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LONG-TIME BEHAVIOR OF EVOLUTION INCLUSION WITH NON-DAMPED IMPULSIVE EFFECTS

Dedicated to Professor I.T. Kiguradze on the occasion of his birthday

Abstract. In this paper we consider an evolution inclusion with impulse effects at fixed moments of time from the point of view of the theory of global attractors. For an upper semicontinuous multivalued term which does not provide the uniqueness of the Cauchy problem, we give sufficient conditions on non-damped multivalued impulse perturbations, which allow us to construct a multivalued non-autonomous dynamical system and prove for it the existence of a compact global attractor in the phase space.

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რეზიუმე. გლობალურ ატრაქტორთა მეთოდით გამოკვლეულია ევოლუციური ჩართვა იმპულსური ზემოქმედებით დროის ფიქსირებულ მომენტებში. ზემოდან ნახევრადუწყვეტი მრავალსახა მარჯვენა მხარისათვის არამილევადი მრავალსახა იმპულსურ შეშფოთებაზე დადებულ გარკვეულ შეზღუდვებში აგებულია არაავტონომიური დინამიური სისტემა და დამტკიცებულია ფაზურ სივრცეში მისი კომპაქტური გლობალური ატრაქტორის არსებობა.

Introduction

One of the possible ways for the description of qualitative behavior of the solutions of evolution problem is the proving of the existence in a phase space of the problem of invariant attracting set, a global attractor. In contrast to finite-dimensional problems, in the case of infinite-dimensional situation of the dissipativity condition of a system does not ensure the existence of the compact attractor, and the resolving of this problem is based essentially on one-parameter semigroups apparatus. This approach was founded in the seventies of the past century by J. Hale and O. A. Ladyzhenskaya. It was then developed by J. Hale [7], [8] for autonomous infinite-dimensional systems generated by equations with delay, but his abstract results concerning the existence of global attractors of dynamical systems mostly coincided with the results due to O. A. Ladyzhenskaya [17], [18], which have been gained in studying the dynamics of solutions of a two-dimensional system of Navier–Stokes equations.

The essence of these results is based on the fact that for the given evolution problem

$$\begin{cases} \partial_t u(t) = F(u(t)), \\ u(0) = u_0 \in E, \end{cases} \tag{1}$$

for which, as is known, it is globally and uniquely solvable in some class W, and $u(t) \in E \ \forall\, t \in \Im_+$, where $\Im$ is a nontrivial semigroup of the additive group $\mathbb{R}$, $\Im_+ = \Im \cap [0, +\infty)$, the one-parametric family of mappings $\{V(t, \cdot\,) : E \mapsto E\}_{t \in \Im_+}$ is constructed, where

$$V(t, u_0) := \big\{ u(t) \,|\ \ u(\,\cdot\,) \text{ is the solution of (1)} \big\}. \tag{2}$$

On the strength that the problem (1) is autonomous, the family of mappings (2) is a semidynamical system, for which the invariant, compact, attracting set in the phase space is found – a global attractor, which is minimal among closed attracting sets and maximal among invariant compact sets.

In the papers of J. Hale [7], [8], O. A. Ladyzhenskaya [17], [18], M. I. Vishik [1], R. Temam [26] and of other mathematicians the existence and properties of global attractors were established in many nonlinear equations of mathematical physics.

Owing to these works, the theory of global attractors of dynamical systems has became almost completed and for a wide class of autonomous well-posed evolution dissipative problems it gives response to the question about the existence of a global attractor, its connectedness, stability, robustness, regularity, structure and dimension.

At the same time, a large class of autonomous problems was left aside, for which there is a global solvability theorem in phase space and there is no uniqueness theorem or it hasn't proved yet. These are the three-dimensional Navier–Stokes system, the three-dimensional Benard system, the system of equations of chemical kinetics under general conditions on parameters, wave equations in the case of nonlinearity of general polynomial form, evolution nonlinear equations with non-Lipschitz function of interface, as well as an evolution inclusion that arises while investigating evolution equations with discontinuous coefficients. The problem of studying dynamics of systems with possible nonuniqueness of a solution was solved in two ways. G. R. Sell [25], M. I. Vishik [5] suggested the concept of a trajectory attractor, in the context of which the dynamical system is constructed in the space of trajectories on the basis of a shift operator. For that (already a single-valued) dynamical system one can find an attracting set, a trajectory attractor. But it is important to note that in the course of this approach the connection with the system‘s phase space has been lost. Another approach proposed in the papers due to J. M. Ball [2], V. S. Melnik [19], [20], assumed a possible nonuniqueness of the solution by introducing a multivalued analogue of the one-parameter semigroup (2).

Let us assume that the problem (1) is globally solved in the class W, $u(t) \in E\ \forall\, t \in \Im_+$. Then correctly defined (multivalued in the general case) is a family of mappings $\{G(t,\,\cdot\,) : E \mapsto 2^E\}_{t\in\Im_+}$, where

$$G(t,u_0) := \big\{u(t)|\ \ u(\,\cdot\,) \in W \text{ is the solution of (1)}\big\}. \tag{3}$$

The family of mappings (3) showing that the conditions

$$\begin{cases} G(0,x) = x & \forall\, x \in E, \\ G(t+s,x) \subset G(t,G(s,x)) & \forall\, x \in E,\ \ t,s \in \Im_+, \end{cases}$$

are fulfilled, is called an m-semiflow.

The global attractor of the m-semiflow in the phase space E is called a compact set Ξ which satisfies the following conditions:

1) $\forall\, t \in \Im_+ \Xi \subset G(t,\Xi)$ (semiinvariance),

2) for any bounded $B \subset E\, dist(G(t,B),\Xi) \to 0,\ t \to +\infty$ (attraction).

As it turned out, the mappings of type (3) occur naturally in the evolution equations without the uniqueness of a solution and also in evolution inclusions. For most of them, the existence of a global attractor was proved.

Eventually, the apparatus of global attractors of one-parameter semigroups turned out to be not an easy-to-use for research of the qualitative behavior of evolution systems, but it admits the generalization of nonautonomous systems. In [4] by V. V. Chepyzhov and M. I. Vishik, such type of generalization was realized by introducing an additional parameter, that was responsible for non-autonomous terms. Moreover, the application for equations with almost periodic in time right-hand part, as well as cascade systems were examined.

This scheme has been generalized in the case of ambiguous solvability by O. V. Kapustyan, V. S. Melnik, J. Valero[10]. The main idea of this approach consists in that for the problem

$$\begin{cases} \partial_t u(t) = F_{\sigma(t)}(u(t)), \\ u(\tau) = u_\tau \in E, \end{cases} \tag{4}$$

it is assumed that a non-autonomous term $\sigma(t)$ belongs to some space Σ, where $\{T(h) : \Sigma \mapsto \Sigma\}_{h\in\Im_+}$ is a semigroup, $\forall\,\sigma \in \Sigma$, $\tau \in \Im$, $u_\tau \in E$, the problem (4) is expected to be globally solvable in some class W_τ, $u(t) \in E$ $\forall\, t \geq \tau$. Thus we can correctly define the mapping (possibly multivalued):

$$U_\sigma(t,\tau,u_\tau) := \big\{u(t)|\ \ u(\,\cdot\,) \in W \text{ is the solution of (4)}\big\}. \tag{5}$$

It describes the dynamics of solutions of problems (4). If the following conditions are fulfilled for (5), $\forall\,\sigma \in \Sigma$

$$\begin{cases} U_\sigma(\tau,\tau,u_\tau) = u_\tau, \\ U_\sigma(t,\tau,u_\tau) \subset U_\sigma(t,s,U_\sigma(s,\tau,u_\tau)) & \forall\, t \geq s \geq \tau, \\ U_\sigma(t+h,\tau+h,u_\tau) \subset U_{T(h)\sigma}(t,\tau,u_\tau) & \forall\, h \in \Im_+, \end{cases}$$

then the family of mappings (5) is called a family of m-processes, for which the global attractor is determined in the phase space E as a compact set Θ_Σ, for which the conditions below are fulfilled:

1) for any bounded $B \subset E$ $\forall\,\tau \in R$ $dist(U_\Sigma(t,\tau,B),\Theta_\Sigma) \to 0$, $t \to +\infty$,

2) Θ_Σ is minimal in a class of closed sets, which satisfies 1).

As it turned out, the dynamics of many classes of evolution problems can be described in terms of global attractors of m-processes. Random ambiguously solvable dynamical systems and evolution inclusions with non-autonomous right-hand part were investigated with the exception of the above-mentioned equations with almost periodic right-hand part and cascade systems. Consequently, such an essential non-autonomous object as evolution equations with impulses perturbations at fixed moments, can likewise be described in terms of non-autonomous dynamical processes. The existence of global attractors for evolution equations with impulsive effects was, for the first time, obtained in [11], [12], but only in the case of damped impulsive effects, that is, when values of impulsive perturbations tend to zero. This fact is essentially used in proving of the existence of global attractor, because in reality it is proved that every element of global attractor belongs to some trajectory of a non-perturbed evolution problem.

In the present article, relying on the theory of impulsive differential equations [24], the authors prove that the evolution inclusion with translation-compact perturbations at fixed moments [13] generates a multivalued dynamical system for which there exists the compact global attractor.

Global Attractors of Multivalued Processes

Let (X,ρ) be a metric space, $\Im_d = \{(t,\tau) \in \Im^2|\ \ t \geq \tau\}$, $P(X)$ be a set of all non-empty subsets of X, $\beta(X)$ be a set of all non-empty, bounded subsets of X, and Σ be some metric space, for which the semigroup $\{T(h) : \Sigma \mapsto \Sigma\}_{h\in\Im_+}$ is defined.

Definition 1. We say that the family of multivalued processes (MP) is defined, $\{U_\sigma : \Im_d \times X \mapsto P(X)\}_{\sigma\in\Sigma}$ $\forall\,\sigma \in \Sigma$, if the following conditions are fulfilled:

1) $U_\sigma(\tau,\tau,x) = x \quad \forall\, x \in X,\, \forall\,\tau \in \Im,$

2) $U_\sigma(t,\tau,x) \subseteq U_\sigma(t,s,U_\sigma(s,\tau,x)) \quad \forall\, t \geq s \geq \tau,\, \forall\, x \in X,$

3) $U_\sigma(t+h,\tau+h,x) \subseteq U_{T(h)\sigma}(t,\tau,x) \quad \forall\, t \geq \tau,\, \forall\, h \in \Im_+,$

where for $A \subset X$, $B \subset \Sigma$ $U_B(t,s,A) = \bigcup\limits_{\sigma\in B}\bigcup\limits_{x\in A} U_\sigma(t,s,x)$.

Definition 2. The compact set $\Theta_\Sigma \subset X$ is called a global attractor of the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ if the following conditions are fulfilled:

1) Θ_Σ is a uniformly attracting set, i.e. $\forall\,\tau \in \mathbb{R}$, $\forall\, B \in \beta(X)$

$$dist(U_\Sigma(t,\tau,B),\Theta_\Sigma) \to 0, \quad t \to +\infty; \tag{6}$$

2) Θ_Σ is a minimal set in the class of all closed uniformly attracting sets.

Theorem 1. *Let the family* MP $\{U_\sigma\}_{\sigma\in\Sigma}$ *satisfy the following conditions:*

1) $\exists\, B_0 \in \beta(X)$ $\forall\, B \in \beta(X)$ $\forall\,\tau \in \Im$ $\exists\, T = T(B,\tau)$ $\forall\, t \geq T$ $U_\Sigma(t,\tau,B) \subset B_0$*;*

2) $\forall\, B \in \beta(X)$ $\forall\,\tau \in \Im$ $\forall\, t_n \to +\infty$ *any* $\xi_n \in U_\Sigma(t_n,\tau,B)$ *is precompact in* X*.*

Then there exists Θ_Σ *which is the global attractor of* MP $\{U_\sigma\}_{\sigma\in\Sigma}$*. If, moreover,* $\forall\, h \in \Im_+$ $T(h)\Sigma = \Sigma$ *and in condition* 3) *from Definition* 1 *the equality is fulfilled, then it suffices to check only the conditions* 1)*,* 2) *from the theorem for* $\tau = 0$*.*

Proof. For any $B \in \beta(X)$, $\tau \in \Im$, let us consider a set

$$\omega_\Sigma(\tau,B) = \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s} U_\Sigma(t,\tau,B)}. \tag{7}$$

Under the condition 2) we find in a standard way that $\omega_\Sigma(\tau,B) \neq \varnothing$ is a compact, attracting set B, i.e.,

$$dist\big(U_\Sigma(t,\tau,B),\omega_\Sigma(\tau,B)\big) \to 0, \quad t \to +\infty,$$

and it is a minimal closed set possessing this property. Then the set

$$\Theta_\Sigma = cl_X\Big(\bigcup_{\tau\in\Im}\ \bigcup_{B\in\beta(X)} \omega_\Sigma(\tau,B)\Big) \tag{8}$$

satisfies the conditions 1), 2) from Definition 2.

Let us prove its compactness. Since $\forall\, B \in \beta(X)$ $\forall\,\tau \in \Im$ $\exists\, T = T(B,\tau)$ $\forall\, t \geq T$ $U_\Sigma(t,\tau,B) \subset B_0$, therefore $\forall\, p \in \Im_+$

$$\begin{aligned} U_\Sigma(t+p,\tau,B) &\subset U_\Sigma\big(t+p,t,U_\Sigma(t,\tau,B)\big) \subset \\ &\subset U_\Sigma(t+p,t,B_0) \subset U_{T(t)\Sigma}(p,0,B_0) \subset U_\Sigma(p,0,B_0). \end{aligned}$$

Thus $\forall\, s \geq T$, $\forall\, p \in \Im_+$

$$\bigcup_{t' \geq s+p} U_\Sigma(t', \tau, B) \subset U_\Sigma(p, 0, B_0).$$

Then $\forall\, s' \in \Im_+$

$$\bigcup_{p \geq s'} \bigcup_{t' \geq s+p} U_\Sigma(t', \tau, B) \subset \bigcup_{p \geq s'} U_\Sigma(p, 0, B_0),$$

$$cl_X\Big(\bigcup_{t' \geq s+s'} U_\Sigma(t', \tau, B)\Big) \subset cl_X\Big(\bigcup_{p \geq s'} U_\Sigma(p, 0, B_0)\Big),$$

$$\bigcap_{s' \geq 0} cl_X\Big(\bigcup_{t' \geq s+s'} U_\Sigma(t', \tau, B)\Big) \subset \bigcap_{s' \geq 0} cl_X\Big(\bigcup_{p \geq s'} U_\Sigma(p, 0, B_0)\Big),$$

$$\bigcap_{s'' \geq s} cl_X\Big(\bigcup_{t' \geq s''} U_\Sigma(t', 0, B)\Big) \subset \omega_\Sigma(0, B_0).$$

Thereby, $\omega_\Sigma(\tau, B) \subset \omega_\Sigma(0, B_0)$, hence $\Theta_\Sigma = \omega_\Sigma(0, B_0)$, and the desired compactness is proved. The second part of the theorem follows from the following inclusions: if $\tau \geq 0$

$$U_\Sigma(t, \tau, B) \subset U_{T(\tau)\Sigma}(t-\tau, 0, B) \subset U_\Sigma(t-\tau, 0, B);$$

if $\tau < 0$

$$U_\Sigma(t, \tau, B) = U_{T(-\tau)\Sigma}(t, \tau, B) = U_\Sigma(t-\tau, 0, B).$$

The theorem is proved. □

The Statement of the Impulsive Problem and the Properties of Solutions

Given a triplet $V \subset H \subset V^*$ of Hilbert spaces with a compact and dense embedding, $\langle \cdot\,, \cdot \rangle$ is a canonical duality between V and V^*. Let us denote by $\|\cdot\|$ and $(\,\cdot\,, \cdot\,)$ the norm and the scalar product in the space H, $\|\cdot\|_V$ is a norm in the space V. Assume that the inequality $\|u\|^2 \leq \alpha\|u\|_V^2$ is fulfilled.

We consider a linear continuous operator $A : V \to V^*$, which for the constants $\lambda_1 > 0, \lambda_2 > 0$ satisfies the following conditions:

$$\forall\, u \in V \;\; \langle Au, u\rangle \geq \lambda_1 \|u\|_V^2, \tag{9}$$

$$\forall\, u, v \in V \;\; |\langle Au, v\rangle| \leq \lambda_2^{\frac{1}{2}} \langle Au, u\rangle^{\frac{1}{2}} \|v\|_V. \tag{10}$$

From the condition (9), we obtain the estimate $|\langle Au, v\rangle| \leq \lambda_2 \|u\|_V \|v\|_V$. Then using Lax–Milgram's lemma, we have that $\exists\, A^{-1} \in L(V^*, V)$ and, moreover, $\|A^{-1}\| \leq \frac{1}{\lambda_1}$, $\|A\| \leq \lambda_2$.

Suppose that the multivalued perturbation $F : H \mapsto P(H)$ satisfies the conditions

$$\forall\, y \in H \;\; F(y) \text{ is convex, closed, bounded subset of } H; \tag{11}$$

F is w-upper semicontinuous (w-u.s.), and has no more than linear growth, i.e.

$$\forall \varepsilon > 0 \ \ \forall y_0 \in H \ \ \exists \delta > 0 \quad y \in O_\delta(y_0), \ \ F(y) \subset O_\varepsilon(F(y_0)); \tag{12}$$

$$\exists C \geq 0 \ \ \forall y \in H \quad \|F(y)\|_+ \leq C(1 + \|y\|). \tag{13}$$

Here, for $B \subset H$, we denote $\|B\|_+ = \sup\limits_{b \in B} \|b\|$.

Consider the problem

$$\begin{cases} \dfrac{dy}{dt} + Ay \in F(y) + h(t), \quad t > \tau, \\ y(\tau) = y_\tau, \end{cases} \tag{14}$$

where $\tau \in \mathbb{R}$, $y_\tau \in H$, the operator A and the multivalued function F satisfy the conditions (9), (10), (11)–(13), $h \in L^2_{loc}(\mathbb{R}, H)$.

Definition 3. By the solution of the problem (14) on (τ, T) is meant the function $y \in L^2(\tau, T; V)$ with $\frac{dy}{dt} \in L^2(\tau, T; V^*)$ such that there exists $f \in L^2(\tau, T; H)$, $f(t) \in F(y(t))$ almost everywhere (a.e.), and

$$\begin{cases} \dfrac{dy}{dt} + Ay = f(t) + h(t), \\ y(\tau) = y_\tau. \end{cases} \tag{15}$$

It is known [6] that for all $\tau \in \mathbb{R}$, $T > \tau$, $y_\tau \in H$ under the conditions (9), (10), (11)–(13) the problem (14) has at least one solution and, moreover, any solution of problem (14) belongs to the space $C([\tau; T]; H)$. Thus, there is a reason to speak about global solvability of (14) on $(\tau, +\infty)$.

For the problem (14), we formulate the following impulsive problem: at fixed time moments $\{\tau_i\}_{i \in Z}$, $\tau_{i+1} - \tau_i \geq \gamma > 0$, every solution of the problem (14) in the phase space H undergoes impulsive perturbation of the form:

$$y(\tau_i + 0) - y(\tau_i) \in g(y(\tau_i)) + \Psi_i, \ \ i \in \mathbb{Z}, \tag{16}$$

where $g : H \mapsto H$ is the given function and $\Psi_i \subset H$ are the given sets.

Then $\forall \tau \in [\tau_i, \tau_{i+1})$, $\forall y_\tau \in H$, the Cauchy problem for (14), (16) is globally solvable in the sense that $\forall y_\tau \in H$ there exists the function $y(\cdot)$, which is the solution of (14) on (τ, τ_{i+1}), $(\tau_{i+1}, \tau_{i+2}), \ldots$, $y(\tau) = y_\tau$, and at the time moments $\{\tau_i, \tau_{i+1}, \ldots\}$, the function $y(\cdot)$ satisfies the relation (16) and is left-continuous.

Let us define some properties of the solution for the problem (14), (16). Towards this end, we consider an auxiliary problem

$$\begin{cases} \dfrac{dy}{dt} + Ay = f(t), \\ y(\tau) = y_\tau. \end{cases} \tag{17}$$

It is known [3], [26] that the problem (17) under the conditions (9), (10) for any $y_\tau \in H$, $T > \tau$, $f \in L^2(\tau, T; H)$ has a unique solution in the Hilbert space

$$W(\tau,T)=\Big\{y|\ \ y\in L^2(\tau,T;V),\ \frac{dy}{dt}\in L^2(\tau,T;V^*)\Big\},$$

which is denoted by $y=I(f,y_\tau)$. Moreover, the function $t\mapsto\|y(t)\|$ is absolutely continuous on $[\tau,T]$ and a.e. on (τ,T) the equality

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|^2+\langle Ay(t),y(t)\rangle=(f(t),y(t)) \tag{18}$$

is valid.

Lemma 1. *We have a sequence of problems* (17) *with right-hand parts* $f_n\in L^2(\tau,T;H)$ *and initial datas* $y_\tau^n\in H$. *Let* $f_n\xrightarrow{w} f$ *in* $L^2(\tau,T;H)$, $y_\tau^n\xrightarrow{w} y_\tau$ *in* H. *Then* $y_n=I(f_n,y_\tau^n)\to y=I(f,y_\tau)$ *in* $C([\delta,T];H)$ $\forall\,\delta\in(\tau,T)$. *If* $y_\tau^n\to y_\tau$ *in* H, *then* $y_n\to y$ *in* $C([\tau,T];H)$.

Proof. From (18), we have an estimation for $\tau\le s\le t\le T$,

$$\|y_n(t)\|^2+2\lambda_1\int_s^t\|y_n(p)\|_V^2\,dp\le\|y_n(s)\|^2+2\int_s^t(f_n(p),y_n(p))\,dp. \tag{19}$$

From (19), due to the boundedness of $\{f_n\}$ in $L^2(\tau,T;H)$, the boundedness of $\{y_\tau^n\}$ in H and (7), we have that $\exists\,M>0\ \forall\,n\ge1$,

$$\sup_{t\in[\tau,T]}\|y_n(t)\|+\int_\tau^T\|y_n(p)\|_V^2\,dp+\int_\tau^T\Big\|\frac{dy_n}{dt}\Big\|_{V^*}^2\,dp\le M. \tag{20}$$

Hence there exists $y\in W(\tau,T)$ such that $y_n\xrightarrow{w} y$ in $W(\tau,T)$. Then under the compactness of the embedding $W(\tau,T)\subset L^2(\tau,T;H)$, we obtain $y_n\to y$ in $L^2(\tau,T;H)$, and it means that $y_n(t)\to y(t)$ in H for almost all $t\in(\tau,T)$, and, besides, $y_n(t_n)\xrightarrow{w} y(t_0)$ in H $\ \forall\,t_n\to t_0\in[\tau,T]$. Hence, in particular, $y=I(f,y_\tau)$.

Let us now consider the functions

$$J_n(t)=\|y_n(t)\|^2-2\int_\tau^t(f_n(p),y_n(p))\,dp,\quad J(t)=\|y(t)\|^2-2\int_\tau^t(f(p),y(p))\,dp.$$

These functions under (19) are monotonous non-increasing, continuous, and $J_n(t)\to J(t)$ a.e. on (τ,T). Then $J_n(t)\to J(t)$ in $C([\delta,T])$ $\ \forall\,\delta\in(\tau,T)$.

Let

$$\max_{t\in[\delta,T]}\|y_n(t)-y(t)\|=\|y_n(t_n)-y(t_n)\|$$

and on some subsequence $t_n\to t_0$.

Thus, under (20),

$$\int_{t_0}^{t_n}\big|(f_n(p),y_n(p))\big|\,dp\le M\int_{t_0}^{t_n}\|y_n(p)\|\,dp\to0,\ \ n\to+\infty.$$

Then

$$\int_{\tau}^{t_n} (f_n(p), y_n(p))\, dp \longrightarrow \int_{\tau}^{t_0} (f(p), y(p))\, dp.$$

Hence, under the weak convergence of $y_n(t_n)$ to $y(t_0)$, we have a system of inequalities

$$J(t_0) \le \underline{\lim} \|y_n(t_n)\|^2 - 2\int_{\tau}^{t_0} (f(p), y(p))\, dp \le$$

$$\le \overline{\lim} \|y_n(t_n)\|^2 - 2\int_{\tau}^{t_0} (f(p), y(p))\, dp \le \overline{\lim} J_n(t_n) = J(t_0).$$

It follows that there exists $\lim\limits_{n\to+\infty} \|y_n(t_n)\| = \|y(t_0)\|$ such that $y_n(t_n) \to y(t_0)$ in H. Hence, on some subsequence, $y_n \to y$ in $C([\delta, T]; H)$. Since (17) has a unique solution, the convergence goes along the whole sequence.

If $y_\tau^n \to y_\tau$, then $J_n(\tau) \to J(\tau)$, hence $J_n \to J$ in $C([\tau, T])$ and, similarly to the previous arguments, we obtain $y_n \to y$ in $C([\tau, T]; H)$. The lemma is proved. □

The following lemma provides us with the sufficient conditions of dissipativity for the impulsive problem (14), (16).

Lemma 2. *Let the conditions*

$$\sup_{i\in Z} \|\Psi_i\|_+ < \infty, \tag{21}$$

$$\exists\, D > 0 \ \ \forall\, u \in H \quad \|g(u)\| \le D(1 + \|u\|), \tag{22}$$

$$\|h\|_+^2 := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|h(s)\|^2\, ds < \infty, \tag{23}$$

$$-\frac{2\lambda_1}{\alpha} + 2C + \frac{1}{\gamma} \ln(1 + (D+1)^2) < 0 \tag{24}$$

be fulfilled. Then

$$\begin{gathered} \exists\, R > 0 \ \ \textit{such that} \ \forall\, r \ge 0 \ \ \forall\, y_0 \in H, \ \ \|y_0\| \le r, \\ \textit{and for any solution} \ \ y(\cdot) \ \ \textit{of the problem (14), (16) on} \ \ (0, +\infty) \\ \textit{with} \ \ y(0) = y_0, \ \ \exists\, T = T(r) \ \ \textit{such that} \ \forall\, t \ge T, \ \ \|y(t)\| \le R. \end{gathered} \tag{25}$$

Proof. From the inequality

$$\frac{d}{dt}\|y(t)\|^2 + \frac{2\lambda_1}{\alpha}\|y(t)\|^2 \le 2C\|y(t)\|^2 + 2C\|y(t)\| + 2\|h(t)\|\,\|y(t)\| \tag{26}$$

and under the condition (24), for a.a. t we have the estimation

$$\frac{d}{dt}\|y(t)\|^2 + \delta\|y(t)\|^2 \le C_1\big(\|h(t)\|^2 + 1\big), \tag{27}$$

where the constants $\delta = \frac{2\lambda_1}{\alpha} - 2C > 0$, $C_1 > 0$ depend only on the constants of the problem (14), (16). Moreover, taking (16) into account, we have

$$\Big| \|y(\tau_i+0)\|^2 - \|y(\tau_i)\|^2 \Big| \le (D+1)^2 \|y(\tau_i)\|^2 + C_2,$$

where the constant $C_2 > 0$ depends only on the constants of the problem (14), (16). It turns out that the function $t \mapsto \|y(t)\|^2$ is the solution of the impulsive problem

$$\begin{cases} \dfrac{d}{dt} \|y(t)\|^2 + \delta \|y(t)\|^2 \le C_1\big(\|h(t)\|^2 + 1\big), \\ \|y(\tau_i+0)\|^2 - \|y(\tau_i)\|^2 \le (D+1)^2 \|y(\tau_i)\|^2 + C_2, \end{cases}$$

and the solutions of this problem at every moment cannot exceed the solutions of the problem

$$\begin{cases} \dfrac{d}{dt} x(t) + \delta x(t) = C_1\big(\|h(t)\|^2 + 1\big), \\ x(\tau_i+0) - x(\tau_i) = (D+1)^2 x(\tau_i) + C_2. \end{cases} \tag{28}$$

For every $x_0 \in \mathbb{R}$, the solution $x(\cdot)$ of the problem (28) with $x(0) = x_0$ is defined by the formula [24]

$$x(t) = e^{-\delta t}\big(1 + (D+1)^2\big)^{i(t,0)} \cdot x_0 + \\ + \int_0^t C_1\big(\|h(p)\|^2 + 1\big) e^{-\delta(t-p)} \big(1 + (D+1)^2\big)^{i(t,p)}\, dp + \\ + C_2 \sum_{0 \le \tau_i < t} e^{-\delta(t-\tau_i)} \big(1 + (D+1)^2\big)^{i(t,\tau_i)},$$

where $i(t,s)$ is a number of points τ_i on $[s,t)$.

By the condition (24), $\exists\, \mu > 0$ such that

$$-\delta + \frac{1}{\gamma} \ln(1 + (D+1)^2) \le -\mu < 0,$$

and $\forall\, t > 0$, we have the inequality

$$\int_0^t \|h(s)\|^2 e^{-\mu(t-s)}\, ds \le \\ \le \int_{t-1}^t \|h(s)\|^2\, ds + e^{-\mu} \int_{t-2}^{t-1} \|h(s)\|^2\, ds + e^{-2\mu} \int_{t-3}^{t-2} \|h(s)\|^2\, ds + \cdots \le \\ \le \|h\|_+^2 (1 - e^{-\mu})^{-1},$$

then for $x_0 = \|y(0)\|^2$, it is easy to get an estimation for all $t \ge 0$

$$\|y(t)\|^2 \le x(t) \le e^{-\mu t} \|y(0)\|^2 + M,$$

from which follows the condition (25). The lemma is proved. □

The Construction of the Semigroup of Translations for Impulsive Systems with Nondamped Perturbations

Let us begin with the presentation of the concept of translation-compact functions [5]. Let $(\mathbb{M}, \rho_{\mathbb{M}})$ be a complete metric space. We consider the space $\mathbb{C}(\mathbb{R}; \mathbb{M})$ of continuous functions from $\mathbb{R}$ to $\mathbb{M}$ with topology of uniform convergence on the compacts, i.e.,

$$\sigma_n \to \sigma \text{ in } \mathbb{C}(\mathbb{R}; \mathbb{M}) \iff \\ \iff \forall\, [t_1, t_2] \subset \mathbb{R}, \quad \max_{t \in [t_1, t_2]} \rho_{\mathbb{M}}(\sigma_n(t), \sigma(t)) \to 0, \quad n \to \infty.$$

The defined topology can be described by using the metric, and with this metric $\mathbb{C}(\mathbb{R}; \mathbb{M})$ will be the complete metric space.

For the fixed $\sigma(\cdot) \in \mathbb{C}(\mathbb{R}; \mathbb{M})$, define the set

$$H(\sigma) := cl_{\mathbb{C}(\mathbb{R};\mathbb{M})}\big\{\sigma(t + \cdot)|\ \ t \in \mathbb{R}\big\}.$$

Definition 4. The function $\sigma(\cdot) \in \mathbb{C}(\mathbb{R}; \mathbb{M})$ is called a translation-compact function (tr.-c.) in $\mathbb{C}(\mathbb{R}; \mathbb{M})$ if $H(\sigma)$ is compact in $\mathbb{C}(\mathbb{R}; \mathbb{M})$.

The concept of the translation-compactness, as the form of generalization of almost periodicity, was presented in [5]. In this paper, an example of translation-compact but not almost periodic function is given.

Lemma 3 ([5])**.** *If $\sigma \in \mathbb{C}(\mathbb{R}; \mathbb{M})$ is tr.-c. function in $\mathbb{C}(\mathbb{R}; \mathbb{M})$, then*

1) *any $\sigma_1(\cdot) \in H(\sigma)$ is also tr.-c. in $\mathbb{C}(\mathbb{R}; \mathbb{M})$, $H(\sigma_1) \subseteq H(\sigma)$;*

2) $\exists\, R > 0\ \forall\, \sigma_1(\cdot) \in H(\sigma)\ \sup_{s \in \mathbb{R}} \rho_{\mathbb{M}}(\sigma_1(s), 0) \le R$;

3) *the translation group $\{T(t)\}_{t \in \mathbb{R}}$, $T(t)\sigma(s) = \sigma(t + s)$, for any $t \in \mathbb{R}$ is continuous in the topology $\mathbb{C}(\mathbb{R}; \mathbb{M})$, and $T(t)H(\sigma) = H(\sigma)$.*

Let us consider the space $L^{2,w}_{loc}(\mathbb{R}; H)$, that is, the space $L^2_{loc}(\mathbb{R}; H)$ with a local weak convergence topology, i.e.,

$$\sigma_n \to \sigma \text{ in } L^{2,w}_{loc}(\mathbb{R}; H) \iff \forall\, [t_1, t_2] \subset \mathbb{R}\ \ \forall\, \eta \in L^2(t_1, t_2; H), \\ \int_{t_1}^{t_2} (\sigma_n(t) - \sigma(t), \eta(t))\, dt \to 0, \quad n \to \infty.$$

In the same way as above, for the function $\sigma \in L^{2,w}_{loc}(\mathbb{R}; H)$ we consider the set

$$H(\sigma) := cl_{L^{2,w}_{loc}(\mathbb{R};H)}\big\{\sigma(t + s)|\ \ t \in \mathbb{R}\big\}.$$

Definition 5. The function $\sigma(\cdot) \in L^{2,w}_{loc}(\mathbb{R}; H)$ is to be called translation-compact (tr.-c.) in $L^{2,w}_{loc}(\mathbb{R}; H)$, if $H(\sigma)$ is compact in $L^{2,w}_{loc}(\mathbb{R}; H)$.

Lemma 4 ([5])**.** *The function $\sigma \in L^{2,w}_{loc}(\mathbb{R}; H)$ is tr.-c. in $L^{2,w}_{loc}(\mathbb{R}; H) \Leftrightarrow \|\sigma\|^2_+ < \infty$.*

Lemma 5 ([5]). *If $\sigma \in L^{2,w}_{loc}(\mathbb{R}; H)$ is tr.-c. in $L^{2,w}_{loc}(\mathbb{R}; H)$, then*

1) *any $\sigma_1(\cdot) \in H(\sigma)$ is also tr.-c. in $L^{2,w}_{loc}(\mathbb{R}; H)$, $H(\sigma_1) \subseteq H(\sigma)$;*

2) *$\forall\, \sigma_1(\cdot) \in H(\sigma)$ $\|\sigma_1\|^2_+ \leq \|\sigma\|^2_+$;*

3) *the translation group $\{T(t)\}_{t\in\mathbb{R}}$, $T(t)\sigma(s) = \sigma(t+s)$, for any $t \in \mathbb{R}$ is continuous in the topology $L^{2,w}_{loc}(\mathbb{R}; H)$, and $T(t)H(\sigma) = H(\sigma)$.*

We proceed to the construction of translation-compact distribution as the generalization of almost periodic distribution [24].

Consider the separable Banach space

$$D = \big\{\varphi \in \mathbb{C}^1(\mathbb{R}) \mid \ D^1\varphi \text{ is absolutely continuous on } \mathbb{R},\ D^j\varphi \in L^1(\mathbb{R}),\ j = 0, 1, 2\big\}$$

with the norm

$$|\varphi|_D := \max_{j=0,1,2} \Big\{ \int_{-\infty}^{+\infty} |\ D^j\varphi(t)|\, dt \Big\}.$$

Let $(X, \|\cdot\|)$ be the Banach space. We consider a subset of the space $L(D, X)$ of all linear continuous operators from D into X for fixed $K > 0$:

$$W_K = \big\{h \in L(D, X) \mid \ \|h\|_{L(D,X)} \leq K\big\}.$$

Lemma 6. *There exists the function ρ_{W_K} on W_K for which the following conditions are fulfilled:*

1) *(W_K, ρ_{W_K}) is the complete metric space;*

2) *$\rho_{W_K}(A_n, A) \to 0 \Longleftrightarrow \forall\, \varphi \in D$ $A_n\varphi \to A\varphi$;*

3) *$\rho_{W_K}(A_1, A_2) \leq L\|A_1 - A_2\|_{L(D,X)}$.*

Proof. Let $\{x_i\}$ be a dense set in D. There is

$$\rho_{W_K}(A, B) = \sum_{i=1}^{\infty} \alpha_i \frac{\|Ax_i - Bx_i\|}{1 + \|Ax_i - Bx_i\|}$$

for $\alpha_i > 0$, $\sum\limits_{i=1}^{\infty} \alpha_i < \infty$, the metric is determined in W_K, and the condition 2) is fulfilled. Moreover, in this formula we always can choose numbers $\{\alpha_i\}$ such that the inequality $\sum\limits_{i=1}^{\infty} \alpha_i \|x_i\| < \infty$ holds. Let us now prove that (W_K, ρ_{W_K}) is a complete metric space.

Indeed, if $\rho_{W_K}(A_n, A_m) \to 0$, then $A_n\varphi - A_m\varphi \to 0$ for any $\varphi \in D$. We put $A\varphi := \lim A_n\varphi$, then A is linear. Thus, $\|A\varphi\| \leq K|\varphi|_D$ under $\|A_n\varphi\| \leq K|\varphi|_D$, so $A \in W_K$. Since $\sum\limits_{i=1}^{\infty} \alpha_i\|x_i\| < \infty$, it follows that $\rho_{W_K}(A, B) \leq L\|A - B\|_{L(D,X)}$. The lemma is proved. □

Next, for any $s \in \mathbb{R}$, we consider the map $T(s) : W_K \mapsto W_K$ such that

$$(T(s)h)\varphi(\cdot) = h\varphi(\cdot - s) \ \ \forall\, h \in W_K, \ \ \forall\, \varphi \in D.$$

It is easy to find that $T(s)W_K = W_K \ \forall\, s \in \mathbb{R}$ and $\{T(s)\}$ is a continuous group in W_K.

Definition 6. The element $h \in W_K$ is called a translation-compact distribution if the function $T(\cdot)h : \mathbb{R} \mapsto W_K$ is translation-compact in $\mathbb{C}(\mathbb{R}; W_K)$.

Here, the set

$$\Sigma_K = cl_{W_K}\big\{T(s)h \mid \ s \in \mathbb{R}\big\} \tag{29}$$

is called a minimal flow which is generated by $h \in W_K$.

Lemma 7. *If $h \in W_K$ is the translation-compact distribution, then Σ_K is compact in W_K and $T(s)\Sigma_K = \Sigma_K$ for any $s \in \mathbb{R}$. If for $h \in W_K$, the mapping $T(\,\cdot\,)h : \mathbb{R} \mapsto W_K$ is uniformly continuous in $\mathbb{R}$ and Σ_K is compact in W_K, then h is the translation-compact distribution.*

Let the sequences $\{f_i\}_{i\in\mathbb{Z}} \subset X$, $\{t_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ be given, and the following conditions be fulfilled:

$$\begin{gathered} \sup_{i\in\mathbb{Z}} \|f_i\| \le K, \ \ \{f_i\}_{i\in\mathbb{Z}} \text{ is precompact in } X, \\ t_i = ai + c_i \ \text{ for } \ a > 0, \ \ \sup_{i\in\mathbb{Z}} |c_i| < \infty, \ \ t_{i+1} - t_i \ge \gamma > 0. \end{gathered} \tag{30}$$

Then $h \in L(D, X)$ is determined by $h = \sum\limits_i f_i \delta_{t_i}$, $h\varphi = \sum\limits_i f_i \varphi(t_i)$ and

$$\|h\varphi\| \le \Big\| \sum_i f_i \varphi(t_i) \Big\| \le K \sum_i |\varphi(t_i)| \, \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \le \frac{2K}{\gamma} \, |\varphi|_D \, .$$

The last inequality is a consequence of the following lemma.

Lemma 8. *If $\varphi \in D$, then the inequality*

$$\sum_i \|\varphi(t_i)\| (t_{k+1} - t_k) \le \int\limits_{\mathbb{R}} \big(\|\varphi(t)\| + \|\varphi'(t)\|\big)\, dt$$

holds.

Proof. The lemma can be considered as already proven if the inequality

$$|\varphi(t)|(t_{k+1} - t_k) \le \int\limits_{t_k}^{t_{k+1}} \big(|\varphi(s)| + |\varphi'(s)|\big)\, ds$$

holds for $k \in \mathbb{Z}$, where $t \in [t_k, t_{k+1}]$.

Summing the following inequalities

$$\int_{t_k}^{t} \varphi'(s)(s-t_k)\,ds = \varphi(t)(t-t_k) - \int_{t_k}^{t} \varphi(s)\,ds,$$

$$\int_{t}^{t_{k+1}} \varphi'(s)(s-t_{k+1})\,ds = -\varphi(t)(t-t_k) + (t_{k+1}-t_k)\varphi(t) - \int_{t}^{t_{k+1}} \varphi(s)\,ds,$$

we get

$$\varphi(t)(t_{k+1}-t_k) = \int_{t_k}^{t_{k+1}} \varphi(s)\,ds + \int_{t_k}^{t} \varphi'(s)(s-t_k)\,ds + \int_{t}^{t_{k+1}} \varphi'(s)(s-t_k-1)\,ds.$$

So,

$$\begin{aligned}
&|\varphi(t)|(t_{k+1}-t_k) \le \\
&\le \int_{t_k}^{t_{k+1}} |\varphi(s)|ds + (t-t_k)\int_{t_k}^{t} |\varphi'(s)|\,ds + (t_k+1-t)\int_{t}^{t_{k+1}} |\varphi'(s)|\,ds \le \\
&\le \int_{t_k}^{t_{k+1}} |\varphi(s)|\,ds + (t-t_k)\int_{t_k}^{t_{k+1}} |\varphi'(s)|\,ds + (t_k+1-t)\int_{t_k}^{t_{k+1}} |\varphi'(s)|\,ds \le \\
&\qquad \le \int_{t_k}^{t_{k+1}} \big(|\varphi(s)| + |\varphi'(s)|\big)\,ds.
\end{aligned}$$

The lemma is proved. □

Denote $W = W_{\frac{2K}{\gamma}}$, $\Sigma = \Sigma_{\frac{2K}{\gamma}}$. Under the conditions that $\{f_i\}_{i\in\mathbb{Z}}$ is precompact, and $\{c_i\}_{i\in\mathbb{Z}}$ is bounded, we can use the following property: for any sequence of integers $\{m_n\}$ there exist sequences $\{m_k\}$ and $\{\widetilde{f}_i\}_{i\in Z} \subset X$, $\{\widetilde{c}_i\}_{i\in Z} \subset \mathbb{R}$ such that for all $i \in \mathbb{Z}$,

$$\|f_{i+m_k} - \widetilde{f}_i\| \to 0, \;\; |c_{i+m_k} - \widetilde{c}_i| \to 0, \;\; k \to \infty. \tag{31}$$

As is known [24], the uniform with respect to $i \in \mathbb{Z}$ convergence in (31) characterizes almost periodic sequences.

Theorem 2. *Let the conditions* (30) *be fulfilled. Then* $h = \sum_i f_i\delta_{t_i}$ *is the translation-compact distribution, and for any* $g \in \Sigma$*, the representation* $g = \sum_i l_i\delta_{\tau_i}$ *holds, and also the sequences* $\{l_i\} \subset X$, $\{\tau_i\} \subset \mathbb{R}$ *satisfy the condition* (30)*. Moreover, if* $g^n = \sum_i l_i^n\delta_{\tau_i^n} \longrightarrow g = \sum_i l_i\delta_{\tau_i}$ *in* Σ*, then* $l_i^n \to l_i$ *in* X*,* $\tau_i^n \to \tau_i$ *in* $\mathbb{R}$ $\forall\, i \in \mathbb{Z}$.

Proof. At our first step, we prove that the mapping $h = \sum_i f_i \delta_{t_i}$ is the translation-compact distribution, if and only if the function $F(t) = \sum_i f_i \varphi(t - t_i)$ is translation-compact in $\mathbb{C}(\mathbb{R}; X)$ for any $\varphi \in D$.

$$\big\|(T(s)h)\varphi - (T(t)h)\varphi\big\| \leq \Big\| \sum_i (f_i \varphi(t_i - s) - f_i \varphi(t_i - t))\Big\| \leq$$
$$\leq \frac{C}{\gamma} |t - s| \sum_i \big\|\varphi'(t_i^*)(t_i - t_{i-1})\big\|$$

for $t > s$, where $t_i^* \in [t_i - t, t_i - s]$. Here, without loss of generality, we assume that $t - s < \gamma$. Then for an arbitrary number i, $t - s < t_i - t_{i-1}$ and $t_i^* \in [t_i - t, t_i - s] \subset [t_{i-1} - s, t_i - s]$. Relying on the proof of Lemma 8, we have

$$|\varphi'(t_i^*)|(t_i - t_{i-1}) \leq \int_{t_{i-1}-s}^{t_i - s} \big(|\varphi'(r)| + |\varphi''(r)|\big)\, dr.$$

Thus,

$$\big\|(T(s)h)\varphi - (T(t)h)\varphi\big\| \leq \frac{2C}{\gamma} |t - s|\, |\varphi|_D,$$

so,

$$\|T(s)h - T(t)h\|_{L(D,X)} \leq \frac{2C}{\gamma} |t - s|,$$
$$\|F(s) - F(t)\| \leq \frac{2C}{\gamma} |t - s|\, |\varphi|_D,$$

and also, the functions $F(\cdot)$, $T(\cdot)h$ are uniformly continuous in $\mathbb{R}$. If h is the translation-compact distribution, then $\{T(s)h| \ \ s \in \mathbb{R}\}$ is precompact in W. Thus, on the basis of Lemma 7, we find that $\{F(s)| \ \ s \in \mathbb{R}\}$ is precompact in X for any $\varphi \in D$, and also, the mapping F is translation-compact in $\mathbb{C}(\mathbb{R}; X)$.

Inversely, let F be the translation-compact in $\mathbb{C}(\mathbb{R}; X)$. We choose $\{\varphi_j\}_{j \geq 1} \subset D$, *supp* $\varphi_j \subset [-\frac{1}{j}, \frac{1}{j}]$, $\varphi_j \geq 0$, $\int_{-\infty}^{+\infty} \varphi_j(t)\, dt = 1$, and consider the mapping F_j which is defined as follows:

$$F_j \varphi = \int_{-\infty}^{+\infty} \sum_i f_i \varphi_j(t - t_i) \varphi(t)\, dt \quad \forall\, \varphi \in D.$$

Then

$$\|F_j\varphi\| = \left\| \int\limits_{-\infty}^{+\infty} \varphi_j(t) \sum_i f_i \varphi(t+t_i)\, dt \right\| \le$$
$$\le C \int\limits_{-\infty}^{\infty} \varphi_j(t) \sum_i |\varphi(t+t_i)|\, dt \le C \sum_i \left|\varphi(\theta_i^j + t_i)\right|,$$

where $\theta_i^j \in [-\frac{1}{j}, \frac{1}{j}]$. Here, without loss of generality, we assume that $\frac{1}{j} < \gamma$. Hence, we have

$$\|F_j\varphi\| \le C \sum_i \left|\varphi(\theta_i^j + t_i)\right| \le \frac{C}{2\gamma} \sum_i |\varphi(\theta_i^j + t_i)|(t_{i+1} - t_{i-1}) \le$$
$$\le \frac{C}{2\gamma} \sum_i \int\limits_{t_{i-1}}^{t_{i+1}} \big(|\varphi(s)| + |\varphi'(s)|\big)\, ds \le \frac{2C}{\gamma}\, |\varphi|_D,$$

by virtue of Lemma 8, i.e. $F_j \in W$. Let us show that F_j is the translation-compact distribution. We start with

$$\big\|(T(t')F_j)\varphi - (T(t'')F_j)\varphi\big\| =$$
$$= \left\| \int\limits_{-\infty}^{+\infty} \sum_i f_i \varphi_j(t-t_i) \big(\varphi(t-t') - \varphi(t-t'')\big)\, dt \right\| \le$$
$$\le C \int\limits_{-\infty}^{+\infty} \varphi_j(t) \sum_i \big|\varphi(t+t_i-t') - \varphi(t+t_i-t'')\big|\, dt \le$$
$$\le C \sum_i \big|\varphi(t^*_{i,j} - t') - \varphi(t^*_{i,j} - t'')\big|,$$

where $t^*_{i,j} \in [t_i - \frac{1}{j}, t_i + \frac{1}{j}]$. Then

$$\big\|(T(t')F_j)\varphi - (T(t'')F_j)\varphi\big\| \le C|t'-t''| \sum_i \big|\varphi'(\theta_i^j))\big|,$$

holds for $t'' < t'$, where

$$\theta_i^j \in [t^*_{i,j} - t', t^*_{i,j} - t''] \subset \left[t_i - \frac{1}{j} - t', t_i + \frac{1}{j} - t''\right] \subset$$
$$\subset \left[t_i - \frac{1}{j} - t', t_i + \frac{1}{j} - t' + |t'-t''|\right] \subset [t_{i-1} - t', t_{i+1} - t'].$$

Hence, if $\frac{1}{j} < \gamma/2$, $|t' - t''| < \gamma/2$, we have the estimation

$$\begin{aligned}&\big\|(T(t')F_j)\varphi - (T(t'')F_j)\varphi\big\| \leq \\ &\qquad \leq \frac{C}{2\gamma}\,|t'-t''|\sum_i \big|\varphi'(\theta_i^j)\big|(t_{i+1}-t_{i-1}) \leq \frac{2C}{\gamma}\,|t'-t''|\,|\varphi|_D\end{aligned}$$

to be fulfilled. Thus, we have proved that $T(\cdot)F_j$ is uniformly continuous. It remains to prove that $\{T(s)F_j| \;\; s \in \mathbb{R}\}$ is a precompact set in W. Let $s_n \to \infty$ be an arbitrary sequence. Since the function $F_j(t) = \sum\limits_i f_i\varphi_j(t-t_i)$ is translation-compact in $\mathbb{C}(\mathbb{R}; X)$, there exists the subsequence (denoted as $\{s_n\}$), and when $R > 0$, the statement

$$\sup_{|t|\leq R}\big\|F_j(t-s_n) - F_j(t-s_m)\big\| \to 0, \;\; n, m \to \infty$$

holds. Note that on the basis of diagonal method we can use the general subsequence s_n for all φ_j Since for all $\varphi \in D$, $\varepsilon > 0$ there exists $R > 0$, and also $\int\limits_{|t|>R} |\varphi(t)|\,dt < \varepsilon$, hence

$$\Bigg|\int\limits_{|t|>R}\sum_i f_i\varphi_j(t+s_n-t_i)\Bigg|\varphi(t)\,|dt| \leq \frac{2C}{\gamma}\,|\varphi_j|_D\int\limits_{|t|>R}|\varphi(t)|\,dt < C(j)\varepsilon.$$

Then

$$\begin{aligned}&\big\|(T(s_n)F_j)\varphi - (T(s_m)F_j)\varphi\big\| = \\ &= \Bigg\|\int\limits_{-\infty}^{\infty}\sum_i f_i\varphi_j(t-t_i)\varphi(t-s_n)\,dt - \int\limits_{-\infty}^{\infty}\sum_i f_i\varphi_j(t-t_i)\varphi(t-s_m)\,dt\Bigg\| = \\ &\quad = \Bigg\|\int\limits_{-\infty}^{\infty}\Big(\sum_i f_i\varphi_j(t+s_n-t_i) - \sum_i f_i\varphi_j(t+s_m-t_i)\Big)\varphi(t)\,dt\Bigg\| \leq \\ &\qquad \leq \Bigg\|\int\limits_{-R}^{R}\sum_i f_i(\varphi_j(t+s_n-t_i) - \varphi_j(t+s_m-t_i))\varphi(t)\,dt\Bigg\| + 2C(j)\varepsilon.\end{aligned}$$

That's why for all $\varepsilon > 0$, $j \geq 1$, $\varphi \in D$ there exists $N = N(\varepsilon, j, \varphi)$ such that $\forall\, m, n \geq N$

$$\big\|(T(s_n)F_j)\varphi - (T(s_m)F_j)\varphi\big\| < \varepsilon.$$

Hence, the set $\{T(s)F_j| \ \ s\in\mathbb{R}\}$ is precompact in W. Relying on Lemma 8, for all $\varphi\in D$

$$\Big\|\int_{-\infty}^{\infty}\sum_i f_i\varphi_j(t-t_i)\varphi(t)\,dt-\sum_i f_i\varphi(t_i)\Big\|=$$

$$=\Big\|\int_{-\infty}^{\infty}\sum_i f_i\varphi_j(t-t_i)(\varphi(t)-\varphi(t_i)\,dt\Big\|\le$$

$$\le C\sum_i\int_{t_i-\frac1j}^{t_i+\frac1j}\frac1j\max_{\theta\in[t_i-\frac1j,t_i+\frac1j]}|\varphi'(\theta)|\varphi_j(t-t_i)\,dt\le\frac{2C}{\gamma}\,\frac1j\,|\varphi|_D,$$

i.e. $\|F_j-h\|_{L(D;X)}\le\frac{2C}{\gamma}\,\frac1j$. Then for all $\varphi\in D$ and $\varepsilon>0$, there exist $j(\varepsilon,\varphi)$ and $N(j,\varepsilon,\varphi)$ such that for any $n,m>N$,

$$\begin{aligned}&\big\|(T(s_n)h)\varphi-(T(s_m)h)\varphi\big\|\le\\&\qquad\le\big\|T(s_n)h-T(s_n)F_j\big\|_{L(D;X)}|\varphi|_D+\\&+\big\|T(s_m)h-T(s_m)F_j\big\|_{L(D;X)}|\varphi|_D+\big\|(T(s_n)F_j)\varphi-(T(s_m)F_j)\varphi\big\|\le\\&\qquad\le\frac{4C}{\gamma}\,\frac1j\,|\varphi|_D+\big\|(T(s_n)F_j)\varphi-(T(s_m)F_j)\varphi\big\|<\varepsilon.\end{aligned}$$

Hence, the set Σ is compact in W. Thus the desired equivalence is proved.

Let us now prove the first statement. We show that the set $\{F(s)| \ \ s\in\mathbb{R}\}$ is precompact in X for all $\varphi\in D$. Let $s_n\to\infty$ be an arbitrary sequence. Then there exists the sequence $\{m_n\}\subset\mathbb{Z}$, such that $|s_n-am_n|\le a$, and on some subsequence $s_n-am_n\to b$, $n\to\infty$. On the basis of $\{m_n\}$, we choose $\{m_k\}\subset\{m_n\}$, $\widetilde{f}_i$, $\widetilde{c}_i$ from (31). Let $\widetilde{t}_i=ai-b+\widetilde{c}_i$. By (31), $\sup\limits_i\|\widetilde{f}_i\|\le C$, $\sup\limits_i|\widetilde{c}_i-b|<\infty$. Moreover, if $t_{i+1}-t_i=a+c_{i+1}-c_i\ge\gamma$, $\widetilde{t}_{i+1}-\widetilde{t}_i=a+\widetilde{c}_{i+1}-\widetilde{c}_i$, then from (31) it follows that $\widetilde{t}_{i+1}-\widetilde{t}_i\ge\gamma$. Thus the sequences $\{\widetilde{f}_i\}\subset X$ and $\{\widetilde{c}_i-b\}\subset\mathbb{R}$ satisfy the conditions (30). Therefore, for any $i\in\mathbb{Z}$, from the convergence $s_k-t_{i+m_k}\to-\widetilde{t}_i$, $k\to\infty$, we have

$$\begin{aligned}&\Big\|\sum_i f_i\varphi(s_k-t_i)-\sum_i\widetilde{f}_i\varphi(-\widetilde{t}_i)\Big\|\le\\&\qquad\le\Big\|\sum_{|i|\le N}\big(f_{i+m_k}\varphi(s_k-t_{i+m_k})-\widetilde{f}_i\varphi(-\widetilde{t}_i)\big)\Big\|+\\&+C\Big\|\sum_{|i|>N}\Big(\big|\varphi(-ai+s_k-am_k-c_{i+m_k})\big|+\big|\varphi(-ai+b-\widetilde{c}_i)\big|\Big)\Big\|.\end{aligned}$$

Then $\forall\,\varepsilon>0$ there exist $N\ge1$, $K(\varepsilon,N)$, such that $\forall\,k\ge K(\varepsilon,N)$

$$\Big\|F(s_k)-\sum_i\widetilde{f}_i\varphi(-\widetilde{t}_i)\Big\|<\varepsilon.$$

Thus, h is the translation-compact distribution.

Consider now an arbitrary element $g \in \Sigma$. Then there exists the sequence $\{s_n\}$, such that $T(s_n)h \to g$ in W, i.e.

$$\big\|(T(s_n)h)\varphi - g\varphi\big\| \to 0, \quad n \to \infty, \quad \forall\, \varphi \in D.$$

Similarly to the above-mentioned, for all $\varphi \in D$ we have

$$(T(s_n)h)\varphi = \sum_i f_i \varphi(t_i - s_n) \longrightarrow \sum_i \widetilde{f}_i \varphi(\widetilde{t}_i).$$

Then (31) yields $\widetilde{h} = \sum\limits_i \widetilde{f}_i \delta_{\widetilde{t}_i} \in W$. Hence, $g = \widetilde{h}$. We have proved the first part of the lemma.

Let now $g^n = \sum\limits_i l_i^n \delta_{\tau_i^n} \longrightarrow g = \sum\limits_i l_i \delta_{\tau_i}$ in Σ. From the previous considerations we have $\|l_i^n\| \le C$, $\{l_i^n\}_{i\in\mathbb{Z}} \subset K$ for any $n \ge 1$, where $K = cl_X\{f_i\}_{i\in\mathbb{Z}}$ is compact in X, and $\tau_i^n = ai + c_i^n$, $\{c_i^n\}_{i\in\mathbb{Z}}$ is uniformly bounded as $n \ge 1$. Then there exists $\{\widetilde{l}_i\}_{i\in\mathbb{Z}} \subset K$, $\{\widetilde{c}_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that $l_i^{n_k} \to \widetilde{l}_i$ in X, $c_i^{n_k} \to \widetilde{c}_i$ in $\mathbb{R}$ $\forall\, i \in \mathbb{Z}$. We put $\widetilde{\tau}_i = ai + \widetilde{c}_i$, $\widetilde{g} = \sum\limits_i \widetilde{l}_i \delta_{\widetilde{\tau}_i}$. Then $\forall\, \varphi \in D$

$$\begin{aligned}
\|g^{n_k}\varphi - \widetilde{g}\varphi\| &= \Big\| \sum_i \big(l_i^{n_k}\varphi(\tau_i^{n_k}) - \widetilde{l}_i \varphi(\widetilde{\tau}_i)\big)\Big\| \le \\
&\le \sum_{i=-N}^{N} \big\| l_i^{n_k}\varphi(\tau_i^{n_k}) - \widetilde{l}_i\varphi(\widetilde{\tau}_i)\big\| + C \sum_{|i|>N} \big(|\varphi(\tau_i^{n_k})| + |\varphi(\widetilde{\tau}_i)|\big).
\end{aligned}$$

Since $\{c_i^n\}_{i\in\mathbb{Z}}$ is uniformly bounded as $n \ge 1$, the estimation $\forall\, k \ge 1$ $C \sum\limits_{|i|>N} \big(|\varphi(\tau_i^{n_k})| + |\varphi(\widetilde{\tau}_i)|\big) < \frac{\varepsilon}{2}$ holds for all $\varepsilon > 0$, where $N \ge 1$. Then for all $\varepsilon > 0$, there exist $k(\varepsilon) \ge 1$ such that $\|g^{n_k}\varphi - \widetilde{g}\varphi\| < \varepsilon$ $\forall\, k \ge k(\varepsilon)$, i.e. $g^{n_k} \to \widetilde{g}$ in Σ. Hence, $\widetilde{g} = g$, and the theorem is proved. □

The Existence of a Global Attractor for a Nonautonomous Impulsive-Perturbed Evolutional Inclusion

Let $\forall\, i \in \mathbb{Z}$

$$\Psi_i = [f_i, g_i] = \big\{\lambda f_i + (1-\lambda) g_i \big|\ \lambda \in [0,1]\big\}, \tag{32}$$

and the sequences $\{\tau_i\}$, $\{f_i\}$, $\{g_i\}$ satisfy the conditions (30), i.e.,

$$\begin{gathered}
\{f_i\}_{i\in\mathbb{Z}} \subset H, \quad \sup_{i\in\mathbb{Z}} \|f_i\| \le K, \quad \{f_i\}_{i\in\mathbb{Z}} \text{ is precompact in } H, \\
\{g_i\}_{i\in\mathbb{Z}} \subset H, \quad \sup_{i\in\mathbb{Z}} \|g_i\| \le K, \quad \{g_i\}_{i\in\mathbb{Z}} \text{ is precompact in } H, \\
\{\tau_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}, \quad \tau_i = ai + c_i, \quad a > 0, \quad \sup_{i\in\mathbb{Z}} |c_i| < \infty, \quad \tau_{i+1} - \tau_i \ge \gamma > 0.
\end{gathered} \tag{33}$$

Let us construct a non-autonomous multivalued dynamical system for (14), (16).

On the basis of Lemmas 4 and 5, the set

$$\Sigma^1 = cl_{L^{2,w}_{loc}(\mathbb{R};H)}\big\{h(t+\cdot\,) \mid \ t \in \mathbb{R}\big\}$$

is compact in $L^{2,w}_{loc}(\mathbb{R};H)$, with the action of continuous group of shifts on it $\{T^1(s) : \Sigma^1 \mapsto \Sigma^1\}_{s\in\mathbb{R}}$, and $\forall\, s \in \mathbb{R}\ T^1(s)\Sigma^1 = \Sigma^1$.

By virtue of Theorem 2, both of the mappings $h = \sum\limits_i f_i\delta_{\tau_i}$ and $p = \sum\limits_i g_i\delta_{\tau_i}$ are translation-compact distributions. Moreover, if the linear continuous mapping is defined by $(h,p) : D \mapsto H^2$, $(h,p)(\varphi) := (h\varphi, p\varphi)$, then it is easy to find that the set

$$\Sigma^2 = cl_{W^2}\big\{T(s)(h,p) \mid \ s \in \mathbb{R}\big\}$$

satisfies the following conditions: Σ^2 is compact in W^2, $T(s)\Sigma^2 = \Sigma^2\ \forall\, s \in \mathbb{R}$, and for all $\sigma^2 \in \Sigma^2$, we have $\sigma^2 = (\widetilde{h},\widetilde{p})$, where $\widetilde{h} = \sum\limits_i \widetilde{f}_i\delta_{\widetilde{\tau}_i}$, $\widetilde{p} = \sum\limits_i \widetilde{g}_i\delta_{\widetilde{\tau}_i}$ and $\{\widetilde{f}_i\}$, $\{\widetilde{g}_i\}$, $\{\widetilde{\tau}_i = ai + \widetilde{c}_i\}$ fulfill (33).

Consider the impulsive problem

$$\begin{cases} \dfrac{du}{dt} + Au \in F(u) + l(t), \quad t > \tau, \\ u(\tau) = u_\tau, \end{cases} \tag{34}$$

$$u(\widetilde{\tau}_i + 0) - u(\widetilde{\tau}_i) \in g(u(\widetilde{\tau}_i)) + \widetilde{\Psi}_i = g(u(\widetilde{\tau}_i)) + [\widetilde{f}_i, \widetilde{g}_i], \ \ i \in \mathbb{Z}, \tag{35}$$

for all $\sigma = (\sigma^1,\sigma^2) \in \Sigma := \Sigma^1 \times \Sigma^2$, where $\sigma^1 = l$, $\sigma^2 = (\widetilde{h},\widetilde{p})$.

The impulsive problem (34), (35) is globally solvable in the sense of solvability of the problem (14), (16). Then for all $\sigma = (\sigma^1,\sigma^2) \in \Sigma$, $\tau \in \mathbb{R}$, $u_\tau \in H$, one can correctly construct the multivalued mapping

$$U_\sigma : \mathbb{R}_d \times H \mapsto P(H),$$

$$U_\sigma(t,\tau,u_\tau) = \big\{u(t) \mid \ u(\cdot) \text{ is the solution of (34), (35), } u(\tau) = u_\tau\big\}. \tag{36}$$

Theorem 3. *Let for the problem* (14), (16) *the conditions* (9)–(13), (21)–(24), (32), (33) *be fulfilled. Then the formula* (36) *defines the family of MP* $\{U_\sigma\}_{\sigma\in\Sigma}$, *for which there exists the compact global attractor in the phase space* H.

Proof. Let us prove that (36) defines the family of processes and $U_\sigma(t+h,\tau+h,x) = U_{T(h)\sigma}(t,\tau,x)$ holds for all $(t,\tau) \in \mathbb{R}_d$, $h \in \mathbb{R}_+$, $x \in H$. Let $\xi \in U_\sigma(t,\tau,x)$. Then $\xi = u(t)$, $u(\cdot)$ is the solution of (34), (35), $u(\tau) = x$. Thus, $\forall\, s \in (\tau,t)\ u(s) \in U_\sigma(s,\tau,x)$. Let $\omega(p) = u(p)$, if $p \geq s$. Then $\omega(\cdot)$ is the solution of (34), (35), $\omega(s) = u(s)$, i.e., $\xi = u(t) = \omega(t) \in U_\sigma(t,s,u(s)) \subset U_\sigma(t,s,U_\sigma(s,\tau,x))$. Let $\xi \in U_\sigma(t+s,\tau+s,x)$. Then $\xi = u(t+s)$, $u(\cdot)$ is the solution of (34), (35), $u(\tau+s) = x$. We put $v(p) = u(p+s)$, $p \geq \tau$. If $\tau+s \in (\widetilde{\tau}_{i-1},\widetilde{\tau}_i]$, then $u(\cdot)$ is the solution of (34) on $(\tau+s,\widetilde{\tau}_i), (\widetilde{\tau}_i,\widetilde{\tau}_{i+1}), \ldots$, such that

$$u(\widetilde{\tau}_j + 0) - u(\widetilde{\tau}_j) \in g(u(\widetilde{\tau}_j)) + \widetilde{\Psi}_j, \ \ j \geq i,$$

holds. Thus, $v(\cdot)$ is the solution of (34) on $(\tau, \widetilde{\tau}_i - s), (\widetilde{\tau}_i - s, \widetilde{\tau}_{i+1} - s), \dots,$ such that

$$v(\widetilde{\tau}_j - s + 0) - v(\widetilde{\tau}_j - s) \in g(v(\widetilde{\tau}_j - s)) + \widetilde{\Psi}_j, \;\; j \geq i,$$

and $v(\tau) = u(\tau + s) = x$ hold. Hence, $\xi = u(t + s) = v(t) \in U_{T(s)\sigma}(t, \tau, x)$. Let $\xi \in U_{T(s)\sigma}(t, \tau, x)$. Then $\xi = u(t)$, $u(\cdot)$ is the solution of (34), (35) with parameter $T(s)\sigma$, $u(\tau) = x$. We put $v(p) := u(p - s)$, $p \geq \tau + s$. Then $v(\cdot)$ is the solution of (34), (35) with parameter σ, $v(\tau + s) = u(\tau) = x$. Thus, $\xi = u(t) = v(t + s) \in U_\sigma(t + s, \tau + s, x)$.

Let us check the conditions of Theorem 1, using both the equality proven above and the equality $T(s)\Sigma = \Sigma$. Since $\{\widetilde{f}_i\}$, $\{\widetilde{g}_i\}$, $\{\widetilde{\tau}_i = ai + \widetilde{c}_i\}$ satisfy (33), and basing on Lemma 5 $\|l\|_+^2 \leq \|h\|_+^2$, from Lemma 2 we can get

$$\exists\, R_0 > 0 \;\; \forall\, r \geq 0 \;\; \exists\, T = T(r) \;\; \forall\, t \geq T(r) \;\; U_\Sigma(t, 0, B_r) \subset B_{R_0} \tag{37}$$

and thus we obtain the uniform dissipativity condition 1) from Theorem 1. Let us show that condition 2) from Theorem 1 holds, that is, the sequence $\xi_n \in U_\Sigma(t_n, 0, B_r)$ is precompact for any $t_n \to \infty$ and $r > 0$. Since

$$\xi_n \in U_{\sigma_n}(t_n, 0, B_r) \subset U_{\sigma_n}(t_n, t_n - \widetilde{t}, U_{\sigma_n}(t_n - \widetilde{t}, 0, B_r) \subset U_{T(t_n)\sigma_n}(\widetilde{t}, 0, B_{R_0}),$$

it remains only to prove that $\xi_n \in U_{(\sigma_n^1, \sigma_n^2)}(\widetilde{t}, 0, u_n^0)$ is precompact in H, when $\widetilde{t} \in (0, \gamma)$, $u_n^0 \to u_0$ weakly in H, $\sigma_n^1 = l_n \to \sigma^1 = l$ in Σ^1, $\sigma_n^2 = (\widetilde{h}^n, \widetilde{p}^n) \to \sigma^2 = (\widetilde{h}, \widetilde{p})$ in Σ^2, $\widetilde{h}^n = \sum\limits_i \widetilde{f}_i^n \delta_{\widetilde{\tau}_i^n}$, $\widetilde{p}^n = \sum\limits_i \widetilde{g}_i^n \delta_{\widetilde{\tau}_i^n}$, $\widetilde{\tau}_{i+1}^n - \widetilde{\tau}_i^n \geq \gamma > 0$, $\widetilde{\tau}_i^n = ai + \widetilde{c}_i^n$. By Theorem 2, $\widetilde{h} = \sum\limits_i \widetilde{f}_i \delta_{\widetilde{\tau}_i}$, $\widetilde{p} = \sum\limits_i \widetilde{g}_i \delta_{\widetilde{\tau}_i}$, where $\widetilde{\tau}_i = ai + \widetilde{c}_i$ and $\widetilde{f}_i^n \to \widetilde{f}_i$, $\widetilde{g}_i^n \to \widetilde{g}_i$ in H, $\widetilde{\tau}_i^n \to \widetilde{\tau}_i$ in $\mathbb{R}$ $\forall\, i \in \mathbb{Z}$. Thus, $\xi_n = u_n(\widetilde{t})$, where $u_n(\cdot)$ is the solution of the problem

$$\begin{cases} \dfrac{du_n}{dt} + Au_n \in F(u_n) + l_n(t), \quad t > 0, \\ u_n(0) = u_n^0, \end{cases} \tag{38}$$

$$u_n(\widetilde{\tau}_i^n + 0) - u_n(\widetilde{\tau}_i^n) \in g(u_n(\widetilde{\tau}_i^n)) + \widetilde{\Psi}_i^n = g(u_n(\widetilde{\tau}_i^n)) + [\widetilde{f}_i^n, \widetilde{g}_i^n], \;\; i \in \mathbb{Z}. \tag{39}$$

There exists no more than one moment of impulsive perturbations $\widetilde{\tau}_i^n$ on $[0, \widetilde{t})$ for any $n \geq 1$, and the number i depends on n, i.e., $i = i(n)$.

If for infinitely many numbers $n \geq 1$ there are no moments of impulsive perturbations on $[0, \widetilde{t})$, then $\{\xi_n = u_n(\widetilde{t})\}$ is precompact by virtue of Lemma 1, estimate (13) and the dissipativity condition (37).

Let $i(n) \in \mathbb{Z}$ to be exist for any $n \geq 1$, such that $\widetilde{\tau}_{i(n)}^n \in [0, \widetilde{t})$. As $\widetilde{\tau}_{i(n)}^n = ai(n) + \widetilde{c}_{i(n)}^n$ and $\{\widetilde{c}_i^n\}$ is uniformly bounded on $n \geq 1$, then there exists $i_0 \in \mathbb{N}$, such that $i(n) \in [-i_0, i_0] \cap \mathbb{Z}$ $\forall\, n \geq 1$. Then $i(n) \equiv i \in [-i_0, i_0] \cap \mathbb{Z}$, for infinitely many $n \geq 1$, and for some subsequence $\{u_n(\cdot)\}$ we have the following impulsive problem:

$$u_n(\widetilde{\tau}_i^n + 0) - u_n(\widetilde{\tau}_i^n) \in g(u_n(\widetilde{\tau}_i^n)) + \widetilde{\Psi}_i^n = g(u_n(\widetilde{\tau}_i^n)) + [\widetilde{f}_i^n, \widetilde{g}_i^n],$$

for fixed $i \in \mathbb{Z}$. Since for any $y_n \in \widetilde{\Psi}_i^n$ $y_n = \lambda_n \widetilde{f}_i^n + (1-\lambda_n)\widetilde{g}_i^n$, $\lambda_n \in [0,1]$, therefore on the subsequence $y_n \to y \in \lambda \widetilde{f}_i + (1-\lambda)\widetilde{g}_i \in \widetilde{\Psi}_i = [\widetilde{f}_i, \widetilde{g}_i]$. Let us consider all possible situations.

If $\widetilde{\tau}_i^n \in (0,\widetilde{t})$ and $\widetilde{\tau}_i^n \to \widetilde{\tau}_i \in (0,\widetilde{t})$, then by Lemma 1 $u_n(\widetilde{\tau}_i^n) \to u(\widetilde{\tau}_i)$, where $u(\cdot)$ is the solution of (14), $u(0) = u_0$. Since

$$\xi_n = u_n(\widetilde{t}) \in U_{\widetilde{\sigma}_n}\big(\widetilde{t}, \widetilde{\tau}_i^n, u_n(\widetilde{\tau}_i^n)\big) \subset U_{T(\widetilde{\tau}_i^n)\widetilde{\sigma}_n}\big(\widetilde{t} - \widetilde{\tau}_i^n, 0, u_n(\widetilde{\tau}_i^n)\big), \tag{40}$$

therefore $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n)$, where $v_n(\cdot)$ is the solution of the following problem:

$$\begin{cases} \dfrac{dv_n}{dt} + Av_n \in F(v_n) + l_n(t + \widetilde{\tau}_i^n), \\ v_n\big|_{t=0} = v_n(0) \in u_n(\widetilde{\tau}_i^n) + g(u_n(\widetilde{\tau}_i^n)) + \widetilde{\Psi}_i^n. \end{cases} \tag{41}$$

Denote $\widetilde{l}_n(t,x) = l_n(t + \widetilde{\tau}_i^n, x) = T^1(\widetilde{\tau}_i^n) l_n(t,x)$. Since $l_n \in \Sigma^1$ is compact in $L^{2,w}_{loc}(\mathbb{R}; H)$ and $T^1(p)\Sigma^1 = \Sigma^1$, $\forall p \in \mathbb{R}$, the subsequence $\widetilde{l}_n \to \widetilde{l}$ in Σ^1. Then $v_n(0) \to v_0$ weakly in H, hence, $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \to v(\widetilde{t} - \widetilde{\tau}_i)$ and $\{\xi_n\}$ is precompact in H, by Lemma 1.

If $\widetilde{\tau}_i^n \in (0,\widetilde{t})$, $\widetilde{\tau}_i^n \searrow 0$ (or $\widetilde{\tau}_i^n = 0$ for infinitely many $n \ge 1$), then $u_n(\widetilde{\tau}_i^n) \to u_0$ weakly in H, by Lemma 1. In a similar way,

$$\xi_n = u_n(\widetilde{t}) \in U_{T(\widetilde{\tau}_i^n)\widetilde{\sigma}_n}\big(\widetilde{t} - \widetilde{\tau}_i^n, 0, u_n(\widetilde{\tau}_i^n)\big),$$

i.e., $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n)$, $v_n(0) \in g\big(u_n(\widetilde{\tau}_i^n) + u_n(\widetilde{\tau}_i^n) + \widetilde{\Psi}_i^n\big)$.

Due to the weak convergence of $u_n(\widetilde{\tau}_i^n)$ to u_0, it is easy to find that the sequence $\{v_n(0)\}$ is bounded in H. Thus the sequence $v_n(0) \to v_0$ converges weakly in H. Then $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \to v(\widetilde{t})$, by lemma 1, hence $\{\xi_n\}$ is precompact in H.

If $\widetilde{\tau}_i^n \in (0,\widetilde{t})$, $\widetilde{\tau}_i^n \nearrow \widetilde{t}$, then

$$\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \in U_{T(\widetilde{\tau}_i^n)\widetilde{\sigma}_n}\big(\widetilde{t} - \widetilde{\tau}_i^n, 0, u_n(\widetilde{\tau}_i^n)\big),$$

where $u_n(\widetilde{\tau}_i^n) \to u(\widetilde{t}) \in U_{(l,\sigma_{s_0})}(\widetilde{t}, 0, u_0)$, $v_n(0) \in g(u_n(\widetilde{\tau}_i^n)) + u_n(\widetilde{\tau}_i^n) + \widetilde{\Psi}_i^n$, $v_n(0) \to v_0$ weakly in H. Since $\widetilde{t} - \widetilde{\tau}_i^n \nearrow 0$, and by Lemma 1 $\xi_n = v_n(\widetilde{t} - \widetilde{\tau}_i^n) \to v_0$ and $\{\xi_n\}$ is precompact in H. Thus the theorem is proved. $\square$

Remark. In many dissipative nonautonomous problems it is expected that for a global attractor $\Theta_\Sigma \subseteq U_\Sigma(t,\tau,\Theta_\Sigma)$. But a trivial example of periodic one-dimensional problem shows that in impulsive problems, in general, this is not true.

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