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Short Communications

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ON THE WELL-POSEDNESS OF ANTIPERIODIC PROBLEM FOR SYSTEMS OF NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH FIXED IMPULSES POINTS

Abstract. The antiperiodic problem for systems of nonlinear impulsive equations with fixed points of impulses actions is considered. The sufficient (among them effective) conditions for the well-posedness of this problem are given.

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Let m_0 be a fixed natural number, ω be a fixed positive real number, and $0 < \tau_1 < \dots < \tau_{m_0} < \omega$ be fixed points (we assume $\tau_0 = 0$ and $\tau_{m_0+1} = \omega$, if necessary). Let $T = \{\tau_l + m\omega : \ l = 1, \dots, m_0; \ m = 0, \pm 1, \pm 2, \dots\}$.

Consider the system of nonlinear impulsive differential equations with fixed impulses points

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ \mathbb{R} \setminus T,$$
$$x(\tau+) - x(\tau-) = I(\tau, x(\tau)) \ \text{ for } \ \tau \in T,$$

under the ω-antiperiodic problem

$$x(t+\omega) = -x(t) \ \text{ for } \ t \in \mathbb{R},$$

where $f = (f_i)_{i=1}^n$ is a vector-function belonging to the Carathéodory class $Car([\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and $I = (I_i)_{i=1}^n : T \times R^n \to \mathbb{R}^n$ is a vector-function such that $I(\tau, \cdot\,)$ is continuous for every $\tau \in T_{m_0}$.

We assume that

$$f(t+\omega, x) = -f(t, -x) \ \text{ and } \ I(\tau+\omega, x) = -I(\tau, -x), \ \ t \in \mathbb{R}, \ \ \tau \in T, \ \ x \in \mathbb{R}^n.$$

In view of this condition, if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given system, then the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will be a solution of the system, as well. Moreover, it is evident that if $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of the given ω-antiperiodic problem, then its restriction on the closed interval $[0, \omega]$ will be a solution of the problem

$$\frac{dx}{dt} = f(t,x) \ \text{ almost everywhere on } \ [0,\omega] \setminus \{\tau_1, \dots, \tau_{m_0}\}, \tag{1}$$
$$x(\tau_l+) - x(\tau_l-) = I(\tau_l, x(\tau_l)) \ \ (l = 1, \dots, m_0); \tag{2}$$
$$x(0) = -x(\omega). \tag{3}$$

Let now $x : [0, \omega] \to \mathbb{R}^n$ be a solution of the system on $[0, \omega]$. By x we designate the continuation of this function on the whole R as a solution of the system (1), (2). As above, the vector-function $y(t) = -x(t+\omega)$ $(t \in \mathbb{R})$ will be the solution of the system (1), (2). On the other hand, according to the equality (3), we have $y(0) = -x(\omega) = x(0)$. Thus, if we assume that the system (1), (2) under the Cauchy condition $x(0) = c$ is uniquely solvable for every $c \in \mathbb{R}^n$, then $x(t+\omega) = -x(t)$ for $t \in \mathbb{R}$, i.e., x is ω-antiperiodic. This means that the set of restrictions of the ω-antiperiodic solutions of the system $(1),(2)$ on $[0,\omega]$ coincides with the set of solutions of the problem $(1),(2);(3)$.

In this connection we consider the boundary value problem $(1),(2);(3)$ on the closed interval $[0,\omega]$. Below, we will give the sufficient conditions guaranteeing the well-posedness of this problem.

Consider a sequence of vector-functions $f_k \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$ $(k=1,2,\dots)$, the sequences of points τ_{lk} $(k=1,2,\dots;\ l=1,\dots,m_0)$, $a<\tau_{1k}<\dots<\tau_{m_0k}<b$, a sequences of operators $I_k:\{\tau_{1k},\dots,\tau_{m_0k}\}\times R^n\to\mathbb{R}^n$ $(k=1,2,\dots)$ such that $I_k(\tau_{lk},\cdot\,)$ $(k=1,2,\dots;\ l=1,\dots,m_0)$ are continuous.

In this paper the sufficient conditions are established which guarantee both the solvability of the impulsive systems $(k=1,2,\dots)$

$$\frac{dx}{dt}=f_k(t,x)\ \text{ almost everywhere on }\ [0,\omega]\setminus\{\tau_{1k},\dots,\tau_{m_0k}\}, \tag{1_k}$$

$$x(\tau_{lk}+)-x(\tau_{lk}-)=I_k(\tau_{lk},x(\tau_{lk}))\ \ (l=1,\dots,m_0) \tag{2_k}$$

under the condition (3) for any sufficient large k and the convergence of its solutions to a solution of the problem $(1),(2);(3)$ as $k\to+\infty$.

We assume that the circumscribed above concept is fulfilled for the problems $(1_k),(2_k);(3)$ $(k=1,2,\dots)$, as well.

The well-posed problem for the linear boundary value problem for impulsive systems with a finite number of impulses points is investigated in [5], where the necessary and sufficient conditions are given for the case. Analogous problems are investigated in [2,12–14] (see also the references therein) for the linear and nonlinear boundary value problems for ordinary differential systems.

Quite a number of issues on the theory of systems of differential equations with impulsive effect (both linear and nonlinear) have been studied sufficiently well (for a survey of the results on impulsive systems see, e.g., [1,3,4,6–10,15–17] and the references therein). But the above-mentioned works, as we know, do not contain the results obtained in the present paper.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R}=\,]-\infty,+\infty[\,$, $\mathbb{R}_+=[0,+\infty[\,$, $[a,b]$ $(a,b\in R)$ is a closed segment.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$, $|X|=(|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_+=\frac{|X|+X}{2}$.

$\mathbb{R}_+^{n\times m}=\big\{(x_{ij})_{i,j=1}^{n,m}:\ x_{ij}\geq 0\ (i=1,\dots,n;\ j=1,\dots,m)\big\}$.

$\mathbb{R}^{(n\times n)\times m}=\mathbb{R}^{n\times n}\times\dots\times\mathbb{R}^{n\times n}$ (m-times).

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n=R_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; $I_{n\times n}$ is the identity $n\times n$-matrix.

$\bigvee\limits_a^b(X)$ is the total variation of the matrix-function $X:[a,b]\to R^{n\times m}$, i.e., the sum of total variations of the latter components; $V(X)(t)=(v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(a)=0$, $v(x_{ij})(t)=\bigvee\limits_a^t(x_{ij})$ for $a<t\leq b$.

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X:[a,b]\to R^{n\times m}$ at the point t (we assume $X(t)=X(a)$ for $t\leq a$ and $X(t)=X(b)$ for $t\geq b$, if necessary).

$\mathrm{BV}([a,b],R^{n\times m})$ is the set of all matrix-functions of bounded variation $X:[a,b]\to R^{n\times m}$ (i.e., such that $\bigvee\limits_a^b(X)<+\infty$).

$C([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all continuous matrix-functions $X:[a,b]\to D$.

Let $T_{m_0}=\{\tau_1,\dots,\tau_{m_0}\}$.

$C([a,b],D;T_{m_0})$, is the set of all matrix-functions $X:[a,b]\to D$ having the one-sided limits $X(\tau_l-)$ $(l=1,\dots,m_0)$ and $X(\tau_l+)$ $(l=1,\dots,m_0)$ whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b]\setminus T_{m_0}$ belong to $C([c,d],D)$.

$C_s([a,b],\mathbb{R}^{n\times m};T_{m_0})$ is the Banach space of all $X\in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

If $y \in C_s([a,b],\mathbb{R};T_{m_0})$ and $r \in]0,+\infty[$, then

$$U(y;r) = \Big\{x \in C_s([a,b],\mathbb{R}^n;T_{m_0}) : \|x-y\|_s < r\Big\}.$$

$D(y,r)$ is the set of all $x \in \mathbb{R}^n$ such that $\inf\{\|x-y(t)\| : t \in [a,b]\} < r$.

$\widetilde{C}([a,b],D)$, where $D \subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a,b] \to D$.

$\widetilde{C}([a,b],D;T_{m_0})$ is the set of all matrix-functions $X : [a,b] \to D$ having the one-sided limits $X(\tau_l-)$ $(l = 1,\dots,m_0)$ and $X(\tau_l+)$ $(l = 1,\dots,m_0)$ whose restrictions to an arbitrary closed interval $[c,d]$ from $[a,b] \setminus T_{m_0}$ belong to $\widetilde{C}([c,d],D)$.

If B_1 and B_2 are the normed spaces, then an operator $g : B_1 \to B_2$ (nonlinear, in general) is positive homogeneous if $g(\lambda x) = \lambda g(x)$ for every $\lambda \in R_+$ and $x \in B_1$.

An operator $\varphi : C([a,b],\mathbb{R}^{n\times m};T_{m_0}) \to R^n$ is called nondecreasing if the inequality $\varphi(x)(t) \le \varphi(y)(t)$ for $t \in [a,b]$ holds for every $x,y \in C([a,b],\mathbb{R}^{n\times m};T_{m_0})$ such that $x(t) \le y(t)$ for $t \in [a,b]$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

$L([a,b],D)$, where $D \subset R^{n\times m}$, is the set of all measurable and integrable matrix-functions $X : [a,b] \to D$.

If $D_1 \subset R^n$ and $D_2 \subset R^{n\times m}$, then $Car([a,b]\times D_1,D_2)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(a) the function $f_{kj}(\,\cdot\,,x) : [a,b] \to D_2$ is measurable for every $x \in D_1$;

(b) the function $f_{kj}(t,\,\cdot\,) : D_1 \to D_2$ is continuous for almost every $t \in [a,b]$, and

$$\sup\{|f_{kj}(\,\cdot\,,x)| : x \in D_0\} \in L([a,b],R;g_{ik}) \ \text{ for every compact } D_0 \subset D_1.$$

$Car^0([a,b]\times D_1,D_2)$ is the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that the functions $f_{kj}(\,\cdot\,,x(\,\cdot\,))$ $(i = 1,\dots,l;\ k = 1,\dots,n)$ are measurable for every vector-function $x : [a,b] \to \mathbb{R}^n$ with bounded total variation.

We say that the pair $\{X;\{Y_l\}_{l=1}^m\}$ consisting of the matrix-function $X \in L([a,b],\mathbb{R}^{n\times n})$ and of a sequence of constant $n\times n$ matrices $\{Y_l\}_{l=1}^m\}$ satisfies the Lappo–Danilevskiĭ condition if the matrices $Y_1,\dots,Y_m$ are pairwise permutable and there exists $t_0 \in [a,b]$ such that

$$\int\limits_{t_0}^{t} X(\tau)\,dX(\tau) = \int\limits_{t_0}^{t} dX(\tau)\cdot X(\tau) \ \text{ for } \ t \in [a,b]$$

and

$$X(t)Y_l = Y_lX(t) \ \text{ for } \ t \in [a,b] \ \ (l = 1,\dots,m).$$

$M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ is the set of all functions $\omega \in Car([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ such that the function $\omega(t,\,\cdot\,)$ is nondecreasing and $\omega(t,0) = 0$ for every $t \in [a,b]$.

By a solution of the impulsive system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ satisfying both the system (1) for a.e. on $[0,\omega] \setminus T_{m_0}$ and the relation (2) for every $l \in \{1,\dots,m_0\}$.

Definition 1. Let $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ and $\ell_0 : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n_+$ be, respectively, a linear continuous and a positive homogeneous operators. We say that a pair (P,J), consisting of a matrix-function $P \in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and a continuous with respect to the last n-variables operator $J : T_{m_0}\times R^n \to \mathbb{R}^n$, satisfies the Opial condition with respect to the pair (ℓ,ℓ_0) if:

(a) there exist a matrix-function $\Phi \in L([0,\omega],\mathbb{R}_+^{n\times n})$ and a constant matrices $\Psi_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ such that

$$|P(t,x)| \le \Phi(t) \ \text{ a.e. on } \ [0,\omega], \ \ x \in \mathbb{R}^n,$$

and

$$|J(\tau_l,x)| \le \Psi_l \ \text{ for } \ x \in \mathbb{R}^n \ \ (l = 1,\dots,m_0);$$

(b)
$$\det(I_{n\times n}+G_l)\neq 0 \ \ (l=1,\dots,m_0) \tag{4}$$
and the problem
$$\frac{dx}{dt}=A(t)\,x \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0}, \tag{5}$$
$$x(\tau_l+)-x(\tau_l-)=G_l\,x(\tau_l) \ \ (l=1,\dots,m_0); \tag{6}$$
$$|\ell(x)|\leq \ell_0(x) \tag{7}$$
has only a trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l,\dots,G_{m_0}$ for which there exists a sequence $y_k\in \widetilde{C}([0,\omega],\mathbb{R}^n;T_{m_0})$ $(k=1,2,\dots)$ such that
$$\lim_{k\to+\infty}\int\limits_0^t P(\tau,y_k(\tau))\,d\tau=\int\limits_0^t A(\tau)\,d\tau \ \text{ uniformly on } \ [0,\omega]$$
and
$$\lim_{k\to+\infty} J(\tau_l,y_k(\tau_l))=G_l \ \ (l=1,\dots,m_0).$$

Remark 1. In particular, the condition (4) holds if
$$\|\Psi_l\|<1 \ \ (l=1,\dots,m_0).$$

As above, we assume that $f=(f_i)_{i=1}^n\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and, moreover, $f(\tau_l,x)$ is arbitrary for $x\in\mathbb{R}^n$ $(l=1,\dots,m_0)$.

Let x^0 be a solution of the problem $(1),(2);(3)$, and r be a positive number. We introduce the following

Definition 2. A solution x^0 is said to be strongly isolated in the radius r if there exist the matrix- and the vector-functions $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ and $q\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$, a continuous with respect to the last n-variables operators J, $H:T_{m_0}\times R^n\to\mathbb{R}^n$, linear continuous operators ℓ and $\widetilde{\ell}$ and a positive homogeneous operator ℓ_0 acting from $C_s([0,\omega],\mathbb{R}^n;T_{m_0})$ into $\mathbb{R}^n$ such that:

(a) the equalities
$$f(t,x)=P(t,x)\,x+q(t,x) \ \text{ for } \ t\in[0,\omega]\setminus T_{m_0}, \ \ \|x-x^0(t)\|<r,$$
$$I(\tau_l,x)=J(\tau_l,x)\,x+H(\tau_l,x) \ \text{ for } \ \|x-x^0(\tau_l)\|<r \ \ (l=1,\dots,m_0)$$
and
$$x(0)+x(\omega)=\ell(x)+\widetilde{\ell}(x) \ \text{ for } \ x\in U(x^0;r)$$
are valid;

(b) the functions $\alpha(t,\rho)=\max\{\|q(t,x)\|: \ \|x\|\leq\rho\}$, $\beta(\tau_l,\rho)=\max\{\|H(\tau_l,x)\|: \ \|x\|\leq\rho\}$ $(l=1,\dots,m_0)$ and $\gamma(\rho)=\sup\{[|\widetilde{l}(x)|-l_0(x)]_+: \ \|x\|_s\leq\rho\}$ satisfy the condition
$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\gamma(\rho)+\int\limits_0^\omega \alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0; \tag{8}$$

(c) the problem
$$\frac{dx}{dt}=P(t,x)\,x+q(t,x) \ \text{ a.e. on } \ [0,\omega]\setminus T_{m_0},$$
$$x(\tau_l+)-x(\tau_l-)=J(\tau_l,x(\tau_l))\,x(\tau_l)+H(\tau_l,x(\tau_l)) \ \ (l=1,\dots,m_0);$$
$$\ell(x)+\widetilde{\ell}(x)=0$$
has no solution different from x^0.

(d) the pair (P,J) satisfies the Opial condition with respect to the pair (ℓ,ℓ_0).

Remark 2. If $\ell(x)\equiv x(0)+x(\omega)$ and $\ell_0(x)\equiv 0$, then we say that the pair (P,J) satisfies the Opial ω-antiperiodic condition. In this case, the condition (7) coincides with the condition (3), and $\widetilde{\ell}(x)\equiv 0$ and $\gamma(\rho)\equiv 0$ in Definitions 1 and 2.

Definition 3. We say that a sequence (f_k, I_k) $(k = 1, 2, \dots)$ belongs to the set $W_r(f, I; x^0)$ if:

(a) the equalities

$$\lim_{k\to+\infty}\int_0^t f_k(\tau,x)\,d\tau = \int_0^t f(\tau,x)\,d\tau \ \text{ uniformly on } \ [0,\omega]$$

and

$$\lim_{k\to+\infty} I_k(\tau_{lk},x) = I(\tau_l,x) \ \ (l=1,\dots,m_0)$$

are valid for every $x \in D(x^0; r)$;

(b) there exists a sequence of functions $\omega_k \in M([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ $(k = 1, 2, \dots)$ such that

$$\sup\Big\{\int_0^\omega \omega_k(t,r)\,dt:\ k=1,2,\dots\Big\} < +\infty, \tag{9}$$

$$\sup\Big\{\sum_{l=1}^{m_0}\omega_k(\tau_{lk},r):\ k=1,2,\dots\Big\} < +\infty; \tag{10}$$

$$\lim_{s\to0+}\sup\Big\{\int_0^\omega \omega_k(t,s)\,dt:\ k=1,2,\dots\Big\} = 0, \tag{11}$$

$$\lim_{s\to0+}\sup\Big\{\sum_{l=1}^{m_0}\omega_k(\tau_{lk},s):\ k=1,2,\dots\Big\} = 0; \tag{12}$$

$$\big\|f_k(t,x)-f_k(t,y)\big\| \le \omega_k\big(t,\|x-y\|\big) \ \text{ for } \ t\in[0,\omega]\setminus T_{m_0}, \ \ x,y\in D(x^0;r) \ \ (k=1,2,\dots),$$
$$\big\|I_k(\tau_{lk},x)-I_k(\tau_{lk},y)\big\| \le \omega_k\big(\tau_{lk},\|x-y\|\big) \ \text{ for } \ x,y\in D(x^0;r) \ \ (l=1,\dots,m_0;\ \ k=1,2,\dots).$$

Remark 3. If for every natural m there exists a positive number ν_m such that

$$\omega_k(t,m\delta) \le \nu_m\omega_k(t,\delta) \ \text{ for } \ \delta>0, \ \ t\in[0,\omega]\setminus T_{m_0} \ \ (k=1,2,\dots),$$

then the estimate (9) follows from the condition (11); analogously, if

$$\omega_k(\tau_{lk},m\delta) \le \nu_m\omega_k(\tau_{lk},\delta) \ \text{ for } \ \delta>0, \ \ (l=1,\dots,m_0;\ \ k=1,2,\dots),$$

then the estimate (10) follows from the condition (12). In particular, the sequences of functions

$$\omega_k(t,\delta) = \max\Big\{\big\|f_k(t,x)-f_k(t,y)\big\|:\ \ x,y\in U\big(0,\|x^0\|+r\big), \ \ \|x-y\|\le\delta\Big\}$$
$$\text{for } \ t\in[0,\omega]\setminus T_{m_0} \ \ (k=1,2,\dots)$$

and

$$\omega_k(\tau_{lk},\delta) = \max\Big\{\big\|I_k(\tau_{lk},x)-I_k(\tau_{lk},y)\big\|:\ \ x,y\in U\big(0,\|x^0\|+r\big), \ \ \|x-y\|\le\delta\Big\}$$
$$(l=1,\dots,m_0;\ k=1,2,\dots)$$

have the latters' properties, respectively.

Definition 4. The problem $(1),(2);(3)$ is said to be $(x^0;r)$-correct if for every $\varepsilon\in]0,r[$ and $(f_k,I_k)_{k=1}^{+\infty}\in W_r(f,I;x^0)$ there exists a natural number k_0 such that the problem $(1_k),(2_k)$ has at last one ω-antiperiodic solution contained in $U(x^0;r)$, and any such solution belongs to the ball $U(x^0;\varepsilon)$ for every $k\ge k_0$.

Definition 5. The problem $(1),(2);(3)$ is said to be correct if it has a unique solution x^0 and it is $(x^0;r)$-correct for every $r>0$.

Theorem 1. *If the problem* $(1),(2);(3)$ *has a solution* x^0*, strongly isolated in the radius* r*, then it is* $(x^0;r)$*-correct.*

Theorem 2. *Let the conditions*

$$\big\|f(t,x)-P(t,x)\,x\big\|\leq\alpha(t,\|x\|)\ \text{ a.e. on } [0,\omega]\setminus T_{m_0},\ \ x\in\mathbb{R}^n, \tag{13}$$

$$\big\|I(\tau_l,x)-J(\tau_l,x)\,x\big\|\leq\beta(\tau_l,\|x\|)\ \text{ for } x\in\mathbb{R}^n\ \ (l=1,\dots,m_0) \tag{14}$$

and

$$\big|x(0)+x(\omega)-\ell(x)\big|\leq\ell_0(x)+\ell_1(\|x\|_s)\ \text{ for } x\in\mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{15}$$

hold, where $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators, the pair (P,J) satisfies the Opial condition with respect to the pair (ℓ,ℓ_0); $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\|\ell_1(\rho)\|+\int\limits_0^{\omega}\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0. \tag{16}$$

Then the problem (1), (2); (3) is solvable. If, moreover, the problem has a unique solution, then it is correct.

Theorem 3. *Let the conditions (13)–(15),*

$$P_1(t)\leq P(t,x)\leq P_2(t)\ \text{ a.e. on } [0,\omega]\setminus\{\tau_1,\dots,\tau_{m_0}\},\ \ x\in\mathbb{R}^n, \tag{17}$$

and

$$J_{1l}\leq J(\tau_l,x)\leq J_{2l}\ \text{ for } x\in\mathbb{R}^n\ \ (l=1,\dots,m_0) \tag{18}$$

hold, where $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$; $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ and $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ are the functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition (16) holds. Let, moreover, the condition (4) hold and the problem (5), (6), (7) have only a trivial solution for every matrix-function $A\in L([0,\omega],\mathbb{R}^{n\times n})$ and constant matrices $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ such that

$$P_1(t)\leq A(t)\leq P_2(t)\ \text{ a.e. on } [0,\omega]\setminus T_{m_0},\ \ x\in\mathbb{R}^n, \tag{19}$$

and

$$J_{1l}\leq G_l\leq J_{2l}\ \text{ for } x\in\mathbb{R}^n\ \ (l=1,\dots,m_0). \tag{20}$$

Then the problem (1), (2); (3) is solvable. If, moreover, the problem has a unique solution, then it is correct.

Remark 4. Theorem 3 is of interest only in the case $P\notin Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, because the theorem immediately follows from Theorem 2 in the case $P\in Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$.

Theorem 4. *Let the conditions (15),*

$$|f(t,x)-P(t)\,x|\leq Q(t)\,|x|+q(t,\|x\|)\ \text{ a.e. on } [0,\omega]\setminus T_{m_0},\ \ x\in\mathbb{R}^n, \tag{21}$$

and

$$|I_l(x)-J_l\,x|\leq H_l\,|x|+h(\tau_l,\|x\|)\ \text{ for } x\in\mathbb{R}^n\ \ (l=1,\dots,m_0) \tag{22}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ and $H_l\in\mathbb{R}^{n\times n}_+$ $(l=1,\dots,m_0)$ are constant matrices, $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, a linear continuous and a positive homogeneous operators; $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\|\ell_1(\rho)\|+\int\limits_0^{\omega}\|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\bigg)=0. \tag{23}$$

holds. Let, moreover, the conditions

$$\det(I_{n\times n}+J_l)\neq0\ \ (l=1,\dots,m_0) \tag{24}$$

and

$$\|H_l\|\cdot\big\|(I_{n\times n}+J_l)^{-1}\big\|<1 \ \ (j=1,2;\ l=1,\dots,m_0) \tag{25}$$

hold and the system of impulsive inequalities

$$\Big|\frac{dx}{dt}-P(t)\,x\Big|\le Q(t)\,x \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \tag{26}$$

$$\big|x(\tau_l+)-x(\tau_l-)-J_l x(\tau_l)\big|\le H_l\,|x(\tau_l)| \ \ (l=1,\dots,m_0) \tag{27}$$

have only a trivial solution satisfying the condition (7)*. Then the problem* (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1. *Let the conditions*

$$|f(t,x)-P(t)\,x|\le q(t,\|x\|) \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x\in\mathbb{R}^n, \tag{28}$$

$$\big|I(\tau_l,x)-J_l\,x\big|\le h(\tau_l,\|x\|) \ \ \text{for } x\in\mathbb{R}^n \ \ (l=1,\dots,m_0) \tag{29}$$

and

$$\big|x(0)+x(\omega)-\ell(x)\big|\le \ell_1(\|x\|_s) \ \ \text{for } x\in \mathrm{BV}([0,\omega],\mathbb{R}^n) \tag{30}$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ are constant matrices satisfying the condition (24)*, $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ is the linear continuous operator; $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1\in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition* (23) *holds. Let, moreover, the problem*

$$\frac{dx}{dt}=P(t)\,x \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \tag{31}$$

$$x(\tau_l+)-x(\tau_l-)=J_l\,x(\tau_l) \ \ (l=1,\dots,m_0); \tag{32}$$

$$\ell(x)=0. \tag{33}$$

have only a trivial solution. Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 5. Let $Y=(y_1,\dots,y_n)$ be a fundamental matrix, with columns $y_1,\dots,y_n$, of the system (31), (32). Then the homogeneous boundary value problem (31), (32); (33) has only a trivial solution if and only if

$$\det(\ell(Y))\neq 0, \tag{34}$$

where $\ell(Y)=(\ell(y_1),\dots,\ell(y_n))$.

If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the fundamental matrix Y $(Y(0)=I_{n\times n})$ of the homogeneous system (31), (32) has the form

$$Y(t)\equiv\exp\bigg(\int\limits_0^t P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t}(I_{n\times n}+J_l).$$

Theorem 5. *Let the conditions*

$$\big|f(t,x)-f(t,y)-P(t)\,(x-y)\big|\le Q(t)|x-y| \ \ \text{a.e. on } [0,\omega]\setminus T_{m_0}, \ \ x,y\in\mathbb{R}^n, \tag{35}$$

$$\big|I(\tau_l,x)-I(\tau_l,y)-J_l\,(x-y)\big|\le H_l|x-y| \ \ \text{for } x,y\in\mathbb{R}^n \ \ (k=l,\dots,m_0) \tag{36}$$

and

$$\big|x(0)-y(\omega)+x(\omega)-y(\omega)-\ell(x-y)\big|\le \ell_0(x-y) \ \ \text{for } x,y\in\mathrm{BV}([0,\omega],\mathbb{R}^n)$$

hold, where $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}^{n\times n}_+)$, $J_l\in\mathbb{R}^{n\times n}$ and $H_l\in\mathbb{R}^{n\times n}_+$ $(l=1,\dots,m_0)$ are constant matrices satisfying the conditions (24) *and* (25)*, $\ell: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n$ and $\ell_0: C_s([0,\omega],\mathbb{R}^n;T_{m_0})\to\mathbb{R}^n_+$ are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the problem* (26), (27); (7) *have only a trivial solution. Then the problem* (1), (2); (3) *is correct.*

Corollary 2. *Let there exist a solution x^0 of the problem* (1), (2); (3) *and a positive number $r > 0$ such that the conditions*

$$\big|f(t,x) - f(t,x^0(t)) - P(t)\,(x - x^0(t))\big| \le Q(t)\big|x - x^0(t)\big| \ \ a.a.\ [0,\omega]\setminus T_{m_0}, \ \ \|x - x^0(t)\| < r,$$

$$\Big|I\big(\tau_l,x\big) - I\big(\tau_l,x^0(\tau_l)\big) - J_l\,(x - x^0(\tau_l))\Big| \le H_l\big|x - x^0(\tau_l)\big| \ \ for\ \ \|x - x^0(\tau_l)\| < r \ \ (l = l,\dots,m_0)$$

and

$$\big|x(0) - x^0(0) + x(\omega) - x^0(\omega) - \ell(x - x^0)\big| \le \ell^*\big(|x - x^0|\big) \ \ for\ \ x \in U(x^0,r)$$

hold, where $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $Q \in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l, H_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the conditions (24) *and* (25), *$\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ and $\ell^* : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}_+^n$ are, respectively, linear continuous and positive homogeneous continuous operators. Let, moreover, the system of impulsive inequalities*

$$\Big|\frac{dx}{dt} - P(t)\,x\Big| \le Q(t)\,x \ \ a.e.\ on\ [0,\omega]\setminus T_{m_0},$$

$$\big|x(\tau_l+) - x(\tau_l-) - J_l\,x(\tau_l)\big| \le H_l\cdot x(\tau_l) \ \ (l = 1,\dots,m_0)$$

have only a trivial solution under the condition

$$|\ell(x)| \le \ell^*(|x|).$$

Then the problem (1), (2); (3) *is $(x^0;r)$-correct.*

Corollary 3. *Let the components of the vector-functions f and I_l $(l = 1,\dots,n)$ have partial derivatives by the last n variables belonging to the Carathéodory class $Car([0,\omega]\times\mathbb{R}^n,\mathbb{R}^n)$. Let, moreover, x^0 be a solution of the problem* (1), (2); (3) *such that the condition*

$$\det\big(I_{n\times n} + G_l(x^0(\tau_l))\big) \ne 0 \ \ (l = 1,\dots,m_0)$$

holds and the system

$$\frac{dx}{dt} = F(t,x^0(t))\,x \ \ almost\ everywhere\ on\ [0,\omega]\setminus T_{m_0},$$

$$x(\tau_l+) - x(\tau_l-) = G_l(x^0(\tau_l))\,x(\tau_l) \ \ (l = 1,\dots,m_0);$$

$$\ell(x) = 0,$$

where $F(t,x) \equiv \frac{\partial f(t,x)}{\partial x}$ and $G_l(x) \equiv \frac{\partial I_l(x)}{\partial x}$, have only a trivial solution under the condition (3). *Then the problem* (1), (2); (3) *is $(x^0;r)$-correct for any sufficiently small r.*

In general, it is quite difficult to verify the condition (34) directly even in the case where one is able to write out the fundamental matrix of the system (31), (32); (33). Therefore it is important to seek for effective conditions which would guarantee the absence of nontrivial ω-antiperiodic solutions of the homogeneous system (31), (32); (33). Below we will give the results concerning the question under consideration. Analogous results have been obtained in [3] for general linear boundary value problems for impulsive systems, and in [14] by T. Kiguradze for the case of ordinary differential equations.

In this connection, we introduce the following operators. For every matrix-function $X \in L([0,\omega],\mathbb{R}^{n\times n})$ and a sequence of constant matrices $Y_k \in \mathbb{R}^{n\times n}$ $(k = 1,\dots,m_0)$ we put

$$\begin{aligned}
&\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_0 = I_n \ \text{ for } \ 0 \le t \le \omega,\\
&\big[(X,Y_1,\dots,Y_{m_0})(0)\big]_i = O_{n\times n} \ \ (i = 1,2,\dots),\\
&\big[(X,Y_1,\dots,Y_{m_0})(t)\big]_{i+1} = \int_0^t X(\tau)\,\big[(X,Y_1,\dots,Y_{m_0})(\tau)\big]_i\,d\tau\\
&\qquad + \sum_{0\le\tau_l<t} Y_l\,\big[(X,Y_1,\dots,Y_{m_0})(\tau_l)\big]_i \ \text{ for } \ 0 < t \le \omega \ \ (i = 1,2,\dots).
\end{aligned} \tag{37}$$

Corollary 4. *Let the conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \int_0^\omega d\mathcal{L}(t)\cdot x(t),$$

$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the condition (24), *$\mathcal{L} \in L([0,\omega],\mathbb{R}^{n\times n})$; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition* (23) *holds. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = -\sum_{i=0}^{k-1}\int_0^\omega d\mathcal{L}(t)\cdot\big[(P,J_l,\dots,J_{m_0})(t)\big]_i$$

is nonsingular and

$$r(M_{k,m}) < 1, \tag{38}$$

where the operators $[(P,J_1,\dots,J_{m_0})(t)]_i$ $(i = 0,1,\dots)$ are defined by (37), *and*

$$M_{k,m} = \big[\big(|P|,|J_1|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \sum_{i=0}^{m-1}\big[\big(|P|,|J_1|,\dots,|J_{m_0}|\big)(\omega)\big]_i\int_0^\omega dV(M_k^{-1}\mathcal{L})(t)\cdot\big[\big(|P|,|J_1|,\dots,|J_{m_0}|\big)(t)\big]_k.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5. *Let the conditions* (28)–(30) *hold, where*

$$\ell(x) \equiv \sum_{j=1}^{n_0}\mathcal{L}_j x(t_j), \tag{39}$$

$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the condition (24), *$t_j \in [0,\omega]$ and $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$, $\mathcal{L} \in L([0,\omega],\mathbb{R}^{n\times n})$, $\ell : C_s([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ is the linear continuous operator; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is a vector-function such that the condition* (23) *holds. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = \sum_{j=1}^{n_0}\sum_{i=0}^{k-1}\mathcal{L}_j\big[(P,J_l,\dots,J_{m_0})(t_j)\big]_i$$

is nonsingular and the inequality (38) *holds, where*

$$M_{k,m} = \big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1}\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_i\Big)\sum_{j=1}^{n_0}|M_k^{-1}\mathcal{L}_j|\cdot\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(t_j)\big]_k.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5 has the following form for $k = 1$ and $m = 1$.

Corollary 6. *Let the conditions* (28)–(30) *hold, where the operator ℓ is defined by* (39), *$P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ are constant matrices satisfying the condition* (24), *$t_j \in [0,\omega]$ and $\mathcal{L}_j \in \mathbb{R}^{n\times n}$ $(j = 1,\dots,n_0)$; $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ and $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ are the vector-functions, nondecreasing in the second variable, and $\ell_1 \in C(\mathbb{R},\mathbb{R}^n_+)$ is the vector-function such that the condition* (23) *holds. Let, moreover,*

$$\det\Big(\sum_{j=1}^{n_0}\mathcal{L}_j\Big) \neq 0 \ \ \text{and} \ \ r(\mathcal{L}_0 A_0) < 1,$$

where

$$\mathcal{L}_0 = I_{n\times n} + \Big|\Big(\sum_{j=1}^{n_0}\mathcal{L}_j\Big)^{-1}\Big|\cdot\sum_{j=1}^{n_0}|\mathcal{L}_j| \ \ \text{and} \ \ A_0 = \int_0^\omega |P(t)|\,dt + \sum_{l=1}^{m_0}|J_l|.$$

Then the problem $(1),(2);(3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 6. If the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the condition (34) has the forms

$$\det\bigg(\int_0^\omega d\mathcal{L}(t)\cdot\exp\bigg(\int_0^t P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t}(I_{n\times n}+J_l)\bigg)\neq 0$$

and

$$\det\bigg(\sum_{j=1}^{n_0}L_j\exp\bigg(\int_0^{t_j}P(\tau)\,d\tau\bigg)\cdot\prod_{0\le\tau_l<t_j}(I_{n\times n}+J_l)\bigg)\neq 0$$

for the operators ℓ defined, respectively, in Corollary 4 and Corollary 5.

By Remark 2, in the case where $\ell(x)\equiv x(0)+x(\omega)$ and $\ell_0(x)\equiv 0$, the results given above have the following forms, respectively.

Theorem 2′. *Let the conditions* (13) *and* (14) *hold, where the pair* (P,J) *satisfies the Opial* ω*-antiperiodic condition,* $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *are the functions, nondecreasing in the second variable, such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int_0^\omega\alpha(t,\rho)\,dt+\sum_{l=1}^{m_0}\beta(\tau_l,\rho)\bigg)=0. \tag{40}$$

Then the problem $(1),(2);(3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 3′. *Let the conditions* (13), (14), (17), (18) *and* (40) *hold, where* $P\in Car^0([0,\omega]\times\mathbb{R}^n,\mathbb{R}^{n\times n})$, $P_i\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_{il}\in\mathbb{R}^{n\times n}$ $(i=1,2;\ l=1,\dots,m_0)$; $\alpha\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+)$ *and* $\beta\in C(T_{m_0}\times[0,\omega],\mathbb{R}_+)$ *are the functions, nondecreasing in the second variable. Let, moreover, the condition* (4) *hold and the problem* $(5),(6);(3)$ *have only a trivial solution for every matrix-function* $A\in L([0,\omega],\mathbb{R}^{n\times n})$ *and constant matrices* $G_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *satisfying the conditions* (19) *and* (20)*. Then the problem* $(1),(2);(3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 4′. *Let the conditions* (21) *and* (22) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are the constant matrices satisfying the conditions* (24) *and* (25), $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$*, and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable, such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\bigg(\int_0^\omega\|q(t,\rho)\|\,dt+\sum_{l=1}^{m_0}\|h(\tau_l,\rho)\|\bigg)=0. \tag{41}$$

Let, moreover, the system of impulsive inequalities $(26),(27)$ *have only a trivial solution satisfying the condition* (3)*. Then the problem* $(1),(2);(3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 1′. *Let the conditions* (28), (29) *and* (40) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ $(l=1,\dots,m_0)$ *are constant matrices satisfying the condition* (24), $q\in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}_+^n)$ *and* $h\in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}_+^{n\times n})$ *are the vector-functions, nondecreasing in the second variable. Let, moreover, the problem* $(31),(32),(3)$ *have only a trivial solution. Then the problem* $(1),(2);(3)$ *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Theorem 5′. *Let the conditions* (35) *and* (36) *hold, where* $P\in L([0,\omega],\mathbb{R}^{n\times n})$, $Q\in L([0,\omega],\mathbb{R}_+^{n\times n})$, $J_l\in\mathbb{R}^{n\times n}$ *and* $H_l\in\mathbb{R}_+^{n\times n}$ $(l=1,\dots,m_0)$ *are constant matrices satisfying the conditions* (24) *and* (25)*. Let, moreover, the problem* $(26),(27);(7)$ *have only a trivial solution. Then the problem* $(1),(2);(3)$ *is correct.*

Corollary 5′. *Let the conditions* (28), (29) *and* (41) *hold, where* $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices satisfying the condition* (24)*;* $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *and* $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable. Let, moreover, there exist natural numbers k and m such that the matrix*

$$M_k = \sum_{i=0}^{k-1}\big[(P,J_l,\dots,J_{m_0})(\omega)\big]_i$$

is nonsingular and the inequality (38) *holds, where*

$$M_{k,m} = \big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_m + \Big(\sum_{i=0}^{m-1}\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_i\Big)|M_k^{-1}|\cdot\big[\big(|P|,|J_l|,\dots,|J_{m_0}|\big)(\omega)\big]_k.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Corollary 5′ has the following form for $k = 1$ and $m = 1$.

Corollary 6′. *Let the conditions* (28), (29) *and* (41) *hold, where* $P \in L([0,\omega],\mathbb{R}^{n\times n})$, $J_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m_0)$ *are constant matrices satisfying the condition* (24)*;* $q \in Car([0,\omega]\times\mathbb{R}_+,\mathbb{R}^n_+)$ *and* $h \in C(T_{m_0}\times\mathbb{R}_+;\mathbb{R}^{n\times n}_+)$ *are the vector-functions, nondecreasing in the second variable. Let, moreover,*

$$r(A_0) < \frac{1}{2}\,,$$

where

$$A_0 = \int_0^\omega |P(t)|\,dt + \sum_{l=1}^{m_0}|J_l|.$$

Then the problem (1), (2); (3) *is solvable. If, moreover, the problem has a unique solution, then it is correct.*

Remark 7. In the conditions of Corollary 6′, if the pair $\{P;\{J_l\}_{l=1}^{m_0}\}$ satisfies the Lappo–Danilevskiĭ condition, then the condition (34) has the form

$$\det\bigg(I_{n\times n} + \exp\bigg(\int_0^\omega P(\tau)\,d\tau\bigg)\cdot\prod_{l=1}^{m_0}(I_{n\times n}+J_l)\bigg) \neq 0.$$

The analogous questions have been investigated in [7, 8] for the system (1), (2) under the general nonlinear boundary condition $h(x) = 0$, where $h : C([0,\omega],\mathbb{R}^n;T_{m_0}) \to \mathbb{R}^n$ is a continuous vector-functional which is nonlinear, in general. The results given in the paper are the particular cases of the results obtained in [7, 8] when $h(x) \equiv x(0) + x(\omega)$.

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