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Short Communication

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ON THE WELL-POSSEDNESS OF GENERAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES

Abstract. The general nonlocal boundary value problems are considered for systems of differential equations with finite and fixed points of impulses. The sufficient conditions, among which are effective spectral ones, are established for the well-posedness of these problems.

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1. Statement of the Problem and Basic Notation

In the present paper, we consider the system of nonlinear impulsive equations with a finite number of impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\}, \tag{1.1}$$

$$x(\tau_l+) - x(\tau_l-) = I_l(x(\tau_l)) \ \ (l=1,\dots,m_0) \tag{1.2}$$

with the general boundary value problem

$$h(x) = 0, \tag{1.3}$$

where $a < \tau_1 < \dots < \tau_{m_0} \le b$ (we will assume $\tau_0 = a$ and $\tau_{m_0+1} = b$, if necessary), $-\infty < a < b < +\infty$, m_0 is a natural number, $f = (f_i)_{i=1}^n$ belongs to Carathéodory class $\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, $I_l = (I_{li})_{i=1}^n : R^n \to \mathbb{R}^n$ $(l=1,\dots,m_0)$ are continuous operators, and $h : C([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n$ is a continuous, nonlinear in general, vector-functional.

Let x_0 be a solution of the problem $(1.1), (1.2); (1.3)$.

Consider a sequence of vector-functions $f_k \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ $(k = 1,2,\dots)$, the sequences of continuous operators $I_{lk} : R^n \to \mathbb{R}^n$ $(k = 1,2,\dots;$ $l = 1,\dots,m_0)$, the sequences τ_{lk} $(k = 1,2,\dots;$ $l = 1,\dots,m_0)$ of points $a < \tau_{1k} < \dots < \tau_{m_0k} \le b$ and the sequence of continuous vector-functionals $h_k : C([a,b],\mathbb{R}^n;\tau_{1k},\dots,\tau_{m_0k}) \to \mathbb{R}^n$ $(k = 1,2,\dots)$.

In this paper the sufficient conditions are given guaranteing both the solvability of the impulsive boundary value problems

$$\frac{dx}{dt} = f_k(t,x) \ \text{ almost everywhere on } \ [a,b]\setminus\{\tau_{1k},\dots,\tau_{m_0k}\}, \tag{1.1$_k$}$$

$$x(\tau_{lk}+) - x(\tau_{lk}-) = I_{lk}(x(\tau_{lk})) \ \ (l = 1,\dots,m_0); \tag{1.2$_k$}$$

$$h_k(x) = 0 \tag{1.3$_k$}$$

$(k = 1,2,\dots)$ for any sufficient large k and the convergence of its solutions to a solution of the problem $(1.1),(1.2);(1.3)$ as $k \to +\infty$.

As is known, the question of the well-possedness for the nonlinear impulsive boundary value problems was not investigated in earlier works. So the statement of the problems under consideration is actual.

The obtained results are analogous to ones given in [12] (see also the references therein) for the general nonlinear boundary value problems for systems of ordinary differential equations. Some results obtained in the paper are more general than already known ones even for ordinary differential case.

The analogous question is investigated in [4], [8] for linear boundary value problems for systems with impulses, and in [1–3], [12–15] for linear and nonlinear boundary value problems for systems of ordinary differential equations. Notice that the necessary and sufficient conditions are obtained for the well-possedness of the linear boundary value problem in [8] for impulsive, and in [1]–[3] for ordinary differential systems.

Quite a number of issues of the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [5]–[7], [9]–[11], [16], [17] and the references therein).

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$; $[a,b]$ $(a,b \in R)$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norms

$$\|X\| = \max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|;$$

$$|X| = (|x_{ij}|)_{i,j}^{n,m}, \ \ [X]_+ = \frac{|X|+X}{2};$$

$$\mathbb{R}_+^{n\times m} = \Big\{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \ge 0 \ (i = 1,\dots,n;\ j = 1,\dots,m)\Big\};$$

$$\mathbb{R}^{(n\times n)\times m} = \mathbb{R}^{n\times n}\times\dots\times\mathbb{R}^{n\times n} \ \ (m\text{-times}).$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = R_+^{n\times 1}$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n \times n$-matrix.

$\overset{b}{\underset{a}{\vee}}(X)$ is the total variation of the matrix-function $X : [a,b] \to R^{n\times m}$, i.e., the sum of total variations of the latters' components; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a) = 0$, $v(x_{ij})(t) = \overset{t}{\underset{a}{\vee}}(x_{ij})$ for $a < t \le b$;

$X(t-)$ and $X(t+)$ are the left and the right limit of the matrix-function $X : [a,b] \to R^{n\times m}$ at the point t (we will assume $X(t) = X(a)$ for $t \le a$ and $X(t) = X(b)$ for $t \ge b$, if necessary);

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [a,b]\big\}.$$

$\mathrm{BV}([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $\overset{b}{\underset{a}{\vee}}(X) < +\infty$);

$C([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all continuous matrix-functions $X : [a,b] \to D$;

$C([a,b], D; \tau_1, \dots, \tau_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$, having the one-sided limits $X(\tau_l-)$ $(l = 1, \dots, m_0)$ and $X(\tau_l+)$ $(l = 1, \dots, m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus \{\tau_1, \dots, \tau_{m_0}\}$ belong to $C([c,d], D)$;

$C_s([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0})$ is the Banach space of all $X \in C([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0})$ with the norm $\|X\|_s$.

If $y \in C_s([a,b], \mathbb{R}; \tau_1, \dots, \tau_{m_0})$ and $r \in]0, +\infty[$, then

$$U(y; r) = \Big\{x \in C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) : \ \|x - y\|_s < r\Big\};$$

$D(y, r)$ is the set of all $x \in \mathbb{R}^n$ such that $\inf\{\|x - y(t)\| : \ t \in [a,b]\} < r$.

$\widetilde{C}([a,b], D)$, where $D \subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$;

$\widetilde{C}([a,b], D; \tau_1, \dots, \tau_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$, having the one-sided limits $X(\tau_l-)$ $(l = 1, \dots, m_0)$ and $X(\tau_l+)$ $(l = 1, \dots, m_0)$, whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus \{\tau_k\}_{k=1}^m$ belong to $\widetilde{C}([c,d], D)$.

If B_1 and B_2 are the normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$.

An operator $\varphi : C([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0}) \to R^n$ is called nondecreasing if for every $x, y \in C([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0})$ such that $x(t) \le y(t)$ for $t \in [a,b]$ the inequality $\varphi(x)(t) \le \varphi(y)(t)$ holds for $t \in [a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D \subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X:[a,b]\to D$.

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $\mathrm{Car}([a,b]\times D_1, D_2)$ is the Carathéodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$ and $k\in\{1,\dots,n\}$:

(a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is measurable for every $x\in D_1$;

(b) the function $f_{kj}(t,\cdot):D_1\to D_2$ is continuous for almost every $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)|:\ x\in D_0\big\}\in L([a,b],R;g_{ik}) \ \text{ for every compact } D_0\subset D_1.$$

$\mathrm{Car}^0([a,b]\times D_1,D_2)$ is the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that the functions $f_{kj}(\,\cdot\,,x(\,\cdot\,))$ $(i=1,\dots,l;\ k=1,\dots,n)$ are measurable for every vector-function $x:[a,b]\to\mathbb{R}^n$ with the bounded total variation.

$M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is the set of all functions $\omega\in\mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ such that the function $\omega(t,\,\cdot\,)$ is nondecreasing, and $\omega(t,0)\equiv 0$.

By a solution of the impulsive system (1.1), (1.2) we understand a continuous from the left vector-function $x\in\widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ satisfying both the system (1.1) for a.e. on $[a,b]\setminus\{\tau_1\dots,\tau_{m_0}\}$ and the relation (1.2) for every $k\in\{1,\dots,m_0\}$.

Definition 1.1. Let $l:C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ be a linear continuous operator, and let $l_0:C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$ be a positive homogeneous operator. We say that a pair $(P,\{J_l\}_{l=1}^{m_0})$, consisting of a matrix-function $P\in\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a finite sequence of continuous operators $J_l=(J_{li})_{i=1}^n:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$, satisfy the Opial condition with respect to the pair (l,l_0) if:

(a) there exist a matrix-function $\Phi\in L([a,b],\mathbb{R}^n_+)$ and constant matrices $\Psi_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that

$$|P(t,x)|\le\Phi(t)\ \text{ a.e. on } [a,b],\ \ x\in\mathbb{R}^n$$

and

$$|J_l(x)|\le\Psi_l\ \text{ for } x\in\mathbb{R}^n\ \ (l=1,\dots,m_0);$$

(b)

$$\det(I_{n\times n}+G_l)\ne 0\ \ (l=1,\dots,m_0) \tag{1.4}$$

and the problem

$$\frac{dx}{dt}=A(t)x\ \text{ a.e. on } [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \tag{1.5}$$

$$x(\tau_l+)-x(\tau_l-)=G_lx(\tau_l)\ \ (l=1,\dots,m_0); \tag{1.6}$$

$$|l(x)|\le l_0(x) \tag{1.7}$$

has only the trivial solution for every matrix-function $A \in L([a,b],\mathbb{R}^{n\times n})$ and constant matrices G_l $(l=1,\dots,m_0)$ for which there exists a sequence $y_k \in \widetilde{C}([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int_a^t P(\tau,y_k(\tau))d\tau = \int_a^t A(\tau)d\tau \ \text{ uniformly on } \ [a,b]$$

and

$$\lim_{k\to+\infty} J_l(y_k(\tau_l)) = G_l \ \ (l=1,\dots,m_0).$$

Remark 1.1. In particular, the condition (1.4) holds if

$$\|\Psi_l\| < 1 \ \ (l=1,\dots,m_0).$$

Below, we will assume that $f=(f_i)_{i=1}^n \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, in addition, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Let x^0 be a solution of the problem $(1.1),(1.2);(1.3)$, and r be a positive number. Let us introduce the following definition.

Definition 1.2. The solution x^0 is said to be strongly isolated in the radius r if there exist, respectively, the matrix– and the vector-functions $P \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q\in\mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$, a finite sequence of continuous operators $J_l:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$ and $h_l:\mathbb{R}^n\to\mathbb{R}^n$ $(l=1,\dots,m_0)$, a linear continuous operator $l: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$, a positive homogeneous operator $l_0: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n_+$, and a continuous operator $\widetilde{l}: C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0})\to\mathbb{R}^n$ such that

(a) the equalities

$$f(t,x) = P(t,x)x + q(t,x) \ \text{ for } \ t\in[a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\}, \ \ \|x-x^0(t)\|<t,$$
$$I_l(x) = J_l(x)x + h_l(x) \ \text{ for } \ \|x-x^0(\tau_l)\|<t \ \ (l=1,\dots,m_0);$$

and

$$h(x) = l(x) + \widetilde{l}(x) \ \text{ for } \ x\in U(x^0;r)$$

are valid;

(b) the functions

$$\alpha(t,\rho) = \max\big\{\|q(t,x)\|: \ \|x\|\le\rho\big\},$$
$$\beta_l(\rho) = \max\big\{\|h_l(x)\|: \ \|x\|\le\rho\big\} \ \ (l=1,\dots,m_0)$$

and

$$\gamma(\rho) = \sup\Big\{\big[|\widetilde{l}(x)| - l_0(x)\big]_+ : \ \|x\|_s\le\rho\Big\}$$

satisfy the condition

$$\limsup_{\rho\to+\infty}\frac{1}{\rho}\left(\gamma(\rho)+\int\limits_a^b\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta_l(\rho)\right)<1;$$

(c) the problem

$$\begin{gathered}\frac{dx}{dt}=P(t,x)x+q(t,x)\ \text{ a.e. on }\ [a,b]\setminus\{\tau_1,\dots,\tau_{m_0}\},\\ x(\tau_l+)-x(\tau_l-)=J_l(x(\tau_l))+h_l(x(\tau_l))\ \ (l=1,\dots,m_0);\\ l(x)+\widetilde{l}(x)=0\end{gathered}$$

has no solution different from x^0.

(d) the pair $(P,\{J_l\}_{l=1}^{m_0})$ satisfy the Opial condition with respect to the pair (l,l_0).

Definition 1.3. We say that a sequence $(f_k,\{I_{lk}\}_{l=1}^{m_0};h_k)$ $(k=1,2,\dots)$ belongs to the set $W_r(f,\{I_l\}_{l=1}^{m_0},h;x^0)$ if

(a) the equalities

$$\lim_{k\to+\infty}\int\limits_a^t f_k(\tau,x)\,\tau=\int\limits_a^t f(\tau,x)d\tau\ \text{ uniformly on }\ [a,b]$$

and

$$\lim_{k\to+\infty}I_{lk}(x)=I_l(x)\ \ (l=1,\dots,m_0)$$

are valid for every $x\in D(x^0;r)$;

(b)

$$\lim_{k\to+\infty}h_k(x)=h(x)\ \text{ uniformly on }\ U(x^0;r);$$

(c) there exist a sequence of functions $\omega_k\in M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ $(k=1,2,\dots)$ and sequences of nondecreasing functions $\omega_{lk}\in C(\mathbb{R}_+,\mathbb{R}_+)$,

$\omega_{lk}(0) = 0$, $(k = 1, 2, \dots; l = 1, \dots, m_0)$ such that

$$\sup\left\{ \int\limits_a^b \omega_k(t, r)\, dt : \ k = 1, 2, \dots \right\} < +\infty, \tag{1.8}$$

$$\sup\left\{ \sum_{l=1}^{m_0} \omega_{lk}(r) : \ k = 1, 2, \dots \right\} < +\infty; \tag{1.9}$$

$$\lim_{s\to 0+} \sup\left\{ \int\limits_a^b \omega_k(t, s)\, dt : \ k = 1, 2, \dots \right\} = 0, \tag{1.10}$$

$$\lim_{s\to 0+} \sup\left\{ \sum_{l=1}^{m_0} \omega_{lk}(s) : \ k = 1, 2, \dots \right\} = 0; \tag{1.11}$$

and

$$\big\|f_k(t, x) - f_k(t, y)\big\| \le \omega_k\big(t, \|x - y\|\big)$$
$$\text{for } \ t \in [a, b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \ \ x, y \in D(x^0; r) \ \ (k = 1, 2, \dots),$$
$$\big\|I_{lk}(x) - I_{lk}(y)\big\| \le \omega_{lk}\big(\|x - y\|\big)$$
$$\text{for } \ x, y \in D(x^0; r) \ \ (l = 1, \dots, m_0; \ \ k = 1, 2, \dots).$$

Remark 1.2. If for every natural m there exists a positive number ν_m such that

$$\omega_k(t, m\delta) \le \nu_m \omega_k(t, \delta)$$
$$\text{for } \ \delta > 0, \ \ t \in [a, b] \setminus \{\tau_1, \dots, \tau_{m_0}\} \ \ (k = 1, 2, \dots),$$

then the estimate (1.8) follows from the condition (1.10); analogously, if

$$\omega_{lk}(m\delta) \le \nu_m \omega_{lk}(\delta) \ \text{ for } \ \delta > 0 \ \ (l = 1, \dots, m_0; \ \ k = 1, 2, \dots),$$

then the estimate (1.9) follows from the condition (1.11). In particular, the sequences of the functions

$$\omega_k(t, \delta) = \max\left\{ \big\|f_k(t, x) - f_k(t, y)\big\| : \ x, y \in U(0, \|x^0\| + r), \ \|x - y\| \le \delta \right\}$$
$$\text{for } \ t \in [a, b] \setminus \{\tau_1, \dots, \tau_{m_0}\} \ (k = 1, 2, \dots)$$

and

$$\omega_{lk}(\delta) = \max\left\{ \big\|I_{lk}(x) - I_{lk}(y)\big\| : \ x, y \in U(0, \|x^0\| + r), \ \|x - y\| \le \delta \right\}$$
$$(l = 1, \dots, m_0; \ \ k = 1, 2, \dots)$$

have, respectively, the latters' properties.

Definition 1.4. The problem (1.1), (1.2); (1.3) is said to be $(x^0; r)$-correct if for every $\varepsilon \in]0, r[$ and $\big((f_k, \{I_{lk}\}_{l=1}^{m_0}; h_k)\big)_{k=1}^{+\infty} \in W_r(f, \{I_l\}_{l=1}^{m_0}, h; x^0)$ there exists a natural number k_0 such that the problem $(1, 1_k)$, (1.2_k); (1.3_k) has at least a solution contained in $U(x^0; r)$ and any such solution belongs to the ball $U(x^0;)$ for every $k \ge k_0$.

Definition 1.5. The problem (1.1), (1.2); (1.3) is said to be correct if it has the unique solution x^0 and it is $(x^0; r)$-correct for every $r > 0$.

Theorem 1.1. *If the problem* (1.1), (1.2)*;* (1.3) *has a solution* x^0*, strongly isolated in the radius* $r > 0$*, then it is* $(x^0; r)$*-correct.*

Theorem 1.2. *Let the conditions*

$$\big\|f(t,x) - P(t,x)x\big\| \le \alpha\big(t, \|x\|\big)$$
$$\text{a.e. on } [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \quad x \in \mathbb{R}^n, \tag{1.12}$$

$$\big\|I_l(x) - J_l(x)x\big\| \le \beta_l\big(\|x\|\big) \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1, \dots, m_0) \tag{1.13}$$

and

$$|h(x) - l(x)| \le l_0(x) + l_1\big(\|x\|_s\big) \ \text{ for } x \in \mathrm{BV}([a,b], \mathbb{R}^n) \tag{1.14}$$

hold, where $\ell : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators, the pair* $(P, \{J_l\}_{l=1}^{m_0})$ *satisfies the Opial condition with respect to the pair* (ℓ, ℓ_0)*;* $\alpha \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}_+)$ *is a function nondecreasing in the second variable, and* $\beta_l \in C([a,b], \mathbb{R}_+)$ $(l = 1, \dots, m_0)$ *and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *are, respectively, nondecreasing functions and vector-function such that*

$$\limsup_{\rho \to +\infty} \frac{1}{\rho} \bigg(\|l_1(\rho)\| + \int\limits_a^b \alpha(t,\rho)\,dt + \sum_{l=1}^{m_0} \beta_l(\rho) \bigg) < 1. \tag{1.15}$$

Then one-valued solvability of the problem (1.1), (1.2)*;* (1.3) *guarantees its correctness.*

Theorem 1.3. *Let the conditions* (1.12)–(1.14)*,*

$$P_1(t) \le P(t,x) \le P_2(t) \ \text{ a.e. on } [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \ \ x \in \mathbb{R}^n,$$

and

$$J_{1l} \le I_k(x) \le J_{2l} \ \text{ for } x \in \mathbb{R}^n \ \ (l = 1, \dots, m_0)$$

hold, where $P \in \mathrm{Car}^0([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n})$*,* $P_i \in L([a,b], \mathbb{R}^{n\times n})$ $(i = 1, 2)$*,* $J_{il} \in \mathbb{R}^{n\times n}$ $(i = 1, 2;\ l = 1, \dots, m_0)$*,* $l : C_s([a,b], \mathbb{R}^{n\times n}; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ *and* $l_0 : C_s([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n_+$ *are, respectively, the linear continuous and the positive homogeneous continuous operators;* $\alpha \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}_+)$ *is a function nondecreasing in the second variable, and* $\beta_l \in C([a,b], \mathbb{R}_+)$ $(l = 1, \dots, m_0)$ *and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}^n_+)$ *are, respectively, nondecreasing functions and vector-function such that the condition* (1.15) *holds. Let, moreover, the condition* (1.4) *hold and the problem* (1.5), (1.6)*;* (1.7) *have only the trivial solution for every matrix-function* $A \in L([a,b], \mathbb{R}^{n\times n})$ *and the constant matrices* $G_l \in \mathbb{R}^{n\times n} (l = 1, \dots, m_0)$ *such that*

$$P_1(t) \le A(t) \le P_2(t) \ \text{ a.e. on } [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \ \ x \in \mathbb{R}^n$$

and

$$J_{1l} \le G_l \le J_{2l} \ \ for \ \ x \in \mathbb{R}^n \ \ (l = 1, \dots, m_0).$$

Then one-valued solvability of the problem (1.1), (1.2); (1.3) *guarantees its correctness.*

Remark 1.3. Theorem 1.3 is interesting only in the case where $P \notin \mathrm{Car}([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n})$, because the theorem immediately follows from Theorem 1.2 in the case where $P \in \mathrm{Car}([a,b] \times \mathbb{R}^n, \mathbb{R}^{n\times n})$.

Theorem 1.4. *Let the conditions* (1.14),

$$\big|f(t,x) - P_0(t)x\big| \le Q(t)|x| + q\big(t, \|x\|\big) \ \ a.e. \ \ on \ \ [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \quad x \in \mathbb{R}^n,$$

and

$$\big|I_l(x) - J_{0l}x\big| \le H_l|x| + h_l\big(\|x\|\big) \ \ for \ \ x \in \mathbb{R}^n \ \ (l = 1, \dots, m_0)$$

hold, where $P_0 \in L([a,b], \mathbb{R}^{n\times n})$, $Q \in L([a,b], \mathbb{R}_+^{n\times n})$, J_{0l} *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are constant matrices,* $l : C_s([a,b], \mathbb{R}^{n\times n}; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ *and* $l_0 : C_s([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, the linear continuous and positive homogeneous continuous operators;* $q \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}_+^n)$ *is a vector-function nondecreasing in the second variable, and* $h_l \in C([a,b], \mathbb{R}_+)$ $(l = 1, \dots, m_0)$ *and* $\ell_1 \in C(\mathbb{R}, \mathbb{R}_+^n)$ *are, respectively, the nondecreasing functions and vector-function such that the condition* (1.15) *holds. Let, moreover, the conditions*

$$\det(I_{n\times n} + J_{0l}) \ne 0 \ \ (l = 1, \dots, m_0) \tag{1.16}$$

and

$$\|H_l\| \cdot \big\|(I_{n\times n} + J_{0l})^{-1}\big\| < 1 \ \ (j = 1, 2; \ \ l = 1, \dots, m_0) \tag{1.17}$$

hold, and the system of impulsive inequalities

$$\Big|\frac{dx}{dt} - P_0(t)x\Big| \le Q(t)x \ \ a.e. \ \ on \ \ [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \tag{1.18}$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_{0l}x(\tau_l)\big| \le H_l x(\tau_l) \ \ (l = 1, \dots, m_0) \tag{1.19}$$

have only the trivial solution under the condition (1.7). *Then one-valued solvability of the problem* (1.1), (1.2); (1.3) *guarantees its correctness.*

Corollary 1.1. *Let the conditions* (1.16)

$$\big|f(t,x) - P_0(t)x\big| \le q\big(t, \|x\|\big) \ \ a.e. \ \ on \ \ [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \ \ x \in \mathbb{R}^n, \tag{1.20}$$

and

$$\big|I_l(x) - J_{0l}x\big| \le h_l\big(\|x\|\big) \ \ for \ \ x \in \mathbb{R}^n \ \ (l = 1, \dots, m_0) \tag{1.21}$$

hold, where $P_0 \in L([a,b], \mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices,* $\ell : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b], \mathbb{R}^{n\times m}; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, the linear continuous and positive homogeneous continuous operators;* $q \in \mathrm{Car}([a,b] \times \mathbb{R}_+, \mathbb{R}_+^n)$ *is a vector-function* *nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are, respectively, the nondecreasing functions and a vector-function such that the condition* (1.15) *holds. Let, moreover,*

$$|h(x) - \ell(x)| \le \ell_1\big(\|x\|_s\big) \ \ \textit{for} \ \ x \in \mathrm{BV}([a,b],\mathbb{R}^n) \tag{1.22}$$

and the impulsive system

$$\frac{dx}{dt} = P_0(t)x \ \ \textit{a.e. on} \ \ [a,b] \setminus \{\tau_1,\dots,\tau_{m_0}\},$$
$$x(\tau_l+) - x(\tau_l-) = J_{0l}x(\tau_l) \ \ (l = 1,\dots,m_0)$$

have only the trivial solution under the condition (1.7)*. Then one-valued solvability of the problem* (1.1), (1.2); (1.3) *guarantees its correctness.*

For every matrix-function $X \in L([a,b],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m_0)$ we introduce the operators

$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \text{ for } \ a \le t \le b,$$
$$\big[(X,Y_1,\dots,Y_{m_0})(a)\big]_i = O_{n\times n} \ \ (i = 1,2,\dots),$$
$$\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int\limits_a^t X(\tau)\cdot\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau +$$
$$+ \sum_{a\le\tau_l<t} Y_l \cdot \big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \text{ for } \ a < t \le b \ \ (i = 1,2,\dots). \tag{1.23}$$

Corollary 1.2. *Let the conditions* (1.16), (1.20)–(1.22) *hold, where*

$$\ell(x) \equiv \int\limits_a^b dL(t)\cdot x(t),$$

$P_0 \in L([a,b],\mathbb{R}^{n\times n})$, $J_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices,* $L \in L([a,b],\mathbb{R}^{n\times n})$, $\ell_0 : C_s([a,b],\mathbb{R}^n;\tau_1,\dots,\tau_{m_0}) \to \mathbb{R}^n_+$ *are the positive homogeneous continuous operators;* $q \in \mathrm{Car}([a,b]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *is a vector-function nondecreasing in the second variable, and* $h_l \in C([a,b],\mathbb{R}_+)$ $(l = 1,\dots,m_0)$ *and* $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ *are, respectively, the nondecreasing functions and a vector-function such that the condition* (1.15) *holds. Let, moreover, there exist natural numbers* k *and* m *such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int\limits_a^b dL(t)\cdot\big[(P_0,G_l,\dots,G_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1,$$

where the operators $[(P_0, G_1, \dots, G_{m_0})(t)]_i$ $(i = 0, 1, \dots)$ *are defined by* (1.23), *and*

$$M_{k,m} = \Big[\big(|P_0|, |G_1|, \dots, |G_{m_0}|\big)(b)\Big]_m + \\ + \sum_{i=0}^{m-1} \Big[\big(|P_0|, |G_1|, \dots, |G_{m_0}|\big)(b)\Big]_i \times \\ \times \int_a^b dV(M_k^{-1}L)(t) \cdot \Big[\big(|P_0|, |G_1|, \dots, |G_{m_0}|\big)(t)\Big]_k.$$

Then one-valued solvability of the problem (1.1), (1.2); (1.3) *guarantees its correctness.*

Theorem 1.5. *Let the conditions* (1.16), (1.17),

$$\big|f(t,x) - f(t,y) - P_0(t)(x-y)\big| \leq Q(t)|x-y| \\ \text{a.e. on } [a,b] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \;\; x, y \in \mathbb{R}^n,$$

$$\big|I_l(x) - I_l(y) - J_{0l} \cdot (x-y)\big| \leq H_k \cdot |x-y| \\ \text{for } x, y \in \mathbb{R}^n \;\; (k = l, \dots, m_0)$$

and

$$\big|h(x) - h(y) - \ell(x-y)\big| \leq \ell_0(x-y) \;\; \text{for } x, y \in \mathrm{BV}([a,b], \mathbb{R}^n)$$

hold, where $P_0 \in L([a,b], \mathbb{R}^{n\times n})$, $Q \in L([a,b], \mathbb{R}_+^{n\times n})$, J_{0k} *and* $H_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m_0)$ *are the constant matrices,* $\ell : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}^n$ *and* $\ell_0 : C_s([a,b], \mathbb{R}^n; \tau_1, \dots, \tau_{m_0}) \to \mathbb{R}_+^n$ *are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities* (1.18), (1.19) *have only the trivial solution under the condition* (1.7). *Then the problem* (1.1), (1.2); (1.3) *is correct.*

Corollary 1.3. *Let the system* (1.1), (1.2); (1.3) *have a unique solution* x^0 *defined on the whole closed interval* $[a,b]$, *where* $h(x) \equiv x(t_0) - c_0$, *and* $t_0 \in [a,b]$ *and* $c_0 \in \mathbb{R}^n$ *are such that* $I_l(c_0) = 0$ *if* $t_0 = \tau_l$ *for some* $l \in \{1, \dots, m_0\}$. *Let, moreover,*

$$\limsup_{\rho \to +\infty} \Big(\inf\big\{\|x + I_l(y)\| : \; \|x\| \geq \rho, \; \|y\| = \rho\big\}\Big) > \|x^0\|_s \tag{1.24}$$

for every $l \in \{1, \dots, m_0\}$ *such that* $\tau_l > t_0$. *Then the problem* (1.1), (1.2); (1.24) *is correct.*

Remark 1.4. It is evident that the condition (1.24) is valid if $I_l(y) \equiv 0$ for every $l \in \{1, \dots, m_0\}$ such that $\tau_l > t_0$. If the last condition is not fulfilled, i.e., $I_l(y) \not\equiv 0$ for some $l \in \{1, \dots, m_0\}$, then the strict inequality (1.24) cannot be replaced by a non-strict one for this l. Below, we will give the corresponding example.

Example. Let $n = 1$, $m_0 > 2$ be an arbitrary natural number, $\tau_l \in]a, b[$ $(l = 1, \dots, m_0)$, $h(x) \equiv x(t_0) - c_0$, $t_0 = b$, $c_0 = 0$; $h_(x) \equiv x(t_k) - c_k$ $(k = 1, 2, \dots)$, $t_k \to b$ $(k \to +\infty)$ and $c_k \to c_0$ $(k \to +\infty)$; $f(t, x) = f_k(t, x) \equiv 0$ $(k = 1, 2, \dots)$; $I_l(x) = I_{kl}(x) \equiv 0$ $(l = 1, \dots, m_0 - 1;\ k = 1, 2, \dots)$;

$$I_{m_0}(x) = \begin{cases} 0 & \text{for } x < 0, \\ (1 + c_{i+1} - c_i)(i - x) - i - c_i & \text{for } x \in [i, i+1[\ \ (i = 0, 1, \dots) \end{cases}$$

and

$$I_{km_0}(x) =$$

$$= \begin{cases} I_{m_0}(x) & \text{for } x \in]-\infty, k-1[\cup]k+1, +\infty[\,, \\ (1 - c_{k-1} - c_k)(k - x) + c_k - k & \text{for } x \in [k-1, k[\,, \\ (1 + c_{k+1} + c_k)(k - x) + c_k - k & \text{for } x \in [k, k+1[\end{cases}$$

$$(k = 1, 2, \dots).$$

Then $x^0(t) \equiv 0$, $\big((f_k, \{I_{lk}\}_{l=1}^{m_0}; h_k)\big)_{k=1}^{+\infty} \in W_r(f, \{I_l\}_{l=1}^{m_0}, h; x^0)$. Moreover, the problem $(1.1_k), (1.2_k); (1.3_k)$ has the unique solution

$$x_k(t) = \begin{cases} c_k & \text{for } a \le t \le \tau_{m_0}, \\ k & \text{for } \tau_{m_0} < t \le b \end{cases}$$

for every natural k. As to the condition (1.24), it is transformed into the equality for $t = \tau_{m_0}$ only.

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