\, dS_y.$$

Now we introduce a special class of vector functions which is a counterpart of the class $Z(\Omega^-)$.

Definition 4.1. We say that a continuous vector function $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^{\top}$ has the property $Z^*(\Omega^-)$ in the domain Ω^-, if the following conditions are satisfied

$$\widetilde{U}^*(x) = \left(u^*(x), \varphi^*(x), \psi^*(x)\right)^{\top} = \mathcal{O}(|x|^{-1}) \ \text{ as } \ |x| \to \infty,$$

$$\vartheta^*(x) = \mathcal{O}(1) \text{ as } |x| \to \infty,$$

$$\lim_{R \to \infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} \vartheta^*(x)\, d\Sigma_R = 0,$$

where Σ_R is a sphere centered at the origin and radius R.

As in the case of usual layer potentials here we have the following

Theorem 4.2. The generalized single and double layer potentials, defined by (4.2) and (4.3), solve the homogeneous differential equation $A^*(\partial)U^* = 0$ in $\mathbb{R}^3 \setminus S$ and possess the property $Z^*(\Omega^-)$.

For an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in Ω^+ one can derive the following integral representation formula

$$W^*(\{U^*\}^+)(x) - V^*(\{\mathcal{P}U^*\}^+)(x) = \begin{cases} U^*(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases} \tag{4.7}$$

Similar representation formula holds also for an arbitrary regular solution to the equation $A^*(\partial)U^*(x) = 0$ in Ω^- which possesses the property $Z^*(\Omega^-)$:

$$-W^*\big(\{U^*\}_S^-\big)(x) + V^*\big(\{\mathcal{P}U^*\}_S^-\big)(x) = \begin{cases} U^*(x), & x \in \Omega^-, \\ 0, & x \in \Omega^+. \end{cases} \qquad (4.8)$$

To derive this representation we denote $\Omega_R^- := B(0, R)\backslash\overline{\Omega^+}$, where $B(0, R)$ is a ball centered at the origin and radius R. Then in view of (4.7) we have

$$U^*(x) = -W_S^*\big(\{U^*\}_S^-\big)(x) + V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x) + \Phi_R^*(x), \;\; x \in \Omega_R^-, \quad (4.9)$$

$$0 = -W_S^*\big(\{U^*\}_S^-\big)(x) + V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x) + \Phi_R^*(x), \;\; x \in \Omega^+, \quad (4.10)$$

where

$$\Phi_R^*(x) := W_{\Sigma_R}^*\big(U^*\big)(x) - V_{\Sigma_R}^*\big(\mathcal{P}U^*\big)(x). \qquad (4.11)$$

Here $V_{\mathcal{M}}^*$ and $W_{\mathcal{M}}^*$ denote the single and double layer potential operators with integration surface $\mathcal{M}$. Evidently

$$A^*(\partial)\Phi_R^*(x) = 0, \;\; |x| < R. \qquad (4.12)$$

In turn, from (4.9) and (4.10) we get

$$\Phi_R^*(x) = U^*(x) + W_S^*\big(\{U^*\}_S^-\big)(x) - V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x), \;\; x \in \Omega_R^-,$$

$$\Phi_R^*(x) = W_S^*\big(\{U^*\}_S^-\big)(x) - V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x), \;\; x \in \Omega^+, \qquad (4.13)$$

whence the equality $\Phi_{R_1}^*(x) = \Phi_{R_2}^*(x)$ follows for $|x| < R_1 < R_2$. We assume that R_1 and R_2 are sufficiently large numbers. Therefore, for an arbitrary fixed point $x \in \mathbb{R}^3$ the following limit exists

$$\Phi^*(x) := \lim_{R\to\infty} \Phi_R^*(x) =$$

$$= \begin{cases} U^*(x) + W_S^*\big(\{U^*\}_S^-\big)(x) - V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x), & x \in \Omega^-, \\ W_S^*\big(\{U^*\}_S^-\big)(x) - V_S^*\big(\{\mathcal{P}U^*\}_S^-\big)(x), & x \in \Omega^+, \end{cases} \qquad (4.14)$$

and $A^*(\partial)\Phi^*(x) = 0$ for all $x \in \Omega^+ \cup \Omega^-$. On the other hand, for arbitrary fixed point $x \in \mathbb{R}^3$ and a number R_1, such that $|x| < R_1$ and $\overline{\Omega^+} \subset B(0, R_1)$, from (4.13) we have

$$\Phi^*(x) = \lim_{R\to\infty} \Phi_R^*(x) = \Phi_{R_1}^*(x).$$

Now from (4.11)–(4.12) we deduce

$$A^*(\partial)\Phi^*(x) = 0 \;\; \forall\, x \in \mathbb{R}^3. \qquad (4.15)$$

Since U^*, W^*, $V^* \in Z^*(\Omega^-)$ we conclude from (4.14) that $\Phi^*(x) \in Z^*(\mathbb{R}^3)$. In particular, we have

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} \Phi^*(x)\, d\Sigma_R = 0. \qquad (4.16)$$

Our goal is to show that

$$\Phi^*(x) = 0 \;\; \forall\, x \in \mathbb{R}^3.$$

Applying the generalized Fourier transform to equation (4.15) we get

$$A^*(-i\xi)\widehat{\Phi}^*(\xi) = 0, \ \ \xi \in \mathbb{R}^3,$$

where $\widehat{\Phi}^*(\xi)$ is the Fourier transform of Φ^*. Taking into account that $\det A^*(-i\xi) \neq 0$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, we conclude that the support of the generalized functional $\widehat{\Phi}^*(\xi)$ is the origin and consequently

$$\widehat{\Phi}^*(\xi) = \sum_{|\alpha| \leq M} c_\alpha \delta^{(\alpha)}(\xi),$$

where α is a multi-index, c_α are arbitrary constant vectors and M is some nonnegative integer. Then it follows that $\Phi^*(x)$ is polynomial in x and due to the inclusion $\Phi^* \in Z^*(\Omega^-)$, $\Phi^*(x)$ is bounded at infinity, i.e., $\Phi^*(x) = const$ in $\mathbb{R}^3$. Therefore (4.16) implies that $\Phi^*(x)$ vanishes identically in $\mathbb{R}^3$. This proves that the formula (4.8) holds.

Theorem 4.3. Let $S \in C^{2,\kappa}$ and $h \in \left[C^{1,\kappa'}(S)\right]^6$ with $0 < \kappa' < \kappa \leq 1$. Then for the double layer potential W^* defined by (4.3) there holds the following formula (generalized Lyapunov–Tauber relation)

$$\left\{\mathcal{P}W^*(h)\right\}^+ = \left\{\mathcal{P}W^*(h)\right\}^- \ \ \text{on} \ S, \quad\quad (4.17)$$

where the operator $\mathcal{P}$ is given by (2.18).

For $h \in [H_2^{\frac{1}{2}}(S)]^6$ the relation (4.17) also holds in the space $[H_2^{-\frac{1}{2}}(S)]^6$.

Proof. Since $h \in \left[C^{1,\kappa'}(S)\right]^6$, evidently $U^* := W^*(h) \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6$. It is clear that the vector U^* is a solution of the homogeneous equation $A^*(\partial)U^*(x) = 0$ in $\Omega^+ \cup \Omega^-$, where the operator $A^*(\partial)$ is defined by (4.1). With the help of (4.7) and (4.8), for the vector function U^* we derive the following representation formula

$$U^*(x) = W^*([U^*]_S)(x) - V^*\big([\mathcal{P}U^*]_S\big)(x), \ \ x \in \Omega^+ \cup \Omega^-, \quad\quad (4.18)$$

where

$$[U^*]_S \equiv \{U^*\}^+ - \{U^*\}^- \ \ \text{and} \ \ [\mathcal{P}U^*]_S \equiv \{\mathcal{P}U^*\}^+ - \{\mathcal{P}U^*\}^- \ \ \text{on} \ S.$$

In view of the equality $U^* = W^*(h)$, from (4.18) we get

$$W^*(h)(x) = W^*([W^*(h)]_S)(x) - V^*([\mathcal{P}W^*(h)]_S)(x), \ \ x \in \Omega^+ \cup \Omega^-.$$

Using the jump relation (4.4), we find

$$[U^*]_S = [W^*(h)]_S = \{W^*(h)\}^+ - \{W^*(h)\}^- = h.$$

Therefore

$$W^*(h)(x) = W^*(h)(x) - V^*([\mathcal{P}W^*(h)]_S)(x), \ \ x \in \Omega^+ \cup \Omega^-,$$

i.e., $V^*(\Phi^*)(x) = 0$ in $\Omega^+ \cup \Omega^-$, where $\Phi^* := [\mathcal{P}W^*(h)]_S$. With the help of the jump relation (4.5) finally we arrive at the equation

$$0 = \{\mathcal{P}V^*(\Phi^*)\}^- - \{\mathcal{P}V^*(\Phi^*)\}^+ =$$
$$= \Phi^* = [\mathcal{P}W^*(h)]_S = \{\mathcal{P}W^*(h)\}^+ - \{\mathcal{P}W^*(h)\}^-$$

on S, which completes the proof for the regular case.

The second part of the theorem can be proved by standard limiting procedure. $\qquad\square$

Let us consider the interior and exterior homogeneous Dirichlet BVPs for the adjoint operator $A^*(\partial)$

$$A^*(\partial)U^* = 0 \ \text{ in } \ \Omega^{\pm}, \tag{4.19}$$

$$\{U^*\}^{\pm} = 0 \ \text{ on } \ S. \tag{4.20}$$

In the case of the interior problem, we assume that either U^* is a regular vector of the class $[C^{1,\kappa'}(\overline{\Omega^+})]^6$ or $U^* \in [W_2^1(\Omega^+)]^6$, while in the case of the exterior problem, we assume that either $U^* \in [C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap Z^*(\Omega^-)$ or $U^* \in [W_{2,loc}^1(\Omega^-)]^6 \cap Z^*(\Omega^-)$.

Theorem 4.4. The interior and exterior homogeneous Dirichlet type BVPs (4.19)–(4.20) have only the trivial solution in the appropriate spaces.

Proof. First we treat the exterior Dirichlet problem. In view of the structure of the operator $A^*(\partial)$, it is easy to see that we can consider separately the BVP for the vector function $\widetilde{U}^* = (u^*, \varphi^*, \psi^*)^{\top}$, constructed by the first five components of the solution vector U^*,

$$\widetilde{A}^*(\partial)\widetilde{U}^*(x) = 0, \ \ x \in \Omega^-, \tag{4.21}$$

$$\{\widetilde{U}^*(x)\}^- = 0, \ \ x \in S, \tag{4.22}$$

where $\widetilde{A}^*(\partial)$ is the 5×5 matrix differential operator, obtained from $A^*(\partial)$ by deleting the sixth column and the sixth row,

$$\widetilde{A}^*(\partial) := \begin{bmatrix} [c_{kjrl}\partial_j\partial_l]_{3\times 3} & [-e_{jkl}\partial_j\partial_l]_{3\times 1} & [-q_{jkl}\partial_j\partial_l]_{3\times 1} \\ [e_{lrj}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l \\ [q_{lrj}\partial_j\partial_l]_{1\times 3} & a_{jl}\partial_j\partial_l & \mu_{jl}\partial_j\partial_l \end{bmatrix}_{5\times 5}. \tag{4.23}$$

With the help of Green's identity in $\Omega_R^- = B(0,R)\backslash\overline{\Omega^+}$, we have

$$\int\limits_{\Omega_R^-} \left[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) \right] dx =$$

$$= -\int\limits_S \{\widetilde{U}^*\}^- \cdot \{\widetilde{P}(\partial, n)\widetilde{U}^*\}^- dS + \int\limits_{\Sigma_R} \widetilde{U}^* \cdot \widetilde{P}(\partial, n)\widetilde{U}^* \, d\Sigma_R, \tag{4.24}$$

where

$$\widetilde{\mathcal{P}}(\partial, n) := \begin{bmatrix} [c_{rjkl}n_j\partial_l]_{3\times 3} & [-e_{lrj}n_j\partial_l]_{3\times 1} & [-q_{lrj}n_j\partial_l]_{3\times 1} \\ [e_{jkl}n_j\partial_l]_{1\times 3} & \varkappa_{jl}n_j\partial_l & a_{jl}n_j\partial_l \\ [q_{jkl}n_j\partial_l]_{1\times 3} & a_{jl}n_j\partial_l & \mu_{jl}n_j\partial_l \end{bmatrix}_{5\times 5}, \tag{4.25}$$

and

$$\widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) = c_{rjkl}\partial_l u_k^* \partial_j u_r^* + \varkappa_{jl}\partial_l \varphi^* \partial_j \varphi^* +$$
$$+ a_{jl}(\partial_l \varphi^* \partial_j \psi^* + \partial_j \psi^* \partial_l \varphi^*) + \mu_{jl}\partial_l \psi^* \partial_j \psi^*. \quad (4.26)$$

Due to the fact that U^* has the property $Z^*(\Omega^-)$, it follows that $\widetilde{U}^* = \mathcal{O}(|x|^{-1})$ and $\partial_j \widetilde{U}^* = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$, $j = 1, 2, 3$. Therefore,

$$\left| \int_{\Sigma_R} \widetilde{U}^* \cdot \widetilde{P}(\partial, n)\widetilde{U}^* \, d\Sigma_R \right| \le$$

$$\le \int_{\Sigma_R} \frac{C}{R^3} \, d\Sigma_R = \frac{C}{R^3} \, 4\pi R^2 = \frac{4\pi C}{R} \to 0 \text{ as } R \to \infty. \quad (4.27)$$

Taking into account that $\widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) \ge 0$, applying the relations (4.21), (4.22), and (4.27), from (4.24) we conclude that $\widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) = 0$. Hence in view of (2.10)-(2.11) it follows that $\widetilde{U}^* = (a \times x + b, b_4, b_5)$, where a and b are arbitrary constant vectors, and b_4 and b_5 are arbitrary scalar constants. Here the symbol $\times$ denotes the cross product operation. Due to the boundary condition (4.22) we get then $a = b = 0$ and $b_4 = b_5 = 0$, from which we derive that $\widetilde{U}^* = 0$. Since $\widetilde{U}^*$ vanishes in Ω^-, from (4.19)–(4.20) we arrive at the following boundary-value problem for ϑ^*,

$$\eta_{kj}\partial_k \partial_j \vartheta^* = 0 \text{ in } \Omega^-, \qquad\qquad (4.28)$$
$$\{\vartheta^*\}^- = 0 \text{ on } S.$$

From boundedness of ϑ^* at infinity and from (4.28) one can derive that $\vartheta^*(x) = C + \mathcal{O}(|x|^{-1})$, where C is an arbitrary constant. In view of $U^* \in Z^*(\Omega^-)$ we have $C = 0$ and $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$, $\partial_j \vartheta^*(x) = \mathcal{O}(|x|^{-2})$, $j = 1, 2, 3$. Therefore we can apply Green's formula

$$\int_{\Omega_R^-} \left[\vartheta^* \, \eta_{kj}\partial_k \partial_j \vartheta^* + \eta_{kj}\partial_k \vartheta^* \, \partial_j \vartheta^* \right] dx =$$

$$= - \int_S \{\vartheta^*\}^- \{\eta_{kj}n_k \partial_j \vartheta^*\}^- \, dS + \int_{\Sigma_R} \vartheta^* \, \eta_{kj}n_k \partial_j \vartheta^* \, d\Sigma_R.$$

Passing to the limit as $R \to \infty$, we get

$$\int_{\Omega^-} \eta_{kj}\partial_k \vartheta^* \partial_j \vartheta^* \, dx = 0.$$

Using the fact that the matrix $[\eta_{kj}]_{3\times 3}$ is positive definite, we conclude that $\vartheta^* = C_1 = const$ and since $\vartheta^*(x) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$, finally we get that $\vartheta^* = 0$ in Ω^-. Thus $U^* = 0$ in Ω^- which completes the proof for the exterior problem.

The interior problem can be treated quite similarly. $\qquad\square$

4.2. **Investigation of the interior Neumann BVP.** First let us treat
the uniqueness question. To this end we consider the homogeneous interior
Neumann-type BVP

$$A(\partial)U(x) = 0, \;\; x \in \Omega^+, \tag{4.29}$$

$$\big\{ \mathcal{T}(\partial, n)U(x) \big\}^+ = 0, \;\; x \in S = \partial\Omega^+. \tag{4.30}$$

It can be shown that a general solution to the problem (4.29)-(4.30) can be
represented in the form (for details see [20])

$$U = \sum_{k=1}^{9} C_k U^{(k)} \;\; \text{in} \;\; \Omega^+, \tag{4.31}$$

where C_k are arbitrary scalar constants and $\{U^{(k)}\}_{k=1}^{9}$ is the basis in the
space of solution vectors of the homogeneous problem (4.29)–(4.30). They
can be constructed explicitly and read as

$$U^{(k)} = \big(\widetilde{V}^{(k)}, 0\big)^\top, \;\; k = \overline{1,8}, \;\; U^{(9)} = \big(\widetilde{V}^{(9)}, 1\big)^\top, \tag{4.32}$$

where $U^{(k)} = (u^{(k)}, \varphi^{(k)}, \psi^{(k)}, \vartheta^{(k)})^\top$, $\widetilde{V}^{(k)} = (u^{(k)}, \varphi^{(k)}, \psi^{(k)})^\top$,

$$\widetilde{V}^{(1)} = (0, -x_3, x_2, 0, 0)^\top, \;\; \widetilde{V}^{(2)} = (x_3, 0, -x_1, 0, 0)^\top,$$

$$\widetilde{V}^{(3)} = (-x_2, x_1, 0, 0, 0)^\top, \;\; \widetilde{V}^{(4)} = (1, 0, 0, 0, 0)^\top,$$

$$\widetilde{V}^{(5)} = (0, 1, 0, 0, 0)^\top, \;\; \widetilde{V}^{(6)} = (0, 0, 1, 0, 0)^\top,$$

$$\widetilde{V}^{(7)} = (0, 0, 0, 1, 0)^\top, \;\; \widetilde{V}^{(8)} = (0, 0, 0, 0, 1)^\top,$$

and $\widetilde{V}^{(9)}$ is defined as

$$\widetilde{V}^{(9)} = (u^{(9)}, \varphi^{(9)}, \psi^{(9)})^\top, \;\; u_k^{(9)} = b_{kq}x_q, \;\; k = 1, 2, 3,$$

$$\varphi^{(9)} = c_q x_q, \;\; \psi^{(9)} = d_q x_q,$$

with the twelve coefficients $b_{kq} = b_{qk}$, c_q and d_q, $k, q = 1, 2, 3$, defined by
the uniquely solvable linear algebraic system of equations

$$c_{rjkl}b_{kl} + e_{lrj}c_l + q_{lrj}d_l = \lambda_{rj}, \;\; r, j = 1, 2, 3,$$

$$-e_{jkl}b_{kl} + \varkappa_{jl}c_l + a_{jl}d_l = p_j, \;\; j = 1, 2, 3,$$

$$-q_{jkl}b_{kl} + a_{jl}c_l + \mu_{jl}d_l = m_j, \;\; j = 1, 2, 3.$$

From (4.31) it follows that U can be alternatively written as

$$U = (\widetilde{V}, 0)^\top + b_6(\widetilde{V}^{(9)}, 1)^\top$$

with $\widetilde{V} = (a \times x + b, b_4, b_5)^\top$, where $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$
are arbitrary constant vectors and b_4, b_5, b_6 are arbitrary scalar constants.

   Now, we start the investigation of the non-homogeneous interior Neu-
mann-type BVP

$$A(\partial)U(x) = 0, \;\; x \in \Omega^+, \tag{4.33}$$

$$\big\{ \mathcal{T}(\partial, n)U(x) \big\}^+ = F(x), \;\; x \in S, \tag{4.34}$$

where $U \in [C^{1,\kappa'}(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ is a sought for vector and $F \in [C^{0,\kappa'}(S)]^6$ is a given vector-function. It is clear that if the problem (4.33)–(4.34) is solvable, then a solution is defined within a summand vector of type (4.31).

We look for a solution to the problem (4.33)–(4.34) in the form of the single layer potential,

$$U(x) = V(h)(x), \quad x \in \Omega^+, \tag{4.35}$$

where $h = (h_1, \ldots, h_6)^\top \in [C^{0,\kappa'}(S)]^6$ is an unknown density. From the boundary condition (4.34) and by virtue of the jump relation (2.21) (see Theorem 2.5) we get the following integral equation for the density vector h

$$[-2^{-1}I_6 + \mathcal{K}]h = F \quad \text{on} \ \ S, \tag{4.36}$$

where $\mathcal{K}$ is a singular integral operator defined by (2.24). Note that $-2^{-1}I_6 + \mathcal{K}$ is a singular integral operator of normal type with index zero (cf. [23]). Now we investigate the null space $\mathrm{Ker}(-2^{-1}I_6 + \mathcal{K})$. To this end, we consider the homogeneous equation

$$[-2^{-1}I_6 + \mathcal{K}]h = 0 \quad \text{on} \ \ S \tag{4.37}$$

and assume that a vector $h^{(0)}$ is a solution to (4.37), i.e., $h^{(0)} \in \mathrm{Ker}(-2^{-1}I_6 + \mathcal{K})$. Since $h^{(0)} \in [C^{0,\kappa'}(S)]^6$, it is evident that the corresponding single layer potential $U_0(x) = V(h^{(0)})(x)$ belongs to the space of regular vector functions and solves the homogeneous equation $A(\partial)U_0(x) = 0$ in Ω^+. Moreover, $\{\mathcal{T}(\partial, n)U_0(x)\}^+ = -2^{-1}h^{(0)} + \mathcal{K}h^{(0)} = 0$ on S due to (4.37), i.e., $U_0(x)$ solves the homogeneous interior Neumann problem. Therefore, in accordance to the above results, we can write $U_0(x) = \sum\limits_{k=1}^{9} C_k U^{(k)}(x)$ in Ω^+, where C_k, $k = \overline{1,9}$, are some constants, and the vectors $U^{(k)}(x)$ are defined by (4.32). Hence we have

$$V(h^{(0)})(x) = \sum_{k=1}^{9} C_k U^{(k)}(x), \quad x \in \Omega^+.$$

If we take into account the jump relation (2.20), we derive that

$$\{V(h^{(0)})(x)\}^+ \equiv \mathcal{H}(h^{(0)})(x) = \sum_{k=1}^{9} C_k U^{(k)}(x), \quad x \in S. \tag{4.38}$$

The operators

$$\mathcal{H} : [H^{-\frac{1}{2}}(S)]^6 \to [H^{\frac{1}{2}}(S)]^6,$$
$$: [C^{0,\kappa'}(S)]^6 \to [C^{1,\kappa'}(S)]^6$$

are invertible ([19], [23]). Therefore from (4.38) we obtain

$$h^{(0)} = \sum_{k=1}^{9} C_k h^{(k)}(x), \quad x \in S,$$

with

$$h^{(k)} := \mathcal{H}^{-1}(U^{(k)}), \quad k = \overline{1,9}. \tag{4.39}$$

Further we show that the system of vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent. Let us assume the opposite. Then there exist constants c_k, $k = \overline{1,9}$, such that $\sum\limits_{k=1}^{9} |c_k| \neq 0$ and the following equation

$$\sum_{k=1}^{9} c_k h^{(k)} = 0 \ \text{ on } \ S$$

holds, i.e., $\sum\limits_{k=1}^{9} c_k \mathcal{H}^{-1}(U^{(k)}) = 0$ on S. Hence we get

$$\mathcal{H}^{-1}\Big(\sum_{k=1}^{9} c_k U^{(k)} \Big) = 0 \ \text{ on } \ S,$$

and, consequently,

$$\sum_{k=1}^{9} c_k U^{(k)}(x) = 0, \quad x \in S. \tag{4.40}$$

Now consider the vector

$$U^*(x) \equiv \sum_{k=1}^{9} c_k U^{(k)}(x), \quad x \in \Omega^+.$$

Since the vectors $U^{(k)}$ are solutions of the homogeneous equation (4.33), in view of (4.40) we have

$$A(\partial)U^*(x) = 0, \quad x \in \Omega^+,$$

$$\{U^*(x)\}^+ = \Big\{ \sum_{k=1}^{9} c_k U^{(k)}(x) \Big\}^+ = 0, \quad x \in S.$$

That is, U^* is a solution of the homogeneous interior Dirichlet problem and in accordance with the uniqueness theorem for the interior Dirichlet BVP we conclude $U^*(x) = 0$ in Ω^+, i.e.,

$$\sum_{k=1}^{9} c_k U^{(k)}(x) = 0, \quad x \in \Omega^+.$$

This contradicts to linear independence of the system $\{U^{(k)}\}_{k=1}^{9}$. Thus, the system of the vectors $\{h^{(k)}\}_{k=1}^{9}$ is linearly independent which implies that

$$\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}) \geq 9.$$

Next we show that

$$\dim \operatorname{Ker}(-2^{-1}I_6 + \mathcal{K}) \leq 9.$$

Let the equation $(-2^{-1}I_6 + \mathcal{K})h = 0$ have a solution $h^{(10)}$ which is not representable in the form of a linear combination of the system $\{h^{(k)}\}_{k=1}^{9}$. Then

the system $\{h^{(k)}\}_{k=1}^{10}$ is linearly independent. It is easy to show that the system of the corresponding single layer potentials $V^{(k)}(x) := V(h^{(k)})(x)$, $k = \overline{1,10}$, $x \in \Omega^+$, is linearly independent as well. Indeed, let us assume the opposite. Then there are constants a_k, such that

$$U(x) := \sum_{k=1}^{10} a_k V^{(k)}(x) = 0, \ \ x \in \Omega^+, \tag{4.41}$$

with $\sum_{k=1}^{10} |a_k| \neq 0$. From (4.41) we then derive that $\{U(x)\}^+ = 0$, $x \in S$. Therefore,

$$\{U\}^+ = \sum_{k=1}^{10} a_k \{V^{(k)}\}^+ = \sum_{k=1}^{10} a_k \mathcal{H}(h^{(k)}) = \mathcal{H}\Big(\sum_{k=1}^{10} a_k h^{(k)}\Big) = 0 \ \ \text{on} \ \ S.$$

Whence, due to the invertibility of the operator $\mathcal{H}$, we get

$$\sum_{k=1}^{10} a_k h^{(k)} = 0 \ \ \text{on} \ \ S.$$

which contradicts to the linear independence of the system $\{h^{(k)}\}_{k=1}^{10}$.

Thus the system $\{V(h^{(k)})(x)\}_{k=1}^{10}$ is linearly independent.

On the other hand, we have

$$A(\partial)V^{(k)}(x) = 0, \ \ x \in \Omega^+,$$

$$\{\mathcal{T}V^{(k)}\}^+ = (-2^{-1}I_6 + \mathcal{K})h^{(k)} = 0, \ \ x \in S,$$

since $h^{(k)}$, $k = \overline{1,10}$, are solutions to the homogeneous equation (4.37). Therefore, the vectors $V^{(k)}$, $k = \overline{1,10}$, are solutions to the homogeneous interior Neumann-type BVP and they can be expressed by linear combinations of the vectors $U^{(j)}$, $j = \overline{1,9}$, defined in (4.32). Whence it follows that the system $\{V^{(k)}\}_{k=1}^{10}$ is linearly dependent and so is the system $\{h^{(k)}\}_{k=1}^{10}$ for an arbitrary solution $h^{(10)}$ of the equation (4.37). Consequently, $\dim \mathrm{Ker}(-2^{-1}I_6 + \mathcal{K}) \leq 9$ implying that $\dim \mathrm{Ker}(-2^{-1}I_6 + \mathcal{K}) = 9$. We can consider the system $h^{(1)}, \ldots, h^{(9)}$ defined in (4.39) as basis vectors of the null space of the operator $-2^{-1}I_6 + \mathcal{K}$. If h_0 is a particular solution to the nonhomogeneous integral equation (4.36), then a general solution of the same equation is represented as

$$h = h_0 + \sum_{k=1}^{9} c_k h^{(k)},$$

where c_k are arbitrary constants.

For our further analysis we need also to study the homogeneous interior Neumann-type BVP for the adjoint operator $A^*(\partial)$, which reads as follows

$$A^*(\partial)U^* = 0 \ \ \text{in} \ \ \Omega^+, \tag{4.42}$$

$$\{\mathcal{P}U^*\}^+ = 0 \ \ \text{on} \ \ S = \partial\Omega^+; \tag{4.43}$$

here the adjoint operator $A^*(\partial)$ and the boundary operator $\mathcal{P}$ are defined by (4.1) and (2.18) respectively.

Note that in the case of the problem (4.42)–(4.43) we get also two separated problems:

a) For the vector function $\widetilde{U}^* \equiv (u^*, \varphi^*, \psi^*)^\top$,

$$\widetilde{A}^*(\partial)\widetilde{U}^* = 0 \ \text{ in } \ \Omega^+, \tag{4.44}$$

$$\{\widetilde{\mathcal{P}}\widetilde{U}^*\}^+ = 0 \ \text{ on } \ S, \tag{4.45}$$

where $\widetilde{A}^*$ and $\widetilde{\mathcal{P}}$ are defined by (4.23) and (4.25) respectively, and

b) For the function $U_6^* \equiv \vartheta^*$

$$\lambda_{rj}\partial_j u_r^* + p_j \partial_j \varphi^* + m_j \partial_j \psi^* + \eta_{jl}\partial_j\partial_l\vartheta^* = 0 \ \text{ in } \ \Omega^+, \tag{4.46}$$

$$\eta_{jl}n_j\partial_l\vartheta^* = 0 \ \text{ on } \ S. \tag{4.47}$$

For a regular solution vector $\widetilde{U}^*$ of the problem (4.44)–(4.45) we can write the following Green's identity

$$\int\limits_{\Omega^+} \left[\widetilde{U}^* \cdot \widetilde{A}^*(\partial)\widetilde{U}^* + \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*) \right] dx = \int\limits_{\partial\Omega^+} \{\widetilde{U}^*\}^+ \cdot \{\widetilde{\mathcal{P}}(\partial, n)\widetilde{U}^*\}^+ \, dS, \tag{4.48}$$

where $\widetilde{\mathcal{E}}$ is given by (4.26). If we take into account the conditions (4.44)–(4.45), from (4.48) we get

$$\int\limits_{\Omega^+} \widetilde{\mathcal{E}}(\widetilde{U}^*, \widetilde{U}^*)\, dx = 0.$$

Hence we have that $\partial_j\varphi^* = 0$, $\partial_j\psi^* = 0$, $j = 1, 2, 3$, and $\partial_l u_k^* + \partial_j u_r^* = 0$ in Ω^+. Therefore, $u^*(x) = a \times x + b$ is a rigid displacement vector, $\varphi^* = b_4$ and $\psi^* = b_5$ are arbitrary constants in Ω^+. It is evident that

$$\lambda_{rj}\partial_j u_r^* = \frac{1}{2}\,\lambda_{rj}(\partial_j u_r^* + \partial_r u_j^*) = 0$$

and $p_j\partial_j\varphi^* = m_j\partial_j\psi^* = 0$. Then from (4.46)–(4.47) we get the following BVP for the scalar function ϑ^*,

$$\eta_{jl}\partial_j\partial_l\vartheta^* = 0 \ \text{ in } \ \Omega^+,$$

$$\eta_{jl}n_j\partial_l\vartheta^* = 0 \ \text{ on } \ S.$$

Using the following Green's identity

$$\int\limits_{\Omega^+} \eta_{jl}\partial_j\partial_l\vartheta^*\,\vartheta^*\, dx = -\int\limits_{\Omega^+} \eta_{jl}\partial_l\vartheta^*\,\partial_j\vartheta^*\, dx + \int\limits_{\partial\Omega^+} \{\eta_{jl}n_j\partial_l\vartheta^*\}^+\{\partial_j\vartheta^*\}^+\, dS,$$

we find

$$\int\limits_{\Omega^+} \eta_{jl}\partial_l\vartheta^*\,\partial_j\vartheta^*\, dx = 0,$$

and by the positive definiteness of the matrix $[\eta_{jl}]_{3\times 3}$ we get $\partial_j\vartheta^* = 0$, $j = \overline{1, 3}$, in Ω^+, i.e., $\vartheta^* = b_6 = const$ in Ω^+. Consequently, a general

solution $U^* = (u^*, \varphi^*, \psi^*, \vartheta^*)^\top$ of the adjoint homogeneous BVP (4.42)–(4.43) can be represented as

$$U^*(x) = \sum_{k=1}^{9} C_k U^{*(k)}(x), \quad x \in \Omega^+,$$

where C_k are arbitrary scalar constants, while

$$U^{*(1)} = (0, -x_3, x_2, 0, 0, 0)^\top, \quad U^{*(2)} = (x_3, 0, -x_1, 0, 0, 0)^\top,$$
$$U^{*(3)} = (-x_2, x_1, 0, 0, 0, 0)^\top, \quad U^{*(4)} = (1, 0, 0, 0, 0, 0)^\top,$$
$$U^{*(5)} = (0, 1, 0, 0, 0, 0)^\top, \quad U^{*(6)} = (0, 0, 1, 0, 0, 0)^\top, \tag{4.49}$$
$$U^{*(7)} = (0, 0, 0, 1, 0, 0)^\top, \quad U^{*(8)} = (0, 0, 0, 0, 1, 0)^\top,$$
$$U^{*(9)} = (0, 0, 0, 0, 0, 1)^\top.$$

As we see, $U^{*(k)} = U^{(k)}$, $k = \overline{1,8}$, where $U^{(k)}$, $k = \overline{1,8}$, is given in (4.32). One can easily check that the system $\{U^{*(k)}\}_{k=1}^{9}$ is linearly independent. As a result we get the following

Proposition 4.5. The space of solutions of the adjoint homogeneous BVP (4.42)–(4.43) is nine dimensional and an arbitrary solution can be represented as a linear combination of the vectors $\{U^{*(k)}\}_{k=1}^{9}$, i.e., the system $\{U^{*(k)}\}_{k=1}^{9}$ is a basis in the space of solutions to the homogeneous BVP (4.42)–(4.43).

Now, we return to equation (4.36) and consider the corresponding homogeneous adjoint equation

$$(-2^{-1}I_6 + \mathcal{K}^*)h^* = 0 \ \text{ on } \ S,$$

where $\mathcal{K}^*$ is the adjoint operator to $\mathcal{K}$ defined by the duality relation,

$$(\mathcal{K}h, h^*)_{L_2(S)} = (h, \mathcal{K}^* h^*)_{L_2(S)}, \quad \forall h, h^* \in [L_2(S)]^6.$$

It is easy to show that the operator $\mathcal{K}^*$ is the same as the operator given by (4.6). In what follows we prove that $\dim \operatorname{Ker} \left(-\frac{1}{2} I_6 + \mathcal{K}^* \right) = 9$.

Indeed, in accordance with Proposition 4.5 we have that $A^*(\partial)U^{*(k)} = 0$ in Ω^+ and $\{\mathcal{P}U^{*(k)}\}^+ = 0$ on S. Therefore from (4.7) we have

$$U^{*(k)}(x) = W^*\big(\{U^{*(k)}\}^+\big)(x), \quad x \in \Omega^+. \tag{4.50}$$

By the jump relations (4.4) we get

$$h^{*(k)} = 2^{-1} h^{*(k)} + \mathcal{K}^* h^{*(k)} \ \text{ on } \ S,$$

where

$$h^{*(k)} := \{U^{*(k)}\}^+, \quad k = \overline{1,9}. \tag{4.51}$$

Whence it follows that

$$\big(-2^{-1} I_6 + \mathcal{K}^* \big) h^{*(k)} = 0, \quad k = \overline{1,9}.$$

By Theorem 4.4 and the relations (4.50) and (4.51) we conclude that the system $\{h^{*(k)}\}_{k=1}^9$ is linearly independent, and therefore

$$\dim \operatorname{Ker}\left(-2^{-1} I_6 + \mathcal{K}^*\right) \geq 9.$$

Now, let $h^{*(0)} \in \operatorname{Ker}\left(-2^{-1} I_6 + \mathcal{K}^*\right)$, i.e., $\left(-2^{-1} I_6 + \mathcal{K}^*\right) h^{*(0)} = 0$. The corresponding double layer potential $U_0^*(x) := W^*(h^{*(0)})(x)$ is a solution to the homogeneous equation $A^*(\partial) U_0^* = 0$ in Ω^+. Moreover, $\{W^*(h^{*(0)})\}^- = -2^{-1} h^{*(0)} + \mathcal{K}^* h^{*(0)} = 0$ on S. Consequently, U_0^* is a solution of the homogeneous exterior Dirichlet BVP possessing the property $Z^*(\Omega^-)$. With the help of the uniqueness Theorem 4.4 we conclude that $W^*(h^{*(0)}) = 0$ in Ω^-. Further, $\{\mathcal{P} W^*(h^{*(0)})\}^+ = \{\mathcal{P} W^*(h^{*(0)})\}^- = 0$ due to Theorem 4.3, and for the vector function U_0^* we arrive at the following BVP,

$$A^*(\partial) U_0^* = 0 \ \text{ in } \ \Omega^+,$$

$$\{\mathcal{P} U_0^*\}^+ = 0 \ \text{ on } \ S.$$

Using Proposition 4.5 we can write

$$U_0^*(x) = W^*(h^{*(0)})(x) = \sum_{k=1}^9 c_k U^{*(k)}(x), \ \ x \in \Omega^+,$$

where c_k are some constants. The jump relation for the double layer potential then gives

$$\left\{W^*(h^{*(0)})(x)\right\}^+ - \left\{W^*(h^{*(0)})(x)\right\}^-$$

$$= h^{*(0)}(x) = \sum_{k=1}^9 c_k \left\{U^{*(k)}(x)\right\}^+ = \sum_{k=1}^9 c_k h^{*(k)}(x), \ \ x \in S,$$

which implies that the system $\{h^{*(k)}\}_{k=1}^9$ represents a basis of the null space $\operatorname{Ker}\left(-2^{-1} I_6 + \mathcal{K}^*\right)$. Whence it follows that $\dim \operatorname{Ker}\left(-2^{-1} I_6 + \mathcal{K}^*\right) = 9$.

Now we can formulate the following basic existence theorem for the integral equation (4.36) and the interior Neumann-type BVP.

Theorem 4.6. Let $m \geq 0$ be a nonnegative integer and $0 < \kappa' < \kappa \leq 1$. Further, let $S \in C^{m+1,\kappa}$ and $F \in [C^{m,\kappa'}(S)]^6$. The necessary and sufficient conditions for the integral equation (4.36) and the interior Neumann-type BVP (4.33)–(4.34) to be solvable read as

$$\int_S F(x) \cdot h^{*(k)}(x) \, dS = 0, \ \ k = \overline{1,9}, \tag{4.52}$$

where the system $\{h^{*(k)}\}_{k=1}^9$ is defined explicitly by (4.51) and (4.49).

If these conditions are satisfied, then a solution vector to the interior Neumann-type BVP is representable by the single layer potential (4.35), where the density vector $h \in [C^{m,\kappa'}(S)]^6$ is defined by the integral equation (4.36).

A solution vector function $U \in [C^{m+1,\kappa'}(\overline{\Omega^+})]^6$ is defined modulo a linear combination of the vector functions $\{U^{(k)}\}_{k=1}^9$ given by (4.32).

Remark 4.7. Similar to the exterior problem, if S is a Lipschitz surface, $F \in \left[H^{-1/2}(S)\right]^6$, and the conditions (4.52) is fulfilled, then

(i) the integral equation (4.36) is solvable in the space $\left[H^{-1/2}(S)\right]^6$;

(ii) the interior Neumann-type BVP (4.33)-(4.34) is solvable in the space $\left[H_2^1(\Omega^+)\right]^6$ and solutions are representable by the single layer potential (4.35), where the density vector $h \in \left[H^{-1/2}(S)\right]^6$ solves the integral equation (4.36);

(iii) A solution $U \in \left[H_2^1(\Omega^+)\right]^6$ to the interior Neumann-type BVP (4.33)-(4.34) is defined modulo a linear combination of the vector functions $\left\{U^{(k)}\right\}_{k=1}^9$ given by (4.32).

Acknowledgements

References

1. M. S. Agranovich, Elliptic singular integro-differential operators. (Russian) *Usp. Mat. Nauk* **20** (1965), No. 5(125), 3–120.

2. M. Avellaneda and G. Harshe, Magnetoelectric effect in piezoelectric / magnetostrictive multilayer (2-2) composites. *J. Intelligent Mater. Syst. Struct.* **5** (1994), 501–513.

3. Y. Benveniste, Magnetoelectric effect in fibrous composites with piezoelectric and piezomagnetic phases. *Phys. Rev. B* **51** (1995), 424–427.

4. W. F. Jr. Brown, Magnetoelastic interaction, *Springer, New York*, 1966.

5. T. Buchukuri, O. Chkadua, D. Natroshvili, and A.-M. Sändig, Solvability and regularity results to boundary-transmission problems for metallic and piezoelectric elastic materials. *Math. Nachr.* **282** (2009), No. 8, 1079–1110.

6. T. Buchukuri, O. Chkadua, D. Natroshvili, and A.-M. Sändig, Interaction problems of metallic and piezoelectric materials with regard to thermal stresses. *Mem. Differential Equations Math. Phys.* **45** (2008), 7–74.

7. T. Buchukuri, O. Chkadua, and D. Natroshvili, Mixed boundary value problems of thermopiezoelectricity for solids with interior cracks, *Integral Equations and Operator Theory.* **64** (2009), No. 4, 495–537.

8. M. Costabel, Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19** (1988), No. 3, 613–626.

9. R. Duduchava, The Green formula and layer potentials, *Integral Equations and Operator Theory.* **41** (2001), No. 2, 127–178.

10. R. Duduchava, D. Natroshvili, and E. Shargorodsky, Boundary value problems of the mathematical theory of cracks. *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **39** (1990), 68–84.

11. R. Duduchava, D. Natroshvili, and E. Shargorodsky, Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts. I. *Georgian Math. J.* **2** (1995), No. 2, 123–140; II. *Georgian Math. J.* **2** (1995), No. 3, 259–276.

12. G. C. Hsiao and W. L. Wendland, Boundary integral equations. Applied Mathematical Sciences, 164. *Springer-Verlag, Berlin*, 2008.

13. L. JENTSCH AND D. NATROSHVILI, Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies. I. *Mem. Differential Equations Math. Phys.* **17** (1999), 7–126.

14. L. JENTSCH AND D. NATROSHVILI, Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies. II. *Mem. Differential Equations Math. Phys.* **18** (1999), 1–50.

15. V. D. KUPRADZE, T. G. GEGELIA, M. O. BASHELEISHVILI, AND T. V. BURCHULADZE, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. Translated from the second Russian edition. Edited by V. D. Kupradze. *North-Holland Series in Applied Mathematics and Mechanics,* 25. *North-Holland Publishing Co., Amsterdam-New York*, 1979.

16. J. Y. LI, Uniqueness and reciprocity theorems for linear thermo-electro-magneto-elasticity. *Quart. J. Mech. Appl. Math.* **56** (2003), No. 1, 35–43.

17. W. MCLEAN, Strongly elliptic systems and boundary integral equations. *Cambridge University Press, Cambridge*, 2000.

18. S. G. MIKHLIN AND S. PRÖSSDORF, Singular integral operators. Translated from the German by Albrecht Bottcher and Reinhard Lehmann. *Springer-Verlag, Berlin*, 1986.

19. M. MREVLISHVILI, Investigation of interior and exterior Dirichlet BVPs of thermo-electro-magneto elasticity theory and regularity results. *Bull. TICMI* (to appear).

20. M. MREVLISHVILI, D. NATROSHVILI, AND Z. TEDIASHVILI, Uniqueness theorems for thermo-electro-magneto-elasticity problems. *Bull. Greek Math. Soc.* **57** (2010), 281–295.

21. C. W. NAN, Magnetoelectric effect in composites of piezoelectric and piezomagnetic phases. *Phys. Rev. B* **50** (1994), 6082–6088.

22. D. NATROSHVILI, Boundary integral equation method in the steady state oscillation problems for anisotropic bodies. *Math. Methods Appl. Sci.* **20** (1997), No. 2, 95–119.

23. D. NATROSHVILI, Mathematical problems of thermo-electro-magneto-elasticity (to appear).

24. D. NATROSHVILI, T. BUCHUKURI, AND O. CHKADUA, Mathematical modelling and analysis of interaction problems for piezoelastic composites. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)* **30** (2006), 159–190.

25. D. NATROSHVILI, O. CHKADUA, AND E. SHARGORODSKY, Mixed problems for homogeneous anisotropic elastic media. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **39** (1990), 133–181.

26. D. G. NATROSHVILI, A. YA. DJAGMAIDZE, AND M. ZH. SVANADZE, Some problems in the linear theory of elastic mixtures. (Russian) With Georgian and English summaries. *Tbilis. Gos. Univ., Tbilisi*, 1986.

27. W. NOWACKIĬ, Electromagnetic effects in solids. (Russian) Translated from the Polish and with a preface by V. A. Shachnev. *Mechanics: Recent Publications in Foreign Science*, 37. "Mir", Moscow, 1986.

28. R. T. SEELEY, Singular integrals and boundary value problems. *Amer. J. Math.* **88** (1966), 781–809.

29. H. TRIEBEL, Interpolation theory, function spaces, differential operators. *North-Holland Mathematical Library,* 18. *North-Holland Publishing Co., Amsterdam–New York*, 1978.

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