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S. Mukhigulashvili

ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_0^+ = [0,+\infty[$, $\mathbb{R}^+ =]0,+\infty[$, $a,b \in \mathbb{R}^+$, $p \geq 1$.

$L_p([a,b])$ is the space of functions $f :]a,b[\to \mathbb{R}$ such that $|f(x)|^p$ is integrable on $[a,b]$, $\|f\|_{L_p} = \int_a^b |f(s)|^p ds$.

$\widetilde{C}_p([a,b])$ is the space of functions $u : [a,b] \to \mathbb{R}$ such that $u' \in L_p([a,b])$, $\|u\|_{\widetilde{C}_p} = |u(a)| + \|u'\|_{L_p}$.

$C(I,\mathbb{R})$ is the space of continuous functions $u : I \to \mathbb{R}$, $\|u\|_C = \sup\{|u(t)| : t \in I\}$.

$\widetilde{C}'_p([a,b])$ is the set of functions $u \in \widetilde{C}_1([a,b])$ such that $u' \in \widetilde{C}_p([a,b])$.

Consider the boundary value problem

$$u''(t) = H(u,u',u'')(t), \quad t \in [a,b] \tag{1}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2}$$

where $H : C([a,b]) \times C([a,b]) \times L_p([a,b]) \to L_p([a,b])$ is a compact operator, i.e., H is continuous and $H(B)$ is precompact for any bounded $B \subset C([a,b]) \times C([a,b]) \times L_p([a,b])$.

Under a solution of equation (1) we mean a function $u \in \widetilde{C}_p([a,b])$ satisfying a.e. equation (1).

Below two theorems on the solvability of the problem (1), (2) are given.

Theorem 1. *Let the inequality*

$$-g(t) \leq H(x,x',z)(t) \cdot \operatorname{sign} x(t), \quad t \in [a,b], \quad (x,z) \in \widetilde{C}'_p([a,b]) \times L_p([a,b]) \tag{3}$$

be fulfilled, where $g \in L_p([a,b])$. Moreover, let for any $r > 0$ there exist $\gamma_r, \alpha_r \in \mathbb{R}^+$ and $f_r \in C(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$\|H(x,x',z)\|_{L_p} \leq \alpha_r \cdot f_r\big(\|z\|_{L_p}\big) \quad for \quad \|x'\|_C \leq r, \quad \|z\|_{L_p} \geq \gamma_r$$

and

$$\liminf_{\rho\to+\infty} \frac{\rho}{f_r(\rho)} > \alpha_r.$$

Then the problem (1), (2) *is solvable.*

Theorem 2. *Let the condition* (3) *be fulfilled. Moreover, let for any $r \in \mathbb{R}^+$, $\alpha \in]0,(b-a)r[$ and $\beta \in]0,\alpha[$ there exist $\gamma_r, c_r \in \mathbb{R}^+$, $l_r, f_r, g_\beta \in C(\mathbb{R}_0^+,\mathbb{R}_0^+)$ and $h_\beta(t) \in L_p([a,b])$ such that*

$$h_\beta(t) > 0 \quad for \quad t \in [a,b], \quad l_r(0) = 0,$$

$$\|H(x,x',z)\|_{L_p} \leq l_r\big(\|x\|_C\big) \cdot f_r\big(\|z\|_{L_p}\big) + c_r \quad for \quad \|x\|_C < \alpha,$$

$$\|x'\|_C \leq r, \quad \|z\|_{L_p} \geq \gamma_r,$$

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$$|H(x,x',z)| \ge h_\beta(t)\cdot g_\beta\Big(\|z\|_{L_p}\Big) \quad for \quad \|x\|_C \ge \alpha, \;\; \|x'\|_C \le r,$$
$$\|z\|_{L_p} \ge \gamma_r, \quad t \in \Big\{t\in[a,b]: \; |x(t)| \ge \beta\Big\},$$

and

$$\liminf_{\rho\to+\infty} \frac{\rho}{f_r(\rho)} > 0, \quad \limsup_{\rho\to+\infty} g_\beta(\rho) = +\infty.$$

Then the problem (1), (2) *is solvable.*

Let us give some examples.
Let

$$G_1 \in L_p\Big([a,b]\times[a,b];\mathbb{R}^+\Big), \quad K(x,y)(t)\cdot \operatorname{sign} x(t) \ge -g(t), \quad t\in[a,b],$$

where

$$K: C([a,b])\times C([a,b]) \to L_p([a,b]), \quad q,g\in L_p([a,b]), \quad k\in\mathbb{N}, \tag{4}$$
$$0 < G_2(t,s) \le g_1(t), \quad (t,s)\in[a,b]\times[a,b], \quad g_1\in L_p([a,b]). \tag{5}$$

Consider the equation

$$u''(t) = u^{2k+1}(t)\int_a^b G_1(t,s)\Big(1+|u'(s)|^\alpha\Big)\Big[\int_a^b G_2(s,\tau)\cdot|u''(\tau)|^p d\tau\Big]^\mu ds + $$
$$+\,K(u,u')(t)+q(t), \tag{6}$$

where $\alpha\in\mathbb{R}_0^+$, p, $\lambda\mu \le 1$. Then according to Theorem 2, the problem (6), (2) is solvable.
Analogously, the equations

$$u''(t) = u^{2k+1}(t)\Big(1+|u'(t)|^\alpha\Big)\Big[\int_a^b G_2(t,s)\cdot|u''(s)|^p ds\Big]^{\|u\|_C+\varepsilon} +$$
$$+\,K(u,u')(t)+q(t), \quad \text{for} \;\; \alpha\in\mathbb{R}_0^+, \;\; \varepsilon < \frac{1}{p}$$

and

$$u''(t) = u^{2k+1}(t)\|u'\|_C\Big[\int_a^b G_2(t,s)\cdot|u''(s)|^{\|u\|_C+\varepsilon} ds\Big] + K(u,u')(t)+q(t),$$

where

$$p \ge (b-a)\int_a^b |g(s)| + |q(s)|ds + \varepsilon, \quad \varepsilon > 0$$

have solutions satisfying the boundary conditions (2).
Suppose now that the conditions (4) are fulfilled, and

$$0 \le G_2(t,s) \le g_1(t), \quad (t,s)\in[a,b]\times[a,b], \quad g_1\in L_p([a,b]),$$
$$\lambda\mu < 1, \quad \lambda \le p, \quad \beta > 0, \quad 0<\alpha<p, \quad g_0\in L_p([a,b]).$$

Then by Theorem 1, the equations

$$u''(t) = u^{2k+1}(t) \int_a^b G_1(t,s) \cdot |u'(s)| \Big[\int_a^b G_2(s,\tau) \cdot |u(\tau)|^\beta \cdot |u''(\tau)|^\lambda d\tau \Big]^\mu ds +$$
$$+ K(u,u')(t) + q(t),$$

$$u''(t) = u^{2k+1}(t) \cdot |u'(t)| \ln \Big(1 + \int_a^b G_2(t,r) |u(\tau)|^\beta \cdot |u''(\tau)|^\alpha d\tau \Big) + K(u,u')(t) + q(t)$$

have solutions satisfying the boundary conditions (2).

References

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Author's address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia