Memoirs on Differential Equations and Mathematical Physics
Volume 63, 2014, 123–140

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ON THE BLOCK SEPARATION OF THE LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM WITH OSCILLATING COEFFICIENTS IN THE RESONANCE CASE

Abstract. For the linear homogeneous differential system with oscillating coefficients the sufficient conditions of existence of linear transformation reducing this system to a block-diagonal form in a resonance case are obtained.

2010 Mathematics Subject Classification. 34C20, 34C25.
Key words and phrases. Linear systems of differential equations, periodic solutions, slowly varying parameters.

რეზიუმე. წრფივი ერთგვაროვანი დიფერენციალური სისტემისათვის რხევადი კოეფიციენტებით დადგენილია ისეთი წრფივი გარდაქმნის არსებობის საკმარისი პირობები, რომელიც მას მიიყვანს უჯრულ-დიაგონალურ სახეზე რეზონანსულ შემთხვევაში.

1. Introduction

In the theory of differential equations of great importance is the problem of separation of a linear homogeneous n-th order differential system into k independent systems of orders $n_1, n_2, \dots, n_k$ $(n_1 + n_2 + \cdots + n_k = n)$, in particular, separation of this system into n independent first-order differential equations (full separation). This problem has been considered, for example, in [1–8]. Obviously, it is impossible in a general case to construct transformations explicitly, leading to a separated system. Such a construction assumes for the initial system to be integrable. Therefore, in these studies there was no attempt to construct such a transformation explicitly; these works established only the conditions of its existence, investigated its properties and possibility for its approximate construction, particularly, in the form of asymptotic series. Of importance is also the question on the belonging of elements of a transforming matrix to the same classes as elements of the matrix of the original system.

In his articles [9–12], the author considers the problem of full separation of the system of the kind

$$\frac{dx}{dt} = (\Lambda(t,\varepsilon) + \mu B(t,\varepsilon,\theta))x, \tag{1}$$

where $\Lambda(t,\varepsilon) = \operatorname{diag}(\lambda_1(t,\varepsilon), \dots, \lambda_n(t,\varepsilon))$, and the functions $\lambda_j(t,\varepsilon)$ $(j = \overline{1,n})$ are, in a definite sense, slowly varying, μ is a small positive parameter, elements of the matrix $B(t,\varepsilon,\theta)$ are represented by absolutely and uniformly convergent Fourier series with slowly varying coefficients and frequency $\varphi(t,\varepsilon) = \frac{d\theta}{dt}$. At the same time, the cases of resonance absence and presence of resonance, including the special case, have been investigated. For each of these cases the conditions were obtained under which the transforming matrix elements have a structure similar to that of the matrix $B(t,\varepsilon,\theta)$. In this article we study the possibility of block separation of the system (1) into two independent systems of smaller dimensions in a resonance case. Such a statement of the problem has some features as compared with the problems considered in [9–12].

2. Basic Notation and Definitions

Let $G = \{t, \varepsilon :\ t \in \mathbf{R},\ \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 1. We say that the function $p(t,\varepsilon)$ is in general complex-valued, belongs to the class $S(m;\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if $t, \varepsilon \in G$ and

(1) $p(t,\varepsilon) \in C^m(G)$ with respect to t;

(2) $d^k p(t,\varepsilon)/dt^k = \varepsilon^k p_k^*(t,\varepsilon)$, $\sup\limits_G |p_k^*(t,\varepsilon)| < +\infty$ $(0 \le k \le m)$.

Slow variation of a function is understood here in a sense of its belonging to the class $S(m;\varepsilon_0)$. As examples of this class of functions may serve in a general case complex-valued bounded together with their derivatives up to the m-th order, inclusive, functions depending on the "slow time" $\tau = \varepsilon t$: $\sin\tau$, $\operatorname{arctg}\tau$, etc.

Definition 2. We say that the function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$, $m \in \mathbf{N}\cup\{0\}$, if this function can be represented as

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon)\exp\big(in\theta(t,\varepsilon)\big),$$

where

(1) $f_n(t,\varepsilon) \in S(m;\varepsilon_0)$, $d^k f_n(t,\varepsilon)/dt^k = \varepsilon^k f_{nk}(t,\varepsilon)$ ($n \in \mathbf{Z}$, $0 \le k \le m$),

(2) $\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum\limits_{k=0}^{m}\sum\limits_{n=-\infty}^{\infty}\sup\limits_{G}|f_{nk}(t,\varepsilon)| < +\infty$,

(3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)\,d\tau$, $\varphi(t,\varepsilon) \in \mathbf{R}^+$, $\varphi(t,\varepsilon) \in S(m;\varepsilon_0)$, $\inf\limits_G \varphi(t,\varepsilon) > 0$.

In particular, if $\varepsilon = 0$: $\varphi = const$, $\theta = \varphi t$, $f_n = const$, then functions of the class $F(m;\varepsilon_0;\theta)$ are transformed into $2\pi/\varphi$-periodic functions of variable t,

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{in\varphi t},$$

such that

$$\sum_{n=-\infty}^{\infty} |f_n| < +\infty.$$

A set of functions of the class $F(m;\varepsilon_0;\theta)$ forms a linear space which transforms into a full normed space by means of the norm $\|\cdot\|_{F(m;\varepsilon_0,\theta)}$. The following chain of inclusions

$$F(0;\varepsilon_0;\theta) \supset F(1;\varepsilon_0;\theta) \supset \cdots \supset F(m;\varepsilon_0;\theta)$$

is valid.

Let there be given two functions of the class $F(m;\varepsilon_0;\theta)$:

$$u(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} u_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),$$

$$v(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} v_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)).$$

We define product of those functions by the formula [13]:

$$(uv)(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} u_{n-s}(t,\varepsilon)v_s(t,\varepsilon)\exp(in\theta(t,\varepsilon)).$$

Obviously, $uv \in F(m;\varepsilon_0;\theta)$. We state some properties of the norm $\|\cdot\|_{F(m;\varepsilon_0;\theta)}$. Let $u, v \in F(m;\varepsilon_0;\theta)$, $k = const$. Then

(1) $\|ku\|_{F(m;\varepsilon_0;\theta)} = |k|\|u\|_{F(m;\varepsilon_0;\theta)}$;

(2) $\|u+v\|_{F(m;\varepsilon_0;\theta)} \le \|u\|_{F(m;\varepsilon_0;\theta)} + \|v\|_{F(m;\varepsilon_0;\theta)}$;

(3) $\|uv\|_{F(m;\varepsilon_0;\theta)} \leq 2^m \|u\|_{F(m;\varepsilon_0;\theta)} \|v\|_{F(m;\varepsilon_0;\theta)}$.

For any $f(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$, we denote:

$$\Gamma_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t,\varepsilon,u) \exp(-inu)\, du.$$

Let $A(t,\varepsilon,\theta) = (a_{jk}(t,\varepsilon,\theta))$ – $(M \times K)$ be the matrix with elements of the class $F(m;\varepsilon_0;\theta)$. We denote:

$$(A)_{jk} = a_{jk} \;\; (j = \overline{1,M}, \;\; k = \overline{1,K}),$$

$$\|A\|^*_{F(m;\varepsilon_0;\theta)} = \max_{1 \leq j \leq M} \sum_{k=1}^{K} \|(A)_{jk}\|_{F(m;\varepsilon_0;\theta)}.$$

3. Statement of the Problem

Consider the following system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= H_1(\varphi)x_1 + \mu(B_{11}(t,\varepsilon,\theta)x_1 + B_{12}(t,\varepsilon,\theta)x_2), \\ \frac{dx_2}{dt} &= H_2(\varphi)x_2 + \mu(B_{21}(t,\varepsilon,\theta)x_1 + B_{22}(t,\varepsilon,\theta)x_2), \end{aligned} \tag{2}$$

where $x_1 = \operatorname{colon}(x_{11}, \ldots, x_{1N_1})$, $x_2 = \operatorname{colon}(x_{21}, \ldots, x_{2N_2})$,

$$H_1(\varphi) = \begin{pmatrix} ip\varphi & 0 & \ldots & 0 & 0 \\ 1 & ip\varphi & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & ip\varphi \end{pmatrix},$$

$$H_2(\varphi) = \begin{pmatrix} ir\varphi & 0 & \ldots & 0 & 0 \\ 1 & ir\varphi & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & ir\varphi \end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively ($N_1 + N_2 = N$); $p, r \in \mathbf{Z}$; $B_{jk}(t,\varepsilon,\theta)$ are the $(N_j \times N_k)$ matrices with elements of the class $F(m;\varepsilon;\theta)$; $\varphi(t,\varepsilon)$ is the function appearing in the definition of class $F(m;\varepsilon;\theta)$; $\mu \in (0,1)$. In this sense, we deal with a resonance case.

We study the problem of existence and properties of the transformation of kind:

$$x_j = L_{j1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1 + L_{j2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \;\; j = 1,2, \tag{3}$$

where the elements of $(N_j \times N_k)$-matrices L_{jk} $(j,k = 1,2)$ belong to the class $F(m-1;\varepsilon_1;\theta)$ $(0 < \varepsilon_1 \leq \varepsilon_0)$, reducing the system (2) to the form:

$$\frac{d\widetilde{x}_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)\widetilde{x}_1, \quad \frac{d\widetilde{x}_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)\widetilde{x}_2, \tag{4}$$

where the elements of $(N_j \times N_j)$-matrices D_{N_j} $(j = 1, 2)$ likewise belong to the class $F(m-1; \varepsilon_1; \theta)$.

4. Auxiliary Results

Lemma 1. *Let there be given a matrix differential equation*

$$\frac{dX}{dt} = \Big(J_M + \sum_{l=1}^{q} P_l(t,\varepsilon,\theta)\mu^l\Big)X - X\Big(J_K + \sum_{l=1}^{q} Q_l(t,\varepsilon,\theta)\mu^l\Big), \qquad (5)$$

where X is $(M \times K)$-matrix, $P_l(t,\varepsilon,\theta)$, $Q_l(t,\varepsilon,\theta)$ $(l = \overline{1,q})$ are matrices of dimensions $(M \times M)$ and $(K \times K)$ respectively with elements from the class $F(m;\varepsilon;\theta)$,

$$J_M = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad J_K = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

are Jordan blocks of dimensions M and K, respectively, whose diagonal elements are equal to zero, $\mu \in (0,1)$.

Then there exists $\mu_0 \in (0,1)$ such that for any $\mu \in (0,\mu_0)$ there exists transformation of the kind

$$X = \Big(E_M + \sum_{l=1}^{q} \Phi_l(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E_K + \sum_{l=1}^{q} \Psi_l(t,\varepsilon,\theta)\mu^l\Big), \qquad (6)$$

where Y is the $(M\times K)$-matrix, E_M, E_K are identity matrices of dimensions M and K respectively, the elements of $(M \times M)$-matrices Φ_l and those of $(K \times K)$-matrices Ψ_l $(l = \overline{1,q})$ belong to the class $F(m;\varepsilon;\theta)$ reducing equation (5) to the form:

$$\frac{dY}{dt} = \Big(J_M + \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l + \varepsilon\sum_{l=1}^{q} \widetilde{U}_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1}W_1(t,\varepsilon,\theta,\mu)\Big)Y -$$
$$- Y\Big(J_K + \sum_{l=1}^{q} V_l(t,\varepsilon)\mu^l + \varepsilon\sum_{l=1}^{q} \widetilde{V}_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1}W_2(t,\varepsilon,\theta,\mu)\Big), \quad (7)$$

where $U_l(t,\varepsilon)$, $V_l(t,\varepsilon)$ $(l = \overline{1,q})$ are the matrices of dimensions $(M \times M)$ and $(K \times K)$, respectively, with elements from the class $S(m;\varepsilon_0)$, $\widetilde{U}_l(t,\varepsilon)$ and $\widetilde{V}_l(t,\varepsilon)$ $(l = \overline{1,q})$ are the matrices of dimensions $(M\times M)$ and $(K\times K)$, respectively, with elements from the class $F(m-1;\varepsilon_0;\theta)$, W_1, W_2 are the matrices of dimensions $(M \times M)$ and $(K \times K)$, respectively, with elements from the class $F(m-1;\varepsilon_0;\theta)$.

Proof. We substitute (6) into the system (5) and require for the transformed system to have the form (7). Then for the matrices Φ_l, Ψ_l $(l = \overline{1,q})$ we obtain the following differential equations:

$$\frac{d\Phi_1}{dt} = J_M\Phi_1 - \Phi_1 J_M + P_1(t,\varepsilon,\theta) - U_1(t,\varepsilon) - \varepsilon\widetilde{U}_1(t,\varepsilon,\theta), \tag{8}$$

$$\frac{d\Psi_1}{dt} = J_K\Psi_1 - \Psi_1 J_K - Q_1(t,\varepsilon,\theta) + V_1(t,\varepsilon) + \varepsilon\widetilde{V}_1(t,\varepsilon,\theta), \tag{9}$$

$$\begin{aligned}\frac{d\Phi_l}{dt} = {} & J_M\Phi_l - \Phi_l J_M + P_l(t,\varepsilon,\theta) + \sum_{\nu=1}^{l-1} P_\nu(t,\varepsilon,\theta)\Phi_{l-\nu} - \\ & - \sum_{\nu=1}^{l-1}\Phi_\nu U_{l-\nu}(t,\varepsilon) - \varepsilon\sum_{\nu=1}^{l-1}\Phi_\nu\widetilde{U}_{l-\nu}(t,\varepsilon,\theta) - \\ & - U_l(t,\varepsilon) - \varepsilon\widetilde{U}_l(t,\varepsilon,\theta), \quad l = \overline{2,q},\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{d\Psi_l}{dt} = {} & J_K\Psi_l - \Psi_l J_K - Q_l(t,\varepsilon,\theta) - \sum_{\nu=1}^{l-1}\Psi_\nu Q_{l-\nu}(t,\varepsilon,\theta) + \\ & + \sum_{\nu=1}^{l-1} V_\nu(t,\varepsilon)\Psi_{l-\nu} + \varepsilon\sum_{\nu=1}^{l-1}\widetilde{V}_\nu(t,\varepsilon,\theta)\Psi_{l-\nu} + \\ & + V_l(t,\varepsilon) + \varepsilon\widetilde{V}_l(t,\varepsilon,\theta), \quad l = \overline{2,q}.\end{aligned} \tag{11}$$

The matrices W_1, W_2 are defined from the equations

$$\Big(E_M + \sum_{l=1}^{q}\Phi_l(t,\varepsilon,\theta)\mu^l\Big)W_1 =$$

$$= \sum_{s=0}^{q-1}\Big[\sum_{\sigma+\delta=s+q+1}(P_\sigma\Phi_\delta - \Phi_\delta U_\sigma)\Big]\mu^s - \varepsilon\sum_{s=0}^{q-1}\Big(\sum_{\sigma+\delta=s+q+1}\Phi_\sigma\widetilde{U}_\delta\Big)\mu^s, \tag{12}$$

$$W_2\Big(E_K + \sum_{l=1}^{q}\Psi_l(t,\varepsilon,\theta)\mu^l\Big) =$$

$$= \sum_{s=0}^{q-1}\Big[\sum_{\sigma+\delta=s+q+1}(-\Psi_\sigma Q_\delta + V_\sigma\Psi_\delta)\Big]\mu^s + \varepsilon\sum_{s=0}^{q-1}\Big(\sum_{\sigma+\delta=s+q+1}\widetilde{V}_\sigma\Psi_\delta\Big)\mu^s, \tag{13}$$

Based on the equations (8)–(11), we set

$$(U_l)_{sM} = \Gamma_0((T_l)_{sM}),$$

$$(\Phi_l)_{sM} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\frac{\Gamma_n((\Phi_l)_{s-1,M} + (T_l)_{sM})}{in\varphi}\; e^{in\theta},$$

$$(\widetilde{U}_l)_{sM} = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\frac{d}{dt}\Big(\frac{\Gamma_n((\Phi_l)_{s-1,M} + (T_l)_{sM})}{in\varphi}\Big)e^{in\theta} - \Big(\sum_{\nu=1}^{l-1}\Phi_\nu\widetilde{U}_{l-\nu}\Big)_{sM},$$

$$(U_l)_{s,M-j} = \Gamma_0((T_l)_{s,M-j}),$$

$$(\Phi_l)_{s,M-j} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n((\Phi_l)_{s-1,M-j} - (\Phi_l)_{s,M-j+1} + (T_l)_{s,M-j})}{in\varphi}\ e^{in\theta},$$

$$(\widetilde{U}_l)_{s,M-j} = -\frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt} \Big(\frac{\Gamma_n((\Phi_l)_{s-1,M-j} - (\Phi_l)_{s,M-j+1} + (T_l)_{s,M-j})}{in\varphi} \Big) e^{in\theta} -$$

$$- \Big(\sum_{\nu=1}^{l-1} \Phi_\nu \widetilde{U}_{l-\nu} \Big)_{s,M-j} \quad (s = \overline{1, M}; \quad j = \overline{1, M-1}),$$

where

$$T_l = P_l + \sum_{\nu=1}^{l-1} P_\nu \Phi_{l-\nu} - \sum_{\nu=1}^{l-1} \Phi_\nu U_{l-\nu} \quad (l = \overline{1, q}).$$

(if $l = 1$, then we assume $\sum\limits_{\nu=1}^{l-1}$ to be equal to zero; if $s = 1$, then we assume $(\Phi)_{s-1,j}$ to be equal to zero),

$$(V_l)_{sK} = \Gamma_0((R_l)_{sK}),$$

$$(\Psi_l)_{sK} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n((\Psi_l)_{s-1,K} + (R_l)_{sK})}{in\varphi}\ e^{in\theta},$$

$$(\widetilde{V}_l)_{sK} = -\frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt} \Big(\frac{\Gamma_n((\Psi_l)_{s-1,K} + (R_l)_{sK})}{in\varphi} \Big) e^{in\theta} -$$

$$- \Big(\sum_{\nu=1}^{l-1} \widetilde{V}_\nu \Psi_{l-\nu} \Big)_{sK},$$

$$(V_l)_{s,K-j} = \Gamma_0((R_l)_{s,K-j}),$$

$$(\Psi_l)_{s,K-j} = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n((\Psi_l)_{s-1,K-j} - (\Psi_l)_{s,K-j+1} + (R_l)_{s,K-j})}{in\varphi} e^{in\theta},$$

$$(\widetilde{V}_l)_{s,K-j} = -\frac{1}{\varepsilon} \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt} \Big(\frac{\Gamma_n((\Psi_l)_{s-1,K-j} - (\Psi_l)_{s,K-j+1} + (R_l)_{s,K-j})}{in\varphi} \Big) e^{in\theta} -$$

$$- \Big(\sum_{\nu=1}^{l-1} \widetilde{V}_\nu \Psi_{l-\nu} \Big)_{s,K-j} \quad (s = \overline{1, K}; \quad j = \overline{1, K-1}),$$

where

$$R_l = -Q_l - \sum_{\nu=1}^{l-1} \Psi_\nu Q_{l-\nu} + \sum_{\nu=1}^{l-1} V_\nu \Psi_{l-\nu} \quad (l = \overline{1, q}).$$

(if $l = 1$, then we set $\sum\limits_{\nu=1}^{l-1}$ to be equal to zero; if $s = 1$, then we set $(\Psi)_{s-1,j}$ to be equal to zero).

Then for sufficiently small values μ, the matrices W_1, W_2 are uniquely defined from equations (12), (13). $\square$

Consider now the matrix differential equation

$$\frac{dX}{dt} = J_M X - X J_K + F(t,\varepsilon,\theta) + \mu(A(t,\varepsilon,\theta)X - \\ - XB(t,\varepsilon,\theta)) - \mu^2 X R(t,\varepsilon,\theta)X, \qquad (14)$$

where X is the $(M \times K)$-matrix, F, A, B, R are matrices of dimensions $(M \times K)$, $(M \times M)$, $(K \times K)$, $(K \times M)$, respectively, whose all elements belong to the class $F(m;\varepsilon;\theta)$.

Lemma 2. *Let equation* (14) *satisfy one of the sets of conditions* I*,* II*,* III*:*

I. (1) $M < K$,

(2) $\sum\limits_{s=1}^{j} \Gamma_0((F)_{s,K-j+s}) \equiv 0,\ j = \overline{1,M}$,

(3) $\inf\limits_{G} |\Gamma_0((B)_{1K})| > 0$;

II. (1) $M = K$,

(2) $\sum\limits_{s=1}^{j} \Gamma_0((F)_{s,K-j+s}) \equiv 0,\ j = \overline{1,M}$,

(3) $\inf\limits_{G} |\Gamma_0((A)_{1M} - (B)_{1M})| > 0$;

III. (1) $M > K$,

(2) $\sum\limits_{s=1}^{j} \Gamma_0((F)_{s,K-j+s}) \equiv 0,\ j = \overline{1,K}$,

(3) $\inf\limits_{G} |\Gamma_0((A)_{1M})| > 0$.

Then there exists $\mu_1 \in]0,1[$ such that for any $\mu \in]0,\mu_1[$ there exists the transformation of the kind

$$X = \sum_{s=0}^{2q-1} \Xi_s(t,\varepsilon,\theta)\mu^s + \Phi(t,\varepsilon,\theta,\mu) Y \Psi(t,\varepsilon,\theta,\mu), \qquad (15)$$

where the elements of $(M \times K)$-matrices Ξ_s $(s = \overline{0,2q-1})$, of $(M \times M)$-matrix Φ and of $(K \times K)$-matrix Ψ belong to the class $F(m;\varepsilon_0,\theta)$ $\forall \mu \in (0,\mu_1)$, reducing the equation (14) *to the form*

$$\frac{dY}{dt} = J_M Y - Y J_K + \Big(\sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l\Big)Y - Y\Big(\sum_{l=1}^{q} V_l(t,\varepsilon)\mu^l\Big) + \\ + \varepsilon\big(\widetilde{U}(t,\varepsilon,\theta,\mu)Y - Y\widetilde{V}(t,\varepsilon,\theta,\mu)\big) + \\ + \mu^{q+1}\big(\widetilde{W}_1(t,\varepsilon,\theta,\mu)Y - Y\widetilde{W}_2(t,\varepsilon,\theta,\mu)\big) + \\ + \varepsilon G(t,\varepsilon,\theta,\mu) + \mu^{2q} H(t,\varepsilon,\theta,\mu) + \mu Y R_1(t,\varepsilon,\theta,\mu) Y, \qquad (16)$$

where the elements of matrices U_l, V_l $(l = \overline{1,q})$ belong to the class $S(m;\varepsilon_0)$, and the elements of matrices $\widetilde{U}$, $\widetilde{V}$, $\widetilde{W_1}$, $\widetilde{W_2}$, G, H, R_1 of the corresponding dimensions belong to the class $F(m-1;\varepsilon_0;\theta)$.

Proof. Along with the equation (14), we consider an auxiliary matrix equation

$$\varphi(t,\varepsilon)\frac{d\Xi}{d\theta} = J_M\Xi - \Xi J_K + F(t,\varepsilon,\theta) + \\ + \mu(A(t,\varepsilon,\theta)\Xi - \Xi B(t,\varepsilon,\theta)) - \mu^2\Xi R(t,\varepsilon,\theta)\Xi, \quad (17)$$

where t, φ are considered as constants. The matrices-functions $F(t,\varepsilon,\theta)$, $A(t,\varepsilon,\theta)$, $B(t,\varepsilon,\theta)$, $R(t,\varepsilon,\theta)$ are 2π-periodic with respect to θ. We construct, according to the Poincare method of small parameter [14], an approximate 2π-periodic with respect to θ solution of the equation (17) in the form of a sum:

$$\Xi = \sum_{s=o}^{2q-1} \Xi_s(t,\varepsilon,\theta)\mu^s. \quad (18)$$

The coefficients Ξ_s are determined from the following chain of linear nonhomogeneous matrix differential equations:

$$\varphi(t,\varepsilon)\frac{d\Xi_0}{d\theta} = J_M\Xi_0 - \Xi_0 J_K + F(t,\varepsilon,\theta), \quad (19)$$

$$\varphi(t,\varepsilon)\frac{d\Xi_1}{d\theta} = J_M\Xi_1 - \Xi_1 J_K + A(t,\varepsilon,\theta)\Xi_0 - \Xi_0 B(t,\varepsilon,\theta), \quad (20)$$

$$\varphi(t,\varepsilon)\frac{d\Xi_2}{d\theta} = J_M\Xi_2 - \Xi_2 J_K + A(t,\varepsilon,\theta)\Xi_1 - \Xi_1 B(t,\varepsilon,\theta) - \\ - \Xi_0 R(t,\varepsilon,\theta)\Xi_0, \quad (21)$$

$$\varphi(t,\varepsilon)\frac{d\Xi_s}{d\theta} = J_M\Xi_s - \Xi_s J_K + A(t,\varepsilon,\theta)\Xi_{s-1} - \Xi_{s-1} B(t,\varepsilon,\theta) - \\ - \sum_{l=0}^{s-2} \Xi_l R(t,\varepsilon,\theta)\Xi_{s-2-l}, \quad s = \overline{3, 2q-1}. \quad (22)$$

First consider the case $M < K$. The condition I.(2) ensures the existence of a 2π-periodic with respect to θ solution $\Xi_0(t,\varepsilon,\theta)$ of the equation (19) having the form

$$\Xi_0(t,\varepsilon,\theta) = C_0(t,\varepsilon) + \widetilde{\Xi}_0(t,\varepsilon,\theta), \quad (23)$$

where $\widetilde{\Xi}_0(t,\varepsilon,\theta)$ is the known matrix whose elements belong to the class $F(m;\varepsilon_0;\theta)$, and $(M \times K)$-matrix $C_0(t,\varepsilon)$ has the form

$$C_0(t,\varepsilon) = \begin{pmatrix} c_{01}(t,\varepsilon) & 0 & \dots & 0 & 0 & \dots & 0 \\ c_{02}(t,\varepsilon) & c_{01}(t,\varepsilon) & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{0M}(t,\varepsilon) & c_{0,M-1}(t,\varepsilon) & \dots & c_{01}(t,\varepsilon) & 0 & \dots & 0 \end{pmatrix},$$

where the scalar functions $c_{01}, \dots, c_{0M}$ are determined from the following system of equations:

$$\sum_{s=1}^{j} \Gamma_0\big((A\Xi_0 - \Xi_0 B)_{s,K-j+s}\big) = 0, \quad j = \overline{1, M}. \tag{24}$$

We represent the matrices A and B in the form

$$A(t,\varepsilon,\theta) = A_0(t,\varepsilon) + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} A_n(t,\varepsilon) e^{in\theta},$$

$$B(t,\varepsilon,\theta) = B_0(t,\varepsilon) + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} B_n(t,\varepsilon) e^{in\theta}.$$

Then it is easy to verify that the system (24) is a system of M linear algebraic equations with respect to the functions $c_{01}, \dots, c_{0M}$:

$$\sum_{s=1}^{j} \big(A_0(t,\varepsilon)C_0 - C_0 B_0(t,\varepsilon)\big)_{s,K-j+s} = h^*_{0j}(t,\varepsilon), \quad j = \overline{1, M}, \tag{25}$$

where $h^*_{01}, \dots, h^*_{0M}$ are the known functions of the class $S(m;\varepsilon_0)$. Determinant of this system has a triangular form, and absolute values of all its diagonal elements are equal to $|(B_0(t,\varepsilon))_{1K}|$. Therefore, the condition I.(3) ensures the existence of a unique solution $c^*_{01}(t,\varepsilon), \dots, c^*_{0M}(t,\varepsilon)$ of the system (25), and this solution belongs to the class $S(m;\varepsilon_0)$.

Using the above found 2π-periodic with respect to θ solution (23) of the equation (19), we construct a 2π-periodic with respect to θ solution of the equation (20) of the form

$$\Xi_1(t,\varepsilon,\theta) = C_1(t,\varepsilon) + \widetilde{\Xi}_1(t,\varepsilon,\theta), \tag{26}$$

where $\widetilde{\Xi}_1(t,\varepsilon,\theta)$ is the known matrix, whose elements belong to the class $F(m;\varepsilon_0;\theta)$, and the $(M \times K)$-matrix $C_1(t,\varepsilon)$ has the form

$$C_1(t,\varepsilon) = \begin{pmatrix} c_{11}(t,\varepsilon) & 0 & \dots & 0 & 0 & \dots & 0 \\ c_{12}(t,\varepsilon) & c_{11}(t,\varepsilon) & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{1M}(t,\varepsilon) & c_{1,M-1}(t,\varepsilon) & \dots & c_{11}(t,\varepsilon) & 0 & \dots & 0 \end{pmatrix}.$$

The scalar functions $c_{11}, \dots, c_{1M}$ are determined from the system of linear algebraic equations

$$\sum_{s=1}^{j} \big(A_0(t,\varepsilon)C_1 - C_1 B_0(t,\varepsilon)\big)_{s,K-j+s} = h^*_{1j}(t,\varepsilon), \quad j = \overline{1, M}, \tag{27}$$

where $h^*_{11}, \dots, h^*_{1M}$ are the known functions of the class $S(m;\varepsilon_0)$. Therefore the condition I.(3) ensures the existence of a unique solution of the system (27), as well. Proceeding just as above, we find a 2π-periodic with respect to θ solutions of the equations (21), (22). The elements of all these solutions belong to the class $F(m;\varepsilon_0,\theta)$.

Consider now the case $M=K$. The condition II.(2) ensures the existence of a 2π-periodic with respect to θ solution of the equation (19) having the form (23), where the $(M\times M)$-matrix $C_0(t,\varepsilon)$ takes the form

$$C_0(t,\varepsilon)=\begin{pmatrix} c_{01}(t,\varepsilon) & 0 & \dots & 0 \\ c_{02}(t,\varepsilon) & c_{01}(t,\varepsilon) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{0M}(t,\varepsilon) & c_{0,M-1}(t,\varepsilon) & \dots & c_{01}(t,\varepsilon) \end{pmatrix}.$$

The scalar functions $c_{01}(t,\varepsilon),\dots,c_{0M}(t,\varepsilon)$ are determined from the following system of linear algebraic equations:

$$\sum_{s=1}^{j}\big(A_0(t,\varepsilon)C_0-C_0B_0(t,\varepsilon)\big)_{s,K-j+s}=g^*_{0j}(t,\varepsilon),\quad j=\overline{1,M}, \tag{28}$$

where $g^*_{01},\dots,g^*_{0M}$ are the known functions of the class $S(m;\varepsilon_0)$. Determinant of this system has a triangular form, and absolute values of all its diagonal elements are equal to $|(A_0(t,\varepsilon)_{1M}-(B_0(t,\varepsilon))_{1M}|$. Therefore the condition II.(3) ensures the existence of a unique solution $c^*_{01}(t,\varepsilon),\dots,c^*_{0M}(t,\varepsilon)$ of the system (28), and this solution belongs to the class $S(m;\varepsilon_0)$.

Thus we have fully determined the 2π-periodic with respect to θ solution of the equation (19). Next, in a full analogy with the case $M<K$, we determined 2π-periodic with respect to θ solutions of the equations (20), (21), (22).

In case $M>K$, the matrix $C_0(t,\varepsilon)$ in (23) is of the form

$$C_0(t,\varepsilon)=\begin{pmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ c_{01}(t,\varepsilon) & 0 & \dots & 0 \\ c_{02}(t,\varepsilon) & c_{01}(t,\varepsilon) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{0K}(t,\varepsilon) & c_{0,K-1}(t,\varepsilon) & \dots & c_{01}(t,\varepsilon) \end{pmatrix}.$$

The scalar functions $c_{01}(t,\varepsilon),\dots,c_{0K}(t,\varepsilon)$ are determined from the following system of linear algebraic equations:

$$\sum_{s=1}^{j}\big(A_0(t,\varepsilon)C_0-C_0B_0(t,\varepsilon)\big)_{s,K-j+s}=f^*_{0j}(t,\varepsilon),\quad j=\overline{1,K}, \tag{29}$$

where $f^*_{01},\dots,f^*_{0M}$ are the known functions of the class $S(m;\varepsilon_0)$. Determinant of this system has a triangular form, and absolute values of all its diagonal elements are equal to $|(A_0(t,\varepsilon)_{1M}|$. Therefore the condition III.(3) ensures the existence of a unique solution $c^*_{01}(t,\varepsilon),\dots,c^*_{0K}(t,\varepsilon)$ of the system (29), and this solution belongs to the class $S(m;\varepsilon_0)$. Next, analogously to the case $M<K$, we determine a 2π-periodic with respect to θ solutions

of the equations (20), (21), (22). The elements of all these solutions belong to the class $F(m;\varepsilon_0;\theta)$.

Substituting in (14)

$$X = \sum_{s=0}^{2q-1} \Xi_s(t,\varepsilon,\theta)\mu^s + \widetilde{X}, \tag{30}$$

where $\widetilde{X}$ is a new unknown matrix, we obtain

$$\begin{aligned}\frac{d\widetilde{X}}{dt} &= J_M\widetilde{X} - \widetilde{X}J_K + \varepsilon G_1(t,\varepsilon,\theta,\mu) + \mu^{2q}H_1(t,\varepsilon,\theta,\mu)+ \\ &+ \Big(\sum_{l=1}^{q} P_l(t,\varepsilon,\theta)\mu^l\Big)\widetilde{X} - \widetilde{X}\Big(\sum_{l=1}^{q} Q_l(t,\varepsilon,\theta)\mu^l\Big)+ \\ &+ \mu^{q+1}\big(W_1^*(t,\varepsilon,\theta,\mu)\widetilde{X} - \widetilde{X}W_2^*(t,\varepsilon,\theta,\mu)\big) + \mu^2\widetilde{X}R(t,\varepsilon,\theta)\widetilde{X}. \end{aligned} \tag{31}$$

By Lemma 1, using the substitution of the kind

$$\widetilde{X} = \Big(E_M + \sum_{l=1}^{q}\Phi_l(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E_K + \sum_{l=1}^{q}\Psi_l(t,\varepsilon,\theta)\mu^l\Big),$$

we reduce the equation (31) to the form (16). □

We introduce the matrices

$$U(t,\varepsilon,\mu) = \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l, \quad V(t,\varepsilon,\mu) = \sum_{l=1}^{q} V_l(t,\varepsilon)\mu^l,$$

where U_l and V_l $(l=\overline{1,q})$ are defined in Lemma 2.

Lemma 3. *Let the equation* (16) *satisfy the following conditions:*

(1) *eigenvalues* $\lambda_{1j}(t,\varepsilon,\mu)$ $(j=\overline{1,M})$ *of the matrix* $J_M+U(t,\varepsilon,\mu)$ *and* $\lambda_{2s}(t,\varepsilon,\mu)$ $(s=\overline{1,K})$ *of the matrix* $J_K+V(t,\varepsilon,\mu)$ *are such that*

$$\inf_G \Big|\mathrm{Re}\,\big(\lambda_{1j}(t,\varepsilon,\mu) - \lambda_{2s}(t,\varepsilon,\mu)\big)\Big| \ge \gamma_0\mu^{q_0}$$
$$(\gamma_0>0,\ \ 0<q_0\le q;\ \ j=\overline{1,M},\ \ s=\overline{1,K});$$

(2) *there exist the* $(M\times M)$*-matrix* $L_1(t,\varepsilon,\mu)$ *and the* $(K\times K)$*-matrix* $L_2(t,\varepsilon,\mu)$ *such that*

  (a) *all elements of these matrices belong to the class* $S(m;\varepsilon_0)\subset F(m;\varepsilon_0;\theta)$*;*

  (b) $\|L_j^{-1}(t,\varepsilon,\mu)\|^*_{F(m\varepsilon_0,\theta)} \le M_1\mu^{-\alpha}$, $M_1\in(0,+\infty)$, $\alpha\in[0,q]$, $j=1,2$*;*

  (c) $L_1^{-1}(J_M+U)L_1 = \Lambda_1(t,\varepsilon,\mu)$, $L_2(J_K+V)L_2^{-1} = \Lambda_2(t,\varepsilon,\mu)$, *where* $\Lambda_1=\mathrm{diag}\,(\lambda_{11},\ldots,\lambda_{1M})$, $\Lambda_2=\mathrm{diag}\,(\lambda_{21},\ldots,\lambda_{2K})$*;*

(3) $q>q_0+\alpha-1/2$.

Then there exist $\mu_2 \in (0,1)$ *and* $K^* \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_2)$ *the matrix differential equation* (16) *has a particular solution* $Y(t,\varepsilon,\theta,\mu)$ *such that all its elements belong to the class* $F(m-1;\varepsilon_1(\mu);\theta)$, *where* $\varepsilon_1(\mu) = \min(\varepsilon_0, K^*\mu^{2q_0+2\alpha-1})$.

Proof. In the equation (16), we perform the substitution

$$Y = \frac{\varepsilon + \mu^{2q}}{\mu^{q_0+2\alpha}}\, L_1(t,\varepsilon,\mu) Z L_2(t,\varepsilon,\mu), \tag{32}$$

where Z is a new unknown $(M \times K)$-matrix. We obtain

$$\begin{aligned}\frac{dZ}{dt} &= \Lambda_1(t,\varepsilon,\mu)Z - Z\Lambda_2(t,\varepsilon,\mu) + \varepsilon\big(\widetilde{U}_1(t,\varepsilon,\theta,\mu)Z - Z\widetilde{V}_1(t,\varepsilon,\theta,\mu)\big) + \\ &+ \mu^{q+1}\big(\widetilde{W}_3(t,\varepsilon,\theta,\mu)Z - Z\widetilde{W}_4(t,\varepsilon,\theta,\mu)\big) + \\ &+ \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\, G_2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\, H_2(t,\varepsilon,\theta,\mu) + \\ &+ \frac{\varepsilon+\mu^{2q}}{\mu^{q_0+2\alpha-1}}\, Z R_2(t,\varepsilon,\theta,\mu) Z,\end{aligned} \tag{33}$$

where

$$G_2 = L_1^{-1} G_1 L_2^{-1}, \;\; H_2 = L_1^{-1} H_1 L_2^{-1},$$
$$\widetilde{U}_1 = L_1^{-1}\widetilde{U} L_1 - \varepsilon^{-1} L_1^{-1}(dL_1/dt), \;\; \widetilde{V}_1 = L_2 \widetilde{U} L_2^{-1} + \varepsilon^{-1}(dL_2/dt)L_2^{-1},$$
$$\widetilde{W}_3 = L_1^{-1}\widetilde{W}_1 L_1, \;\; \widetilde{W}_4 = L_2 \widetilde{W}_2 L_2^{-1}, \;\; R_2 = L_2 R_1 L_1.$$

All elements of these matrices belong to the class $F(m-1;\varepsilon_0;\theta)$.

Owing to the formulas for matrices G_2, H_2, $\widetilde{U}_1$, $\widetilde{V}_1$, $\widetilde{W}_3$, $\widetilde{W}_4$ and the condition 2(b) of the lemma, there exists $K_2 \in (0,+\infty)$ such that

$$\|G_2\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{2\alpha}}\,, \quad \|H_2\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{2\alpha}}\,,$$
$$\|\widetilde{U}_1\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,, \quad \|\widetilde{V}_1\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,,$$
$$\|\widetilde{W}_3\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,, \quad \|\widetilde{W}_4\|_{F(m-1;\varepsilon;\theta)} \le \frac{K_2}{\mu^{\alpha}}\,, \quad \|R_2\|_{F(m-1;\varepsilon;\theta)} \le K_2.$$

Along with the equation (33), we consider the matrix linear differential equation

$$\begin{aligned}\frac{dZ_0}{dt} &= \Lambda_1(t,\varepsilon,\mu)Z_0 - Z_0\Lambda_2(t,\varepsilon,\mu) + \\ &+ \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\, G_2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\, H_2(t,\varepsilon,\theta,\mu).\end{aligned} \tag{34}$$

It is easy to see that this equation is a system of MK independent scalar first-order differential equations

$$\frac{d(Z_0)_{js}}{dt} = \big(\lambda_{1j}(t,\varepsilon,\mu) - \lambda_{2s}(t,\varepsilon,\mu)\big)d(Z_0)_{js} + \\ + \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\big(G_2(t,\varepsilon,\theta,\mu)\big)_{js} + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\big(H_2(t,\varepsilon,\theta,\mu)\big)_{js}, \quad (35) \\ j=\overline{1,M}, \quad s=\overline{1,K}.$$

In [13], it has been shown that the conditions of the lemma provide us with the existence of a unique particular solution $(Z_0(t,\varepsilon,\theta,\mu))_{js}$ $(j=\overline{1,M}$, $s=\overline{1,K})$ of the system (35), which belongs to the class $F(m-1;\varepsilon_0;\theta)$, and in addition, there exists $K_0 \in (0,+\infty)$ such that

$$\|(Z_0)_{js}\|_{F(m-1;\varepsilon_0;\theta)} \le \\ \le \frac{K_0}{\mu^{q_0}}\Big(\frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\|(G_2)_{js}\|_{F(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\|(H_2)_{js}\|_{F(m-1;\varepsilon_0;\theta)}\Big).$$

Hence the equation (34) has a particular solution $Z_0(t,\varepsilon,\theta,\mu)$ all elements of which belong to the class $F(m-1;\varepsilon_0;\theta)$ and, in addition, there exists $K_1 \in (0,+\infty)$ such that

$$\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)} \le \\ \le \frac{K_1}{\mu^{q_0}}\Big(\frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}\|G_2\|^*_{F(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}\|H_2\|^*_{F(m-1;\varepsilon_0;\theta)}\Big). \quad (36)$$

We seek for a solution of the equation (33) all elements of which belong to the class $F(m-1;\varepsilon_1;\theta)$, by using the iterative method, identifying $Z_0(t,\varepsilon,\theta,\mu)$ as an initial approximation, and subsequent iterations are defined as a solutions all elements of which belong to the class $F(m-1;\varepsilon_1;\theta)$ of linear inhomogeneous matrix differential equations

$$\frac{dZ_{\nu+1}}{dt} = \Lambda_1(t,\varepsilon,\mu)Z_{\nu+1} - Z_{\nu+1}\Lambda_2(t,\varepsilon,\mu) + \\ + \frac{\varepsilon\mu^{q_0+2\alpha}}{\varepsilon+\mu^{2q}}G_2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+2\alpha+q_0}}{\varepsilon+\mu^{2q}}H_2(t,\varepsilon,\theta,\mu) + \\ + \varepsilon\big(\widetilde{U}_1(t,\varepsilon,\theta,\mu)Z_\nu - Z_\nu\widetilde{V}_1(t,\varepsilon,\theta,\mu)\big) + \\ + \mu^{q+1}\big(\widetilde{W}_3(t,\varepsilon,\theta,\mu)Z_\nu - Z_\nu\widetilde{W}_4(t,\varepsilon,\theta,\mu)\big) + \\ + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0+2\alpha-1}}Z_\nu R_2(t,\varepsilon,\theta,\mu)Z_\nu, \quad \nu=0,1,2,\ldots. \quad (37)$$

Denote

$$\Omega = \Big\{Z \in F(m-1;\varepsilon_0;\theta):\ \|Z - Z_0\|^*_{F(m-1;\varepsilon_0;\theta)} \le d\Big\}.$$

Using a technique known as contraction mapping principle [15], it is not difficult to show that if

$$K_1K_2\Big(\frac{\varepsilon+\mu^{q+1}}{\mu^{q_0+\alpha}}\,2^m\big(\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)}+d\big)+ \\ +\frac{\varepsilon+\mu^{2q}}{\mu^{2q_0+2\alpha-1}}\,2^{2m-2}\big(\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)}+d\big)^2\Big)\le d_0<d, \quad (38)$$

all iterations (37) belong to Ω. And if

$$K_1K_2\Big(\frac{\varepsilon+\mu^{q+1}}{\mu^{q_0+\alpha}}\,2^m+\frac{\varepsilon+\mu^{2q}}{\mu^{2q_0+2\alpha-1}}\,2^{2m-1}\big(\|Z_0\|^*_{F(m-1;\varepsilon_0;\theta)}+d\big)\Big)<1, \quad (39)$$

then the process (37) converges to a solution of the equation (33) all elements of which belong to class the $F(m-1;\varepsilon_1;\theta)$. The inequalities (38), (39) hold due to the conditions (3) of lemma for sufficiently small μ and $\varepsilon/\mu^{2q_0+2\alpha-1}$. Therefore $\varepsilon_1(\mu)=K^*\mu^{2q_0+2\alpha-1}$, where K^* is a sufficiently small constant. □

The following lemma is an immediate consequence of the above one.

Lemma 4. *Let the equation* (14) *satisfy all conditions of Lemma* 2*, and the equation* (16) *obtained from* (14) *by means of the transformation* (15) *satisfy the conditions of Lemma* 3*. Then there exists* $\mu_3\in(0,1)$*,* $K_3\in(0,+\infty)$ *such that for any* $\mu\in(0,\mu_3)$ *the equation* (14) *has a particular solution which belongs to the class* $F(m-1;\varepsilon_2(\mu);\theta)$*, where* $\varepsilon_2(\mu)=K_2\mu^{2q_0+2\alpha-1}$*, and* q_0*,* α *are defined in Lemma* 2*.*

5. The Basic Results

Getting back to the system (2), we make transformation

$$x_1=e^{ip\theta}y_1,\quad x_2=e^{ir\theta}y_2. \quad (39)$$

We obtain

$$\begin{aligned}\frac{dy_1}{dt}&=J_{N_1}y_1+\mu\big(\widetilde{B}_{11}(t,\varepsilon,\theta)y_1+\widetilde{B}_{12}(t,\varepsilon,\theta)y_2\big),\\ \frac{dy_2}{dt}&=J_{N_2}y_2+\mu\big(\widetilde{B}_{21}(t,\varepsilon,\theta)y_1+\widetilde{B}_{22}(t,\varepsilon,\theta)y_2\big),\end{aligned} \quad (40)$$

where

$$J_{N_1}=\begin{pmatrix}0&0&\dots&0&0\\1&0&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots\\0&0&\dots&0&0\\0&0&\dots&1&0\end{pmatrix},\quad J_{N_2}=\begin{pmatrix}0&0&\dots&0&0\\1&0&\dots&0&0\\ \dots&\dots&\dots&\dots&\dots\\0&0&\dots&0&0\\0&0&\dots&1&0\end{pmatrix}$$

are the Jordan blocks of dimensions N_1 and N_2, respectively, whose diagonal elements are equal to zero, and all elements of matrices $\widetilde{B}_{jk}(t,\varepsilon,\theta)$ belong to the class $F(m;\varepsilon_0;\theta)$.

In the system (40) we make the transformation

$$y_1 = z_1 + \mu Q_{12}(t,\varepsilon,\theta,\mu)z_2, \quad y_2 = \mu Q_{21}(t,\varepsilon,\theta,\mu)z_1 + z_2. \tag{41}$$

Having required for the conditions of block diagonality for the above transformed system, we obtain for $(N_j \times N_k)$-matrices Q_{jk} the following system of the form

$$\begin{aligned} \frac{dQ_{jk}}{dt} &= J_{N_j}Q_{jk} - Q_{jk}J_{N_k} + \widetilde{B}_{jk}(t,\varepsilon,\theta) + \\ &\quad + \mu\big(\widetilde{B}_{jj}(t,\varepsilon,\theta)Q_{jk} - Q_{jk}\widetilde{B}_{kk}(t,\varepsilon,\theta)\big) - \mu^2 Q_{jk}\widetilde{B}_{kj}Q_{jk}, \\ &\qquad j,k = 1,2 \ \ (j \neq k). \end{aligned} \tag{42}$$

Then for the N_1-vector z_1 and N_2-vector z_2 we obtain the system

$$\frac{dz_1}{dt} = D_{N_1}(t,\varepsilon,\theta,\mu)z_1, \quad \frac{dz_2}{dt} = D_{N_2}(t,\varepsilon,\theta,\mu)z_2, \tag{43}$$

where

$$\begin{aligned} D_{N_1} &= J_{N_1} + \mu\widetilde{B}_{11}(t,\varepsilon,\theta) + \mu^2\widetilde{B}_{12}(t,\varepsilon,\theta)Q_{21}(t,\varepsilon,\theta,\mu), \\ D_{N_2} &= J_{N_1} + \mu\widetilde{B}_{22}(t,\varepsilon,\theta) + \mu^2\widetilde{B}_{21}(t,\varepsilon,\theta)Q_{12}(t,\varepsilon,\theta,\mu) \end{aligned} \tag{44}$$

are matrices of dimensions $(N_1 \times N_1)$ and $(N_2 \times N_2)$, respectively.

It is easy to see that the system (42) is divided into two independent equations, each of which has the form (14). Therefore, by Lemma 4, the following theorem is true.

Theorem. *Let each of the equations* (42) *satisfy all conditions of Lemma* 4. *Then there exists* $\mu_4 \in (0,1)$, $K_4 \in (0,+\infty)$ *such that for any* $\mu \in (0,\mu_4)$ *there exists the transformation of kind* (3) *with coefficients from the class* $F(m-1;\varepsilon_4(\mu);\theta)$, *where* $\varepsilon_4(\mu) = K_4\mu^{2q_0+2\alpha-1}$ (q_0 *and* α *are defined in Lemma* 2), *reducing the system* (2) *to a block-diagonal form* (4). *The matrices* D_{N_1}, D_{N_2} *are defined in terms of the expressions* (44).

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(Received 20.10.2013)

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