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ON THE QUESTION OF SOLVABILITY OF THE PERIODIC BOUNDARY VALUE PROBLEM FOR A SYSTEM OF LINEAR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Let ω be a positive number, $A = (a_{ik})_{i,k=1}^n : R \to R^{n\times m}$ and $g = (g_i)_{i=1}^n : R \to R^n$ be a matrix function and a vector function from $BV_\omega^{n\times m}$ and BV_ω^n, respectively.

We consider the ω-periodic boundary value problem

$$dx(t) = dA(t)\cdot x(t) + dg(t), \quad x(0) = x(\omega). \tag{1}$$

The use will be made of the following notation and definitions: $R =]-\infty, +\infty[$; $R^{n\times m}$ is a set of all real $n\times m$-matrices; I is the identity $n\times n$-matrix; $R^n = R^{n\times 1}$. $BV_\omega^{n\times m}$ is the set of all matrix functions $X : R \to R^{n\times m}$ such that $X(t+\omega) = X(t) + X(\omega)$ for $t \in R$, and the restriction on $[0,\omega]$ of every its components has bounded total variation; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point $t \in R$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

If $g : R \to R$ is nondecreasing, $x : R \to R$ and $s < t$, then

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s),$$

where $\int_{]s,t[} x(\tau)dg(\tau)$ is the Lebesque–Stieltjes integral over the open interval $]s,t[$ with respect to the measure μ_g corresponding to g, (if $s = t$, then $\int_s^t x(\tau)dg(\tau) = 0$).

$$L_\omega(g) = \Big\{p \in BV_\omega : \int_0^\omega |p(t)|dg(t) < \infty\Big\}.$$

A vector function $x = (x_i)_{i=1}^n \in BV_\omega^n$ is a solution of the problem (1) if it is ω-periodic and

$$x_i(t) = x_i(s) + \sum_{k=1}^n \int_s^t x_k(\tau)da_{ik}(\tau) \quad \text{for} \quad s \le t \ \ (i = 1,\dots,n).$$

Let natural numbers m and $n_1,\dots,n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$, nondecreasing functions $c_{lj} : [0,\omega] \to R$ $(l = 1,2;\ j = 1,\dots,m)$, functions $\alpha_{lj} \in L_\omega(c_{lj})$ $(l = 1,2;\ j = 1,\dots,m)$ and matrix functions $P_{lj} = (p_{ljik})_{i,k=1}^n$ $(l = 1,2;\ j = 1,\dots,m)$, $p_{ljik} \in L_\omega(c_{lj})$

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$(i,k = n_{j-1}+1,\dots,n_j)$ be such that $a_{ik}(t) \equiv 0$ $(i = n_{j-1}+1,\dots,n_j;\ k = n_j+1,\dots,n;\ j = 1,\dots,m-1)$,

$$a_{ik}(t) - \frac{1}{2}\Big(\sum_{0<\tau\le t}\ \sum_{\sigma=n_{j-1}+1}^{n_j} d_1 a_{\sigma i}(\tau)\cdot d_1 a_{\sigma k}(\tau) -$$

$$-\sum_{0\le\tau<t}\ \sum_{\sigma=n_{j-1}+1}^{n_j} d_2 a_{\sigma i}(\tau)\cdot d_2 a_{\sigma k}(\tau)\Big) =$$

$$= b_{1jik}(t) - b_{2jik}(t) \quad \text{for} \quad t\in[0,\omega] \quad (i,k = n_{j-1}+1,\dots,n_j;\ \ j = 1,\dots,m),$$

$$(-1)^{l+1}\sigma_j \sum_{i,k=n_{j-1}+1}^{n_j} p_{ljik}(t)x_i x_k \ge \alpha_{lj}(t) \sum_{i=n_{j-1}+1}^{n_j} x_i^2$$

for $\mu_{c_{lj}}$ almost everywhere $t\in[0,\omega]$, $(x_i)_{i=1}^n \in R^n$ $(l = 1,2;\ \ j = 1,\dots,m)$,

where $\sigma_j \in \{-1,1\}$, $b_{ljik}(t) \equiv \int_0^t p_{ljik}(\tau)dc_{lj}(\tau)$ $(i\ne k)$ and b_{ljii} is such that

$$(-1)^{l+1}\sigma_j(b_{ljii}(t) - b_{ljii}(s) - \int_s^t p_{ljii}(\tau)dc_{lj}(\tau)) \ge 0 \quad \text{for} \quad 0\le s\le t\le\omega.$$

Then we shall say that

$$A \in Q_\omega\Big(m, (n_j; c_{1j}, c_{2j}; \alpha_{1j}, \alpha_{2j}; P_{1j}, P_{2j})_{j=1}^m\Big). \tag{2}$$

Theorem. *Let there exist natural numbers m and $n_1,\dots,n_m$ $(0 = n_0 < n_1 < \dots < n_m = n)$, functions c_{lj} and α_{lj} $(l = 1,2;\ j = 1,\dots,m)$ and matrix functions $P_{lj} = (p_{ljik})_{i,k=1}^n$ such that (2) holds. Let, moreover,*

$$\det(I + (-1)^k d_k A(t)) \ne 0, \quad (1+\sigma_j)d_1 c_j(t) + (1-\sigma_j)d_2 c_j(t) < 2,$$
$$(1-\sigma_j)d_1 c_j(t) + (1+\sigma_j)d_2 c_j(t) \ne -2$$

and

$$\exp\Big(c_j(\omega) - \sum_{0<\tau\le\omega} d_1 c_j(\tau) - \sum_{0\le\tau<\omega} d_2 c_j(\tau)\Big) >$$

$$> \frac{1}{2}\Big[(1+\sigma_j)\prod_{0<\tau\le\omega}\big(1 - d_1 c_j(\tau)\big)\prod_{0\le\tau<\omega}\big(1 + d_2 c_j(\tau)\big)^{-1} +$$

$$+(1-\sigma_j)\prod_{0<\tau\le\omega}\big(1 + d_1 c_j(\tau)\big)^{-1}\prod_{0\le\tau<\omega}\big(1 - d_2 c_j(\tau)\big)\Big],$$

for every $t\in[0,\omega]$ and $j\in\{1,\dots,m\}$, where

$$c_j(t) \equiv 2\sum_{l=1}^{2}\int_0^t \alpha_{lj}(\tau)dc_{lj}(\tau).$$

Then the problem (1) has one and only one solution.

The analogous question has been considered in [1] for a system of linear ordinary differential equations.

References

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