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A PRIORI ESTIMATES OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES HIGHER ORDER DIFFERENTIAL INEQUALITIES AND SYSTEMS OF DIFFERENTIAL INEQUALITIES

Dedicated to the blessed memory of professor A. Razmadze

Abstract. For singular in phase variables higher order nonlinear differential inequalities and systems of nonlinear differential inequalities a priori estimates of solutions satisfying nonlinear boundary conditions of a certain type are established.

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Introduction

The boundary value problems for singular in phase variables second order differential equations attract attention of many mathematicians and are the subject of various investigations (see, e.g., [1–4, 6, 10, 12–14, 16, 17] and references therein). As for the singular in phase variables higher order differential equations and differential systems, for them only the initial and two-point problems [7,9], the Nikoletti perturbed problem [8] and the Kneser type problem [15] are studied.

The construction of the theory of boundary value problems for singular in phase variables differential equations and systems requires a priori estimates of solutions of singular in phase variables higher order differential inequalities and systems of differential inequalities, satisfying different nonlinear boundary conditions. The present paper contains such estimates.

We have used the following notation.

$x = (x_i)_{i=1}^n$ and $X = (x_{ik})_{i,k=1}^n$ are the n-dimensional vector column and the $n \times n$-matrix with the components x_i and x_{ik} $(i, k = 1, \dots, n)$ and the norms

$$\|x\| = \sum_{i=1}^n |x_i|, \quad \|X\| = \sum_{i,k=1}^n |x_{ik}|;$$

$r(X)$ is the spectral radius of the matrix X;

$\mathbb{R}_+ = [0, +\infty[\,$, $\mathbb{R}_{0+} =]0, +\infty[\,$;

$\mathbb{R}^n$ is the n-dimensional real Euclidean space;

$\mathbb{R}_{0+}^n = \big\{(x_i)_{i=1}^n \in \mathbb{R}^n :\ x_1 > 0, \dots, x_n > 0\big\}$;

$\widetilde{C}([a,b];\mathbb{R})$ is the space of absolutely continuous functions $u : [a,b] \to \mathbb{R}$;

$\widetilde{C}^m([a,b];\mathbb{R})$ is the space of m-times continuously differentiable functions $u : [a,b] \to \mathbb{R}$ whose derivative of m-th order is absolutely continuous;

$\widetilde{C}^m([a,b];\mathbb{R}_{0+}^n)$ is the set of vector functions $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}_{0+}^n$ with absolutely continuous components $u_i : [a,b] \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$.

1. Higher Order Differential Inequalities

In a finite interval $[a,b]$ we consider the n-th order differential inequality

$$g_0\big(t, u(t), \dots, u^{(n-1)}(t)\big) \le u^{(n)}(t) \le \\ \le \sum_{k=1}^n g_k\big(t, u(t), \dots, u^{(n-1)}(t)\big) u^{(k-1)}(t) \quad (1.1)$$

with the boundary conditions

$$\alpha_i u^{(i-1)}(b) \le u^{(i-1)}(a) \le \beta_i u^{(i-1)}(b) + \beta_0 \ \ (i = 1, \dots, n). \quad (1.2)$$

Here $g_k : [a,b] \times \mathbb{R}_{0+}^n \to \mathbb{R}_+$ $(k = 0, \dots, n)$ are integrable in the first argument and continuous and nonincreasing in the last n arguments functions, α_i $(i = 1, \dots, n)$ and β_i $(i = 0, \dots, n)$ are constants such that

$$0 < \alpha_i \le \beta_i < 1 \ \ (i = 1, \dots, n), \ \ \beta_0 > 0. \quad (1.3)$$

We are mainly interested in the case where the differential inequality (1.1) is singular in phase variables, i.e., in the case when there exists a set of positive measure $I \subset [a,b]$ such that

$$\lim_{x_1+\cdots+x_n\to 0} g_k(t,x_1,\dots,x_n) = +\infty \ \text{ for } \ t \in I \ \ (k=0,\dots,n).$$

A function $u \in \widetilde{C}^{n-1}([a,b];\mathbb{R})$ is said to be a **solution of the differential inequality** (1.1) if

$$u^{(i-1)}(t) > 0 \ \text{ for } \ a \le t \le b \ \ (i=1,\dots,n)$$

and almost everywhere on $[a,b]$ the inequality (1.1) is fulfilled.

A solution of the differential inequality (1.1) satisfying the boundary conditions (1.2) is called a **solution of the problem** (1.1)**,** (1.2).

Before we give a theorem containing a priori estimates of solutions of the above-mentioned problem, we prove a simple lemma dealing with estimates of solutions of the differential inequality

$$u^{(n)}(t) \ge 0, \tag{1.4}$$

satisfying the boundary conditions (1.2).

Lemma 1.1. *An arbitrary solution u of the problem* (1.4), (1.2) *admits the estimates*

$$\gamma_{0k}\ell \le u^{(k-1)}(t) \le \gamma_k(\ell+\beta_0) \ \ \textit{for} \ \ a \le t \le b \ \ (k=1,\dots,n), \tag{1.5}$$

where

$$\gamma_k = (b-a)^{n-k}\prod_{i=k}^{n}(1-\beta_i)^{-1} \ \ (k=1,\dots,n), \tag{1.6}$$

$$\gamma_{0k} = (b-a)^{n-k}\prod_{i=k}^{n}\frac{\alpha_i}{1-\alpha_i} \ \ (k=1,\dots,n), \tag{1.7}$$

and

$$\ell = \int_a^b u^{(n)}(s)\,ds. \tag{1.8}$$

Proof. In view of (1.2), (1.8), we have

$$u^{(n-1)}(b) = u^{(n-1)}(a) + \ell \ge \alpha_n u^{(n-1)}(b) + \ell,$$
$$u^{(n-1)}(b) \le \beta_n u^{(n-1)}(b) + \beta_0 + \ell,$$

and hence

$$u^{(n-1)}(b) \ge \frac{1}{1-\alpha_n}\,\ell, \quad u^{(n-1)}(b) \le \frac{1}{1-\beta_n}\,(\beta_0+\ell).$$

If along with this we take into account the inequality (1.4), it becomes obvious that

$$u^{(n-1)}(t) \geq u^{(n-1)}(a) \geq \alpha_n u^{(n-1)}(b) \geq \\ \geq \gamma_{0n}\ell, \quad u^{(n-1)}(t) \leq u^{(n-1)}(b) \leq \gamma_n(\beta_0+\ell) \ \text{ for } \ a \leq t \leq b.$$

This, according to the induction law and notations (1.6) and (1.7), results in the estimate (1.5). ☐

Theorem 1.1. *If along with* (1.3) *the conditions*

$$\int\limits_a^b g_0(s,x,\dots,x)\,ds > 0 \ \textit{ for } \ x>0, \tag{1.9}$$

$$\lim_{x\to+\infty} \sum_{k=1}^n \gamma_k \int\limits_a^b g_k(s,x,\dots,x)\,ds < 1 \tag{1.10}$$

are fulfilled, then there exist positive constants δ and ρ such that an arbitrary solution of the problem (1.1), (1.2) *admits the estimates*

$$\delta \leq u^{(k-1)}(t) \leq \rho \ \textit{ for } \ a\leq t\leq b \ \ (k=1,\dots,n). \tag{1.11}$$

Proof. By the inequality (1.10), there exists a positive number x_0 such that

$$\Big(1+\frac{\beta_0}{x_0}\Big)\sum_{k=1}^n \gamma_k \int\limits_a^b g_k(s,x_0,\dots,x_0)\,ds < 1. \tag{1.12}$$

Suppose

$$\gamma_0 = \min\big\{1,\gamma_{01},\dots,\gamma_{0n}\big\}, \quad \gamma = \max\big\{\gamma_1,\dots,\gamma_n\big\}, \\ \rho = \Big(\frac{x_0}{\gamma_0}+\beta_0\Big)\gamma,$$

and

$$\delta = \gamma_0 \int\limits_a^b g_0(s,\rho,\dots,\rho)\,ds.$$

Owing to (1.9), it is clear that $\delta>0$.

Let u be an arbitrary solution of the problem (1.1), (1.2), and let ℓ be the number given by the equality (1.8). Then by Lemma 1.1, the inequalities (1.5) are valid. On the other hand, it follows from (1.1) and (1.5) that

$$\ell \leq (\ell+\beta_0)\sum_{k=1}^n \gamma_k \int\limits_a^b g_k\big(s,\ell\gamma_0,\dots,\ell\gamma_0\big)\,ds \tag{1.13}$$

and

$$\ell \geq \int\limits_a^b g_0\big(s,(\ell+\beta_0)\gamma,\dots,(\ell+\beta_0)\gamma\big)\,ds, \tag{1.14}$$

since g_k $(k = 0, \dots, n)$ are nonincreasing in the last n arguments functions.

Our aim is to prove that u admits the estimates (1.11). Let us first show that

$$\ell < \frac{x_0}{\gamma_0}\,. \tag{1.15}$$

Assume the contrary that

$$\ell \geq \frac{x_0}{\gamma_0}\,.$$

Then $\ell \geq x_0$. Thus taking into account the inequality (1.12), from the inequality (1.13) we find

$$\ell \leq \ell\Big(1 + \frac{\beta_0}{x_0}\Big) \sum_{k=1}^{n} \gamma_k \int\limits_a^b g_k(s, x_0, \dots, x_0)\, ds < \ell.$$

The obtained contradiction proves the validity of the estimate (1.15).

According to (1.5), (1.14) and (1.15), we have

$$u^{(k-1)}(t) < \Big(\frac{x_0}{\gamma_0} + \beta_0\Big)\gamma = \rho \ \text{ for } \ a \leq t \leq b \ \ (k = 1, \dots, n)$$

and

$$u^{(k-1)}(t) \geq \ell\gamma_0 \geq \gamma_0 \int\limits_a^b g_0(s, \rho, \dots, \rho)\, ds = \delta \ \text{ for } \ a \leq t \leq b.$$

Consequently, the estimates (1.11) are valid. □

As an example, we consider the differential inequality

$$\begin{aligned} p_0(t) q_0\big(u(t), \dots, u^{(n-1)}(t)\big) \leq u^{(n)}(t) \leq \\ \leq p(t) q\big(u(t), \dots, u^{(n-1)}(t)\big) + \sum_{k=1}^{n} p_k(t) u^{(k-1)}(t), \end{aligned} \tag{1.16}$$

where $p_k : [a, b] \to \mathbb{R}_+$ $(k = 0, \dots, n)$, $p : [a, b] \to \mathbb{R}_+$ are integrable functions, and $q_0 : \mathbb{R}^n_{0+} \to \mathbb{R}_{0+}$, $q : \mathbb{R}^n_{0+} \to \mathbb{R}_{0+}$ are continuous and nonincreasing in all variables functions.

Corollary 1.1. *If*

$$\int\limits_a^b p_0(s)\, ds > 0, \quad \sum_{k=1}^{n} \gamma_k \int\limits_a^b p_k(s)\, ds < 1, \tag{1.17}$$

then there exist positive constants δ and ρ such that an arbitrary solution of the problem (1.16), (1.2) *admits the estimates* (1.11)*.*

Proof. Let

$$\begin{aligned} &g_0(t, x_1, \dots, x_n) = p_0(t) q_0(x_1, \dots, x_n), \\ &g_k(t, x_1, \dots, x_n) = \frac{p(t)}{n x_k}\, q(x_1, \dots, x_n) + p_k(t) \ \ (k = 1, \dots, n). \end{aligned}$$

Then the differential inequality (1.16) takes the form (1.1). On the other hand, by virtue of (1.17), the functions $g_k : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(k = 0, \dots, n)$ satisfy the conditions (1.9) and (1.10). If now we apply Theorem 1.1, then validity of Corollary 1.1 becomes evident. $\square$

Note that in the conditions of Theorem 1.1 or Corollary 1.1, the differential inequality under consideration may have singularities of arbitrary orders in phase variables. For example, In Corollary 1.1 as q_0 and q we can take the functions

$$q_0(x_1, \dots, x_n) = \ell_{01} \prod_{i=1}^n x_i^{-\lambda_{0i}} \exp\Big(\ell_{02} \prod_{j=1}^n x_j^{-\mu_{0j}}\Big),$$

$$q(x_1, \dots, x_n) = q_0(x_1, \dots, x_n) + \ell_1 \prod_{i=1}^n x_i^{-\lambda_i} \exp\Big(\ell_2 \prod_{j=1}^n x_j^{-\mu_j}\Big),$$

where λ_{0i}, λ_i, μ_{0i}, μ_i $(i = 1, \dots, n)$, ℓ_{0k}, ℓ_k $(k = 1, 2)$ are positive constants.

2. First Order Differential Inequalities

Let us consider the differential inequality

$$\sigma\big(u'(t) - p(t)u(t) - q\big(t, u(t)\big)\big) \geq 0 \tag{2.1}$$

with the boundary condition

$$\sigma\big(u(a) - \alpha u(b) - \alpha_0\big) \geq 0, \tag{2.2}$$

where $p : [a,b] \to \mathbb{R}$ is an integrable function, $q : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ is an integrable in the first argument and continuous and nonincreasing in the second argument function, $\sigma \in \{-1, 1\}$, $\alpha > 0$ and $\alpha_0 \geq 0$ are constants.

An absolutely continuous function $u : [a,b] \to \mathbb{R}_{0+}$ is said to be a **solution of the problem** (2.1), (2.2) if it satisfies the condition (2.2) and almost everywhere on $[a,b]$ satisfies the differential inequality (2.1).

Along with (2.1), (2.2), we consider the boundary value problem of periodic type:

$$v'(t) = p(t)v(t) + q\big(t, v(t)\big), \tag{2.3}$$

$$v(a) = \alpha v(b) + \alpha_0. \tag{2.4}$$

The following theorem holds.

Theorem 2.1. *If*

$$\alpha \exp\Big(\int_a^b p(s)\, ds\Big) < 1 \tag{2.5}$$

and

$$\int_a^b q(s,x)\, ds > 0 \quad \textit{for} \ \ x > 0, \tag{2.6}$$

then the problem (2.3), (2.4) *has a unique solution* v, *and an arbitrary solution* u *of the problem* (2.1), (2.2) *admits the estimate*

$$\sigma\big(u(t)-v(t)\big) \geq 0 \ \ \textit{for} \ \ a \leq t \leq b. \tag{2.7}$$

To prove the theorem, we need the following simple lemma.

Lemma 2.1. *Let* $t_0 \in [a,b[$ *and* $c > 0$. *Then the differential equation* (2.1) *under the initial condition*

$$v(t_0) = c \tag{2.8}$$

has a unique solution v *in the interval* $[t_0, b]$, *and an arbitrary solution* u *of the differential inequality* (2.1), *satisfying the condition*

$$\sigma\big(u(t_0) - c\big) \geq 0,$$

admits the estimate

$$\sigma\big(u(t)-v(t)\big) \geq 0 \ \ \textit{for} \ \ t_0 \leq t \leq b. \tag{2.9}$$

Proof. The unique solvability of the problem (2.1), (2.8) in the interval $[t_0, b]$ follows from the fact that $c > 0$ and the function $q : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ is nonincreasing in the second argument.

Applying now Lemma 4.3 from [5], the validity of the estimate (2.9) becomes evident. □

Proof of Theorem 2.1. For the sake of definiteness we assume that $\sigma = 1$ since the case where $\sigma = -1$ is considered analogously.

If $q : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ is a continuous and nonincreasing in the second argument function, then by Theorem 7 of [11], the conditions (2.5) and (2.6) guarantee the unique solvability of the problem (2.3), (2.4). If, however, q is integrable in the first and continuous and nonincreasing in the second argument, then using the method of proving of the above-mentioned theorem, we can show that the conditions (2.5) and (2.6) again guarantee the existence of a unique solution v of the problem (2.3), (2.4).

Let u be an arbitrary solution of the problem (2.1), (2.2). If

$$u(a) \geq v(a),$$

then by Lemma 2.1, the estimate (2.7) is valid.

To prove the theorem, it remains to show that the inequality

$$u(a) < v(a) \tag{2.10}$$

cannot take place.

Assume the contrary that the inequality (2.10) is valid. Then either

$$u(t) < v(t) \ \ \text{for} \ \ a < t < b, \tag{2.11}$$

or there exists $t_0 \in \,]a,b[$ such that

$$u(t_0) \geq v(t_0). \tag{2.12}$$

Let the inequality (2.11) be fulfilled. Then in view of (2.1), almost everywhere on $[a,b]$ the inequality

$$u'(t) \geq p(t)u(t) + q\big(t, v(t)\big) \tag{2.13}$$

is fulfilled since q is the nonincreasing in the second argument function.

Put

$$w(t) = v(t) - u(t).$$

Then in view of the conditions (2.2), (2.4), (2.10) and (2.13),we have

$$0 < w(a) \leq \alpha w(b)$$

and

$$w'(t) \leq p(t)w(t) \text{ for almost all } t \in [a,b].$$

From these inequalities with regard for the condition (2.5) we find

$$w(b) \leq \exp\bigg(\int\limits_a^b p(s)\,ds\bigg) w(a) \leq \alpha \exp\bigg(\int\limits_a^b p(s)\,ds\bigg) w(b) < w(b).$$

The obtained contradiction proves that the inequality (2.11) cannot take place. Consequently, for some $t_0 \in]a,b[$ the inequality (2.12) is fulfilled.

By Lemma 2.1, the function u admits the estimate (2.9). From (2.4), (2.9) and (2.10), we find

$$u(a) < v(a) = \alpha v(b) + \alpha_0 \leq \alpha u(b) + \alpha_0,$$

which contradicts the inequality (2.2). The obtained contradiction proves that the inequality (2.10) cannot take place. Thus the theorem is proved. $\square$

In conclusion of this section we consider the problem

$$\sigma\big(u'(t) - p(t)u(t) + q\big(t, u(t)\big)\big) \leq 0, \tag{2.14}$$

$$\sigma\big(u(a) - \alpha u(b) + \alpha_0\big) \leq 0, \tag{2.15}$$

and the differential equation

$$v'(t) = p(t)v(t) - q\big(t, v(t)\big) \tag{2.16}$$

with the boundary condition

$$v(a) = \alpha v(b) - \alpha_0. \tag{2.17}$$

As above we assume that $p : [a,b] \to \mathbb{R}$ is an integrable function, and $q : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ is an integrable in the first and continuous and nonincreasing in the second argument function, $\sigma \in \{-1,1\}$, $\alpha > 0$ and $\alpha_0 \geq 0$.

On the basis of Theorem 2.1, the following statement can be proved.

Theorem 2.2. *If along with* (2.6) *the inequality*

$$\alpha \exp\Big(\int_a^b p(s)\,ds\Big) > 1 \tag{2.18}$$

is fulfilled, then the problem (2.16), (2.17) *has a unique solution* v, *and an arbitrary solution* u *of the problem* (2.14), (2.15) *admits the estimate* (2.7).

If $q(t,x) \equiv q(t)$, then the differential inequalities (2.1), (2.14) and the differential equations (2.3) and (2.16) have the following forms

$$\sigma\big(u'(t) - p(t)u(t) - q(t)\big) \geq 0, \tag{2.19}$$

$$\sigma\big(u'(t) - p(t)u(t) + q(t)\big) \leq 0, \tag{2.20}$$

$$v'(t) = p(t)v(t) + q(t), \tag{2.21}$$

$$v'(t) = p(t)v(t) - q(t). \tag{2.22}$$

It is easy to see that for the unique solvability of the problem (2.21), (2.4) (of the problem (2.22), (2.17)) it is necessary and sufficient the inequality

$$1 - \alpha \exp\Big(\int_a^b p(s)\,ds\Big) \neq 0 \tag{2.23}$$

to be fulfilled.

Let the inequality (2.23) hold. Put

$$\Delta(p,\alpha) = 1 - \alpha \exp\Big(\int_a^b p(s)\,ds\Big), \tag{2.24}$$

$$g(p,\alpha)(t,s) = \\
= \begin{cases} \dfrac{1}{\Delta(p,\alpha)} \exp\Big(\displaystyle\int_s^t p(\tau)\,d\tau\Big) & \text{for } a \leq s \leq t \leq b, \\ \dfrac{\alpha}{\Delta(p,\alpha)} \exp\Big(\displaystyle\int_a^b p(\tau)\,d\tau + \int_s^t p(\tau)\,d\tau\Big) & \text{for } a \leq t < s \leq b. \end{cases} \tag{2.25}$$

Then the solution of the problem (2.21), (2.22) admits the representation

$$v(t) = \frac{\alpha_0}{\Delta(p,\alpha)} \exp\Big(\int_a^t p(\tau)\,d\tau\Big) + \int_a^b g(p,\alpha)(t,s)q(s)\,ds,$$

and the solution of the problem (2.22), (2.17) admits the representation

$$v(t) = -\frac{\alpha_0}{\Delta(p,\alpha)} \exp\Big(\int_a^t p(s)\,ds\Big) - \int_a^b g(p,\alpha)(t,s)q(s)\,ds.$$

On the other hand, in view of the fact that the number α is positive, (2.24) and (2.25) imply

$$\Delta(p,\alpha)g(p,\alpha)(t,s) > 0 \quad \text{for} \quad a \le s \le t \le b. \tag{2.26}$$

If along with this we take into account the fact that the function q is nonnegative, then it becomes evident that Theorems 2.1 and 2.2 yield the following propositions.

Corollary 2.1. *If the inequality* (2.5) (*the inequality* (2.18)) *is fulfilled, then an arbitrary solution of the problem* (2.19), (2.2) (*of the problem* (2.20), (2.15)) *admits the estimate* (2.7), *where*

$$v(t) = \frac{\alpha_0}{|\Delta(p,\alpha)|} \exp\bigg(\int_a^t p(s)\,ds\bigg) + \int_a^b \big|g(p,\alpha)(t,s)\big| q(s)\,ds \quad \textit{for} \quad a \le t \le b.$$

Lemma 2.2. *Let p be a constant sign function, satisfying the condition* (2.23). *Then*

$$\int_a^b \big|g(p,\alpha)(t,s)p(s)\big|\,ds \le \frac{\alpha+1+|\alpha-1|}{2}\,\bigg|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\bigg| \quad \textit{for} \quad a \le t \le b, \tag{2.27}$$

$$\int_a^b \big|g(p,\alpha)(t,s)p(s)\big|\,ds \ge \frac{\alpha+1-|\alpha-1|}{2}\,\bigg|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\bigg| \quad \textit{for} \quad a \le t \le b. \tag{2.28}$$

Proof. Due to the fact that p is of constant sign and the condition (2.26), there exists a number $\sigma_0 \in \{-1,1\}$ such that

$$\int_a^b \big|g(p,\alpha)(t,s)p(s)\big|\,ds = \sigma_0 w(t) \quad \text{for} \quad a \le t \le b, \tag{2.29}$$

where

$$w(t) = \int_a^b g(p,\alpha)(t,s)p(s)\,ds.$$

On the other hand, in view of the equalities (2.24) and (2.25), we find

$$w(t) = \frac{1-\alpha}{\Delta(p,\alpha)} \exp\bigg(\int_a^t p(s)\,ds\bigg) - 1.$$

Hence it is clear that

$$\min\big\{|w(a)|,|w(b)|\big\} \le |w(t)| \le \max\big\{|w(a)|,|w(b)|\big\}.$$

However,

$$w(a) = -\frac{\alpha\Delta(p,1)}{\Delta(p,\alpha)}\,, \quad w(b) = -\frac{\Delta(p,1)}{\Delta(p,\alpha)}\,.$$

Thus,

$$\min\{\alpha,1\}\left|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\right| \leq |w(t)| \leq \max\{\alpha,1\}\left|\frac{\Delta(p,1)}{\Delta(p,\alpha)}\right| \text{ for } a \leq t \leq b,$$

according to which from the equality (2.29) it follows the estimates (2.27) and (2.28). ☐

3. Systems of Differential Inequalities

In this section, we establish a priori estimates of solutions of the system of differential inequalities

$$\begin{aligned} q_i\big(t,u_i(t)\big) \leq \sigma_i\big(u_i'(t)-p_i(t)u_i(t)\big) \leq \\ \leq \sum_{k=1}^n p_{ik}\big(t,u_1(t)+\cdots+u_n(t)\big)u_k(t)+ \\ + q_0\big(t,u_1(t),\dots,u_n(t)\big) \;\; (i=1,\dots,n), \end{aligned} \tag{3.1}$$

satisfying the boundary conditions

$$\sigma_i\big(u_i(a)-\alpha_i u_i(b)\big) \geq 0, \quad \sigma_i\big(u_i(a)-\beta_i u_i(b)\big) \leq \beta_0 \;\; (i=1,\dots,n). \tag{3.2}$$

Here

$$\begin{gathered} \sigma_i \in \{-1,1\}, \;\; \alpha_i>0, \;\; \beta_i>0, \\ \sigma_i(\beta_i-\alpha_i)>0 \;\; (i=1,\dots,n), \;\; \beta_0>0, \end{gathered} \tag{3.3}$$

$p_i:[a,b]\to\mathbb{R}$ $(i=1,\dots,n)$ are integrable functions, $q_i:[a,b]\times\mathbb{R}_{0+}\to\mathbb{R}_+$ and $p_{ik}:[a,b]\times\mathbb{R}_{0+}\to\mathbb{R}_+$ $(i,k=1,\dots,n)$ are integrable in the first and continuous and nonincreasing in the second argument functions, and $q_0:[a,b]\times\mathbb{R}^n_{0+}\to\mathbb{R}_+$ is an integrable in the first and continuous and nonincreasing in the last n arguments function.

A vector function $(u_i)_{i=1}^n:[a,b]\to\mathbb{R}^n_{0+}$ with absolutely continuous components $u_i:[a,b]\to\mathbb{R}_{0+}$ $(i=1,\dots,n)$ **is said to be a solution of the system** (3.1) if it satisfies that system almost everywhere on $[a,b]$.

A solution of the system (3.1), satisfying the boundary conditions (3.2), **is said to be a solution of the problem** (3.1), (3.2).

We investigate the problem (3.1), (3.2) in the case, where

$$\int_a^b q_i(s,x)\,ds>0 \text{ for } x>0 \;\; (i=1,\dots,n) \tag{3.4}$$

and

$$\sigma_i\bigg(\beta_i\exp\bigg(\int_a^b p_i(s)\,ds\bigg)-1\bigg)<0 \;\; (i=1,\dots,n). \tag{3.5}$$

Let g be the operator given by the equalities (2.24) and (2.25). Suppose

$$h_{ik}(x) =$$

$$= \max\left\{ \int_a^b \big|g(p_i,\beta_i)(t,s)\big| p_{ik}(s,x)\,ds : \ a \le t \le b \right\} \ \ (i,k = 1,\dots,n) \quad (3.6)$$

and

$$H(x) = \big(h_{ik}(x)\big)_{i,k=1}^{n} \ \text{ for } \ x > 0. \quad (3.7)$$

Theorem 3.1. *Let along with* (3.3)–(3.5) *the condition*

$$\lim_{x\to+\infty} r(H(x)) < 1 \quad (3.8)$$

be fulfilled. Then there exist positive constants δ and ρ such that an arbitrary solution $(u_i)_{i=1}^{n}$ of the problem (3.1), (3.2) *admits the estimates*

$$\delta \le u_i(t) \le \rho \ \text{ for } \ a \le t \le b \ \ (i = 1,\dots,n). \quad (3.9)$$

To prove this theorem, along with the results from Section 2 we need the following lemma.

Lemma 3.1. *Let $h_{ik} : \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i,k = 1,\dots,n)$ be nonincreasing functions, and h_i $(i = 1,\dots,n)$ be nonnegative constants. Let, moreover, there exist a positive number x_0 such that*

$$r(H(x_0)) < 1, \quad (3.10)$$

where H is a matrix function given by the equality (3.7)*. Then arbitrary positive numbers $x_1,\dots,x_n$, satisfying the system of inequalities*

$$x_i \le \sum_{k=1}^{n} h_{ik}(x_1 + \dots + x_n)x_k + h_i \ \ (i = 1,\dots,n), \quad (3.11)$$

satisfy the inequality

$$\sum_{i=1}^{n} x_i \le x_0 + \big\|(E - H(x_0))^{-1}\big\| \sum_{i=1}^{n} h_i \quad (3.12)$$

as well, where E is a unit $n \times n$-matrix, and $(E - H(x_0))^{-1}$ is a matrix, inverse to the matrix $E - H(x_0)$.

Proof. Assume the contrary that

$$\sum_{i=1}^{n} x_i > x_0 + \big\|(E - H(x_0))^{-1}\big\| \sum_{i=1}^{n} h_i. \quad (3.13)$$

Then from (3.10) we have

$$x_i \le \sum_{k=1}^{n} h_{ik}(x_0)x_k + h_i \ \ (i = 1,\dots,n)$$

since h_{ik} $(i,k = 1,\dots,n)$ are nonincreasing functions. Consequently,

$$(E - H(x_0))\overline{x} \le \overline{h}, \quad (3.14)$$

where

$$\overline{x} = (x_i)_{i=1}^n, \quad \overline{h} = (h_i)_{i=1}^n.$$

The nonnegativeness of the matrix $H(x_0)$ and the condition (3.10) guarantee the nondegeneracy of the matrix $E - H(x_0)$ and the nonnegativeness of the matrix $(E - H(x_0))^{-1}$.

If we multiply both sides of the inequality (3.14) by $(E - H(x_0))^{-1}$, we obtain

$$\overline{x} \leq (E - H(x_0))^{-1}\overline{h}.$$

Thus

$$\sum_{i=1}^n x_i \leq (E - H(x_0))^{-1} \sum_{i=1}^n h_i,$$

which contradicts the inequality (3.13). The obtained contradiction proves the validity of the estimate (3.12). □

Proof of Theorem 3.1. According to the condition (3.8), there exists a positive number x_0 such that the inequality (3.10) holds.

(3.3) and (3.5) imply

$$\sigma_i\bigg(\alpha_i \exp\bigg(\int\limits_a^b p_i(s)\,ds\bigg) - 1\bigg) < 0 \quad (i = 1, \dots, n). \tag{3.15}$$

On the other hand, by virtue of Theorems 2.1, 2.2 and the conditions (3.4) and (3.15) for any $i \in \{1, \dots, n\}$ the problem

$$\begin{aligned} &v_i'(t) = p_i(t)v(t) + \sigma_i q_i(t, v_i(t)), \\ &v_i(a) = \alpha_i v_i(b) \end{aligned}$$

has a unique solution v_i.

Put

$$\delta_i = \min\big\{v_i(t) : \ a \leq t \leq b\big\} \quad (i = 1, \dots, n),$$

$$\begin{aligned} h_i = {} & \frac{\beta_0}{|\Delta(p_i, \beta_i)|} \exp\bigg(\int\limits_a^b |p_i(s)|\,ds\bigg) + \\ & + \max\bigg\{\int\limits_a^b \big|g(p_i, \beta_i)(t,s)\big| q_0(s, \delta_1, \dots, \delta_n)\,ds : \ a \leq t \leq b\bigg\} \quad (i = 1, \dots, n), \end{aligned} \tag{3.16}$$

$$\delta = \min\{\delta_1, \dots, \delta_n\}, \quad \rho = x_0 + \big\|(E - H(x_0))^{-1}\big\| \sum_{i=1}^n h_i. \tag{3.17}$$

Let $(u_i)_{i=1}^n$ be a solution of the problem (3.1), (3.2). Our aim is to prove that this solution admits the estimates (3.9).

For each $i \in \{1, \dots, n\}$ the function u_i is a solution of the problem

$$\sigma_i\big(u_i'(t) - p_i(t)u_i(t)\big) \geq q_i(t, u_i(t)),$$
$$\sigma_i\big(u_i(a) - \alpha_i u_i(b)\big) \geq 0.$$

Hence by virtue of the conditions (3.4), (3.15) and Theorems 2.1 and 2.2 it follows that

$$u_i(t) \geq v_i(t) \;\text{ for }\; a \leq t \leq b$$

and, consequently,

$$u_i(t) \geq \delta_i \;\text{ for }\; a \leq t \leq b \;\; (i = 1, \dots, n). \tag{3.18}$$

According to (3.1), (3.2), and (3.18), for each $i \in \{1, \dots, n\}$ the function u_i is a solution of the problem

$$\sigma_i\big(u_i'(t) - p_i(t)u_i(t)\big) \leq \sum_{k=1}^{n} p_{ik}(t, x_1 + \dots + x_n)x_k + q_0(t, \delta_1, \dots, \delta_n),$$
$$\sigma_i\big(u_i(a) - \beta_i(t)u_i(b)\big) \leq \beta_0,$$

where

$$x_k = \max\big\{u_k(t) : a \leq t \leq b\big\} \;\; (k = 1, \dots, n). \tag{3.19}$$

Hence by virtue of the condition (3.5) and Corollary 2.1 it follows that

$$u_i(t) \leq \sum_{k=1}^{n} \bigg(\int_a^b \big|g(p_i, \beta_i)(t, s)\big| p_{ik}(s, x_1 + \dots + x_n)\, ds\bigg) x_k +$$
$$+ \frac{\beta_0}{|\Delta(p_i, \beta_i)|} \exp\bigg(\int_a^t p_i(s)\, ds\bigg) +$$
$$+ \int_a^b \big|g(p_i, \beta_i)(t, s)\big| q_0(s, \delta_1, \dots, \delta_n)\, ds \;\text{ for }\; a \leq t \leq b.$$

If along with this estimate we take into account the notations (3.6) and (3.16), then it becomes clear that the numbers $x_1, \dots, x_n$ satisfy the system of inequalities (3.11). By Lemma 3.1 these numbers satisfy the inequality (3.12) as well.

Due to (3.17) and (3.19), the estimates (3.12) and (3.18) result in the estimates (3.9). ☐

Corollary 3.1. *Let the functions p_i $(i = 1, \dots, n)$ are of constant sign,*

$$p_{ik}(t, x) \equiv |p_i(t)| p_{0ik}(x) \;\; (i, k = 1, \dots, n), \tag{3.20}$$

and let along with (3.3)–(3.5) *the condition*

$$\lim_{x \to +\infty} r(H_0(x)) < 1 \tag{3.21}$$

be fulfilled, where $p_{0ik} : \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i,k = 1,\dots,n)$ *are nonincreasing functions and*

$$H_0(x) = \left(\frac{\beta_i + 1 + |\beta_i - 1|}{2}\,\Big|\frac{\Delta(p_i,1)}{\Delta(p_i,\beta_i)}\Big| p_{0ik}(x)\right)_{i,k=1}^{n}, \tag{3.22}$$

and Δ *is a functional, given by the equality* (2.24)*. Then there exist positive constants* δ *and* ρ *such that an arbitrary solution* $(u_i)_{i=1}^n$ *of the problem* (3.1)*,* (3.2) *admits the estimates* (3.9)*.*

Proof. By Lemma 2.2, the estimates

$$\int\limits_a^b \big|g(p_i,\beta_i)(t,s)p_i(s)\big|\,ds \leq$$

$$\leq \frac{\beta_i + 1 + |\beta_i - 1|}{2}\,\Big|\frac{\Delta(p_i,1)}{\Delta(p_i,\beta_i)}\Big| \ \text{ for } \ a \leq t \leq b \ \ (i = 1,\dots,n)$$

are valid, according to which (3.6) and (3.20) result in the inequalities

$$h_{ik}(x) \leq \frac{\beta_i + 1 + |\beta_i - 1|}{2}\,\Big|\frac{\Delta(p_i,1)}{\Delta(p_i,\beta_i)}\Big| p_{0ik}(x) \ \text{ for } \ x > 0 \ \ (i,k = 1,\dots,n).$$

Hence in view of (3.7) and (3.22) it is obvious that

$$H(x) \leq H_0(x) \ \text{ for } \ x > 0$$

and, consequently,

$$r(H(x)) \leq r(H_0(x)) \ \text{ for } \ x > 0.$$

Thus the inequalities (3.21) yield the inequality (3.8).

If now we apply Theorem 3.1, then the validity of Corollary 3.1 becomes evident. □

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