Memoirs on Differential Equations and Mathematical Physics

Volume 71, 2017, 51–68

Giorgi Dekanoidze and Sergo Kharibegashvili

ON THE GLOBAL SOLVABILITY OF THE FIRST DARBOUX PROBLEM FOR ONE CLASS OF NONLINEAR SECOND ORDER HYPERBOLIC SYSTEMS

Abstract. The first Darboux problem for one class of nonlinear second order hyperbolic systems is considered. The questions of the existence, uniqueness and smoothness of a global solution of this problem are considered.

2010 Mathematics Subject Classification. 35L51, 35L71.

Key words and phrases. Nonlinear hyperbolic system, first Darboux problem, existence, uniqueness and smoothness of a global solution.

რეზიუმე. მეორე რიგის არაწრფივ ჰიპერბოლურ სისტემათა ერთი კლასისათვის განხილულია დარბუს პირველი ამოცანა. გამოკვლეულია ამ ამოცანის გლობალური ამონახსნის არსებობის, ერთადერთობის და სიგლუვის საკითხები.

1 Statement of the problem

In a plane of variables x and t we consider the hyperbolic second order system of the type

$$Lu := u_{tt} - u_{xx} + A(x,t)u_x + B(x,t)u_t + C(x,t)u + f(x,t,u) = F(x,t), \tag{1.1}$$

where A, B, C are the given square n-th order matrices, $f = (f_1, \dots, f_n)$ and $F = (F_1, \dots, F_n)$ are the given and $u = (u_1, \dots, u_n)$ is an unknown vector functions, $n \geq 2$.

By D_T we denote an angular domain lying in the characteristic angle $\{(x,t) \in \mathbb{R}^2 :\ t > |x|\}$ and bounded both by the characteristic segment $\gamma_{1,T} :\ x = t,\ 0 \leq t \leq T$, and by the noncharacteristic segments $\gamma_{2,T} :\ x = 0,\ 0 \leq t \leq T$, and $\gamma_{3,T} :\ t = T,\ 0 \leq x \leq T$.

For system (1.1) in the domain D_T, we consider the boundary value problem which is formulated as follows: find in the domain D_T a solution $u = u(x,t)$ of system (1.1) by the boundary conditions

$$u\big|_{\gamma_{i,T}} = \varphi_i, \ \ i = 1, 2, \tag{1.2}$$

where φ_i, $i = 1, 2$, are the given on $\gamma_{i,T}$ vector functions satisfying at their common point $O = O(0,0)$ the agreement condition $\varphi_1(O) = \varphi_2(O)$. When $T = \infty$, we have $D_\infty :\ \ t > |x|,\ x > 0$, and $\gamma_{1,\infty} :\ x = t,\ 0 \leq t < \infty$, $\gamma_{2,\infty} :\ x = 0,\ 0 \leq t < \infty$. In a scalar case, where $n = 1$, problem (1.1), (1.2) is known as the first Darboux problem.

If in a linear case for a scalar hyperbolic equation the boundary value problems, in particular, the Goursat and Darboux problems, are well studied [4, 6, 7, 10, 15, 16], there arise additional difficulties and new effects in passing to a hyperbolic system. First this has been observed by A. Bitsadze [5] who constructed examples of second order hyperbolic systems for which the corresponding homogeneous characteristic problem had a finite number, and in some cases, an infinite set of linearly independent solutions. Later on, these problems for linear second order hyperbolic systems became a subject of investigations (see [8,9]). In this direction, the work [3] is also noteworthy, in which by simple examples the effect of lowest terms on the well-posedness of the problems under consideration has been revealed. As is shown in [1,2,11–13], the presence of a nonlinear term in a scalar hyperbolic equation may affect the well-posedness of the Darboux problem, when in one case this problem is globally solvable and in other cases there may arise the so-called blow-up solutions. It should be noted that the above-mentioned works do not contain linear terms involving the first order derivatives, since their presence causes difficulties in investigating the problem, and not only of technical character.

In the present work, we investigate the Darboux problem for the nonlinear system (1.1) in the presence of lowest terms involving the first order derivatives. The results obtained here are new even in the case when (1.1) is a scalar hyperbolic equation.

Definition 1.1. Let $A, B, C, F \in C(\overline{D}_T)$, $f \in C(\overline{D}_T \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,T})$, $i = 1, 2$. The vector function u is said to be a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T, if $u \in C(\overline{D}_T)$ and there exists a sequence of vector functions $u^m \in C^2(\overline{D}_T)$ such that $u^m \to u$ and $Lu^m \to F$ in the space $C(\overline{D}_T)$, and $u^m|_{\gamma_{i,T}} \to \varphi_i$ in the space $C^1(\gamma_{i,T})$, $i = 1, 2$, as $m \to \infty$.

Remark 1.1. Obviously, the classical solution $u \in C^2(\overline{D}_T)$ of problem (1.1), (1.2) is likewise a generalized solution of that problem of the class C in the domain D_T. Moreover, if a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then this solution will likewise be a classical solution of that problem. It should also be noted that a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T satisfies the boundary conditions (1.2) in an ordinary classical sense. In case $\varphi_2 = 0$ in Definition 1.1, we will assume that $u^m \in C^2_0(\overline{D}_T; \gamma_{2,T}) := \{v \in C^2(\overline{D}_T) :\ v|_{\gamma_{2,T}} = 0\}$.

Definition 1.2. Let $A, B, C, F \in C(\overline{D}_\infty)$, $f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1, 2$. We say that problem (1.1), (1.2) is locally solvable in the class C, if there exists the number $T_0 = T_0(F, \gamma, \gamma_2) > 0$ such that for any $T < T_0$, problem (1.1), (1.2) has at least one generalized solution of the class C in the domain D_T.

Definition 1.3. Let $A, B, C, F \in C(\overline{D}_\infty)$, $f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1, 2$. We say that problem (1.1), (1.2) is globally solvable in the class C, if for any positive number T, problem (1.1), (1.2) has at least one generalized solution of the class C in the domain D_T.

Definition 1.4. Let $A, B, C, F \in C(\overline{D}_\infty)$, $f \in C(\overline{D}_\infty \times \mathbb{R}^n)$ and $\varphi_i \in C^1(\gamma_{i,\infty})$, $i = 1, 2$. The vector function $u \in C(\overline{D}_\infty)$ is said to be a global generalized solution of problem (1.1), (1.2) of the class C, if for any positive number T, the vector function $U|_{D_T}$ is a generalized solution of that problem of the class C in the domain D_T.

2 A priori estimate of a solution of problem (1.1), (1.2)

Let us consider the following conditions imposed on the vector function $f = f(x,t,u)$:

$$\|f_i(x,t,u)\| \leq M_1 + M_2\|u\|, \ \ (x,t,u) \in \overline{D}_T \times \mathbb{R}^n, \ \ i = 1, 2 \dots, n, \tag{2.1}$$

where $M_j = M_j(T) = const \geq 0$, $j = 1, 2$, $\|u\| = \sum\limits_{i=1}^{n} |u_i|$.

Assume

$$M_0 = \sup_{(x,t)\in\overline{D}_T} \max_{1\leq i,j\leq n} \Big(\max\big\{|A_{i,j}(x,t)|, |B_{i,j}(x,t)|, |C_{i,j}(x,t)|\big\}\Big).$$

Lemma 2.1. *Let $F \in C(\overline{D}_T)$, $\varphi_1 \in C^1(\gamma_{1,T})$, $\varphi_2 = 0$, and the vector function $f \in C(\overline{D}_T \times \mathbb{R}^n)$ satisfy condition (2.1). Then for a generalized solution $u = u(x,t)$ of problem (1.1), (1.2) of the class C in the domain D_T the a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \leq c_1\|F\|_{C(\overline{D}_T)} + c_2\|\varphi_1\|_{C^1(\gamma_{1,T})} + c_3 \tag{2.2}$$

is valid, where the nonnegative constants $c_i = c_i(M_0, M_1, M_2, T)$, $i = 1, 2, 3$, are independent of u, F and φ_1, where $c_i > 0$, $i = 1, 2$, and

$$\|u\|_{C(\overline{D}_T)} = \sum_{i=1}^{n} \|u_i\|_{C(\overline{D}_T)}, \quad \|F\|_{C(\overline{D}_T)} = \sum_{i=1}^{n} \|F_i\|_{C(\overline{D}_T)},$$

$$\|\varphi_1\|_{C^1(\gamma_{1,T})} = \sum_{i=1}^{n} \|\varphi_{1i}\|_{C^1(\gamma_{1,T})}.$$

Proof. Let $u = u(x,t)$ be a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T. Then, according to Definition 1.1 and Remark 1.1, the vector function $u \in C(\overline{D}_T)$ and there exists a sequence of vector functions $u^m \in C^2_0(\overline{D}_T, \gamma_{2,T})$ such that

$$\lim_{m\to\infty} \|u^m - u\|_{C(\overline{D}_T)} = 0, \quad \lim_{m\to\infty} \|Lu^m - F\|_{C(\overline{D}_T)} = 0, \tag{2.3}$$

$$\lim_{m\to\infty} \big\|u^m\big|_{\gamma_{1,T}} - \varphi_1\big\|_{C^1(\gamma_{1,T})} = 0. \tag{2.4}$$

Consider the vector function $u^m \in C^2_0(\overline{D}_T, \gamma_{2,T})$ as a solution of the problem

$$Lu^m = F^m, \tag{2.5}$$

$$u^m\big|_{\gamma_{1,T}} = \varphi^m_1, \quad u^m\big|_{\gamma_{2,T}} = 0. \tag{2.6}$$

Here

$$F^m = Lu^m, \quad \varphi^m_1 = u^m\big|_{\gamma_{1,T}}. \tag{2.7}$$

Multiplying both parts of system (2.5) scalarwise by $\frac{\partial u^m}{\partial t}$ and integrating over the domain $D_\tau := \{(x,t) \in D_T : \ t < \tau\}$, $0 < \tau \leq T$, we have

$$\begin{aligned}\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt - \int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt + \int\limits_{D_\tau}\Big(A(x,t)\,\frac{\partial u^m}{\partial x}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt \\ + \int\limits_{D_\tau}\Big(B(x,t)\,\frac{\partial u^m}{\partial t}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt + \int\limits_{D_\tau}\Big(C(x,t)u^m, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt \\ + \int\limits_{D_\tau}\Big(f(x,t,u^m), \frac{\partial u^m}{\partial t}\Big)\,dx\,dt = \int\limits_{D_\tau}\Big(F^m, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt,\end{aligned} \tag{2.8}$$

where $(v,w)=\sum\limits_{i=1}^{n} v_iw_i$ is a scalar product in the space $\mathbb{R}^n$, $v=(v_1,\dots,v_n), w=(w_1,\dots,w_n)\in\mathbb{R}^n$.

Integrating by parts and applying Green's formula, we obtain

$$\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt=\frac{1}{2}\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\nu_t\,ds, \tag{2.9}$$

$$\begin{aligned}-\int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt&=-\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\nu_x\,ds+\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial^2 u^m}{\partial t\partial x}\Big)\,dx\,dt\\
&=-\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\nu_x\,ds+\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial x}\Big)\,dx\,dt\\
&=-\int\limits_{\partial D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial t}\Big)\nu_x\,ds+\frac{1}{2}\int\limits_{D_\tau}\Big(\frac{\partial u^m}{\partial x},\frac{\partial u^m}{\partial x}\Big)\nu_t\,ds,\end{aligned} \tag{2.10}$$

where $\nu=(\nu_x,\nu_t)$ is the unit vector of the outer normal to the boundary ∂D_τ of the domain D_τ.

Taking into account the fact that $\partial D_\tau=\gamma_{1,\tau}\cup\gamma_{2,\tau}\cup\omega_\tau$, where $\gamma_{i,\tau}=\gamma_{i,\tau}\cap\{t\le\tau\}$, $i=1,2$, and $\omega_\tau=\partial D_\tau\cap\{t=\tau\}=\{t=\tau,\ 0\le x\le\tau\}$, we have

$$(\nu_x,\nu_t)\big|_{\gamma_{1,\tau}}=\Big(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\Big), \tag{2.11}$$

$$(\nu_x,\nu_t)\big|_{\gamma_{2,\tau}}=(-1,0),\quad (\nu_x,\nu_t)\big|_{\omega_\tau}=(0,1), \tag{2.12}$$

$$(\nu_x^2-\nu_t^2)\big|_{\gamma_{1,\tau}}=0, \tag{2.13}$$

$$\nu_t\big|_{\gamma_{1,\tau}}<0. \tag{2.14}$$

In view of (2.11)–(2.14) and the fact that $u^m|_{\gamma_{2,T}}=0$, from (2.9) and (2.10) we arrive at

$$\begin{aligned}\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\nu_t\,ds\\
&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}(u^m_{it})^2\Big)\,dx+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\sum_{i=1}^{n}(u^m_{it})^2\Big)\nu_t\,ds,\end{aligned} \tag{2.15}$$

$$\begin{aligned}-\int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}(u^m_{ix})^2\Big)\,dx\\
&\quad+\frac{1}{2}\int\limits_{\gamma_{1,\tau}}\Big(\sum_{i=1}^{n}(u^m_{ix})^2\Big)\nu_t\,ds-\int\limits_{\gamma_{1,\tau}}\Big(\sum_{i=1}^{n}u^m_{ix}u^m_{it}\Big)\nu_x\,ds.\end{aligned} \tag{2.16}$$

By virtue of (2.13), it follows from (2.15) and (2.16) that

$$\begin{aligned}&\frac{1}{2}\int\limits_{D_\tau}\frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt-\int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2},\frac{\partial u^m}{\partial t}\Big)\,dx\,dt\\
&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}\big((u^m_{ix})^2+(u^m_{it})^2\big)\Big)\,dx+\int\limits_{\gamma_{1,\tau}}\frac{1}{2\nu_t}\Big(\sum_{i=1}^{n}\big[(u^m_{ix}\nu_t-u^m_{it}\nu_x)^2+(u^m_{it})^2(\nu_t^2-\nu_x^2)\big]\Big)\,ds\\
&=\frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^{n}\big((u^m_{ix})^2+(u^m_{it})^2\big)\Big)\,dx+\int\limits_{\gamma_{1,\tau}}\frac{1}{2\nu_t}\Big(\sum_{i=1}^{n}(u^m_{ix}\nu_t-u^m_{it}\nu_x)^2\Big)\,ds.\end{aligned} \tag{2.17}$$

Since $(\nu_t\,\frac{\partial}{\partial x}-\nu_x\,\frac{\partial}{\partial t})$ is the derivative to the tangent, i.e., it is an inner differential operator on $\gamma_{1,\tau}$, taking into account (2.6), we find that

$$\big|(u^m_{ix}\nu_t-u^m_{it}\nu_x)\big|_{\gamma_{1,\tau}}\big|\le\|\varphi^m_{1i}\|_{C^1(\gamma_{1,\tau})}\le\|\varphi^m_{1i}\|_{C^1(\gamma_{1,T})}. \tag{2.18}$$

In view of (2.18) and the fact that $\nu_t|_{\gamma_{1,\tau}} = -\frac{1}{\sqrt{2}}$, (2.17) yields

$$\frac{1}{2}\int\limits_{D_\tau} \frac{\partial}{\partial t}\Big(\frac{\partial u^m}{\partial t}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt - \int\limits_{D_\tau}\Big(\frac{\partial^2 u^m}{\partial x^2}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt$$
$$\geq \frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^n \big((u_{ix}^m)^2 + (u_{it}^m)^2\big)\Big)\,dx - \frac{1}{\sqrt{2}}\int\limits_{\gamma_{1,\tau}} \sum_{i=1}^n \|\varphi_i^m\|^2_{C^1(\gamma_{1,\tau})}\,ds$$
$$\geq \frac{1}{2}\int\limits_{\omega_\tau}\Big(\sum_{i=1}^n \big((u_{ix}^m)^2 + (u_{it}^m)^2\big)\Big)\,dx - \frac{\operatorname{mes}\gamma_{1,T}}{\sqrt{2}} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})}. \tag{2.19}$$

Let $E = E(x,t) \in C(\overline{D}_T)$ be a square matrix of order n and $u, v \in \mathbb{R}^n$.
If $m_0 = \sup\limits_{(x,t)\in\overline{D}_T} \max\limits_{1\leq i,j\leq n} |E_{ij}(x,t)|$, then

$$|(E(x,t)u,v)| \leq m_0\Big(\sum_{i=1}^n |u_i|\Big)\Big(\sum_{i=1}^n |v_i|\Big)$$
$$\leq \frac{1}{2}\,m_0\Big(\sum_{i=1}^n |u_i|\Big)^2 + \frac{1}{2}\,m_0\Big(\sum_{i=1}^n |v_i|\Big)^2 \leq \frac{n}{2}\,m_0\sum_{i=1}^n |u_i|^2 + \frac{n}{2}\,m_0\sum_{i=1}^n |v_i|^2. \tag{2.20}$$

Analogously, in view of condition (2.1), we have

$$|(f(x,t,u),v)| \leq (M_1 + M_2\|u\|)\sum_{i=1}^n |v_i|$$
$$\leq \frac{1}{2}\,(M_1 + M_2\|u\|)^2 + \frac{1}{2}\Big(\sum_{i=1}^n |v_i|\Big)^2 \leq M_1^2 + M_2^2\Big(\sum_{i=1}^n |u_i|\Big)^2 + \frac{1}{2}\,\Big(\sum_{i=1}^n |v_i|\Big)^2$$
$$\leq M_1^2 + M_2^2 n\sum_{i=1}^n |u_i|^2 + \frac{n}{2}\,\Big(\sum_{i=1}^n |v_i|^2\Big). \tag{2.21}$$

Taking into account inequalities (2.20), (2.21) and the definition of the number M_0, we obtain

$$\bigg|\int\limits_{D_\tau}\Big(A(x,t)\,\frac{\partial u^m}{\partial x}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt + \int\limits_{D_\tau}\Big(B(x,t)\,\frac{\partial u^m}{\partial t}, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt$$
$$+\int\limits_{D_\tau}\Big(C(x,t)u^m, \frac{\partial u^m}{\partial t}\Big)\,dx\,dt + \int\limits_{D_\tau}\Big(f(x,t,u^m), \frac{\partial u^m}{\partial t}\Big)\,dx\,dt\bigg|$$
$$\leq \int\limits_{D_\tau}\bigg(\frac{n}{2}\,M_0\sum_{i=1}^n \Big|\frac{\partial u_i^m}{\partial x}\Big|^2 + \frac{n}{2}\,M_0\sum_{i=1}^n \Big|\frac{\partial u_i^m}{\partial t}\Big|^2\bigg)\,dx\,dt$$
$$+\int\limits_{D_\tau}\bigg(nM_0\sum_{i=1}^n \Big|\frac{\partial u_i^m}{\partial t}\Big|^2\bigg)\,dx\,dt + \int\limits_{D_\tau}\bigg(\frac{n}{2}\,M_0\sum_{i=1}^n |u_i^m|^2 + \frac{n}{2}\,M_0\sum_{i=1}^n \Big|\frac{\partial u_i^m}{\partial t}\Big|^2\bigg)\,dx\,dt$$
$$+\int\limits_{D_\tau}\bigg(M_1^2 + M_2^2 n\sum_{i=1}^n |u_i^m|^2 + \frac{n}{2}\sum_{i=1}^n \Big|\frac{\partial u_i^m}{\partial t}\Big|^2\bigg)\,dx\,dt$$
$$\leq M_1^2 \operatorname{mes} D_\tau + \Big(M_2^2 n + \frac{n}{2}\,M_0\Big)\int\limits_{D_\tau}\sum_{i=1}^n |(u_i^m)^2|\,dx\,dt$$
$$+\frac{n}{2}\,M_0\int\limits_{D_\tau}\sum_{i=1}^n \Big|\frac{\partial u_i^m}{\partial x}\Big|^2\,dx\,dt + \Big(2nM_0 + \frac{n}{2}\Big)\int\limits_{D_\tau}\Big|\frac{\partial u_i^m}{\partial x}\Big|^2\,dx\,dt$$

$$\le M_1^2 \operatorname{mes} D_\tau + \Big(M_2^2 n + 2nM_0 + \frac{n}{2}\Big) \int\limits_{D_\tau} \sum_{i=1}^n \bigg((u_i^m)^2 + \Big|\frac{\partial u_i^m}{\partial x}\Big|^2 + \Big|\frac{\partial u_i^m}{\partial t}\Big|^2\bigg)\, dx\, dt$$

$$= M_3 + M_4 \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\, dx\, dt, \tag{2.22}$$

where

$$M_3 = M_1^2 \operatorname{mes} D_\tau, \quad M_4 = M_2^2 n + 2nM_0 + \frac{n}{2}\,. \tag{2.23}$$

By virtue of (2.19) and (2.22), it follows from (2.8) that

$$\int\limits_{D_\tau} \Big(F^m, \frac{\partial u^m}{\partial t}\Big)\, dx\, dt$$

$$\ge \frac{1}{2} \int\limits_{\omega_\tau} \Big(\sum_{i=1}^n \big((u_{ix}^m)^2 + (u_{it}^m)^2\big)\, dx - \frac{1}{\sqrt{2}} \operatorname{mes} \gamma_{1,T} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})}$$

$$- M_3 - M_4 \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\, dx\, dt,$$

whence, owing to the fact that

$$\Big(F^m, \frac{\partial u^m}{\partial t}\Big) \le \frac{1}{2} \sum_{i=1}^n (F_i^m)^2 + \frac{1}{2} \sum_{i=1}^n (u_{it}^m)^2,$$

we get

$$\frac{1}{2} \int\limits_{\omega_\tau} \Big(\sum_{i=1}^n \big((u_{ix}^m)^2 + (u_{it}^m)^2\big)\, dx \le M_4 \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\, dx\, dt$$

$$+ \frac{1}{\sqrt{2}} \operatorname{mes} \gamma_{1,T} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_3 + \frac{1}{2} \int\limits_{D_\tau} \sum_{i=1}^n (u_{it}^m)^2\, dx\, dt + \frac{1}{2} \int\limits_{D_\tau} \sum_{i=1}^n (F_i^m)^2\, dx\, dt$$

$$\le \Big(M_4 + \frac{1}{2}\Big) \int\limits_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\, dx\, dt$$

$$+ \frac{1}{2} \int\limits_{D_\tau} \sum_{i=1}^n (F_i^m)^2\, dx\, dt + \frac{1}{\sqrt{2}} \operatorname{mes} \gamma_{1,T} \sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_3. \tag{2.24}$$

Since $u_i^m(0,t) = 0$, $i = 1, \dots, n$, we have

$$u_i^m(x,\tau) = \int\limits_0^x u_{ix}^m(\sigma,\tau)\, d\sigma, \quad 0 \le x \le \tau.$$

Hence, taking into account the Schwartz inequality, we get

$$(u_i^m)^2(x,\tau) \le \int\limits_0^x 1^2\, d\sigma \int\limits_0^x (u_{ix}^m)^2(\sigma,\tau)\, d\sigma \le x \int\limits_0^\tau (u_{ix}^m)^2(\sigma,\tau)\, d\sigma \le T \int\limits_{\omega_\tau} (u_{ix}^m)^2\, d\sigma. \tag{2.25}$$

Arguing analogously and taking into account (2.6), we obtain

$$u_i^m(x,\tau) = \varphi_{1i}^m + \int\limits_x^\tau u_{it}^m(x,s)\, ds$$

and, consequently,

$$
\begin{aligned}
(u_i^m)^2(x,\tau) &\le 2(\varphi_{1i}^m)^2(x) + 2\Big(\int_x^\tau u_{it}^m(x,s)\,ds\Big)^2 \le 2(\varphi_{1i}^m)^2(x) + 2\int_x^\tau 1^2\,dt \int_x^\tau (u_{it}^m)^2(x,t)\,dt \\
&= 2(\varphi_{1i}^m)^2(x) + 2(\tau - x)\int_x^\tau (u_{it}^m)^2(x,t)\,dt \le 2(\varphi_{1i}^m)^2(x) + 2T\int_x^\tau (u_{it}^m)^2(x,t)\,dt. \qquad (2.26)
\end{aligned}
$$

Integration of inequality (2.26) yields

$$
\begin{aligned}
\int_{\omega_\tau} (u_i^m)^2\,dx &= \int_0^\tau (u_i^m)^2(x,\tau)\,dx \\
&\le 2\int_x^\tau (\varphi_{1i}^m)^2(x)\,dx + 2T\int_0^\tau \Big[\int_x^\tau (u_{it}^m)^2(x,t)\,dt\Big]\,dx = 2\int_0^\tau (\varphi_{1i}^m)^2(x)\,dx + 2T\int_{D_\tau} (u_{it}^m)^2\,dx\,dt \\
&\le 2\tau\|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,\tau})} + 2T\int_{D_\tau} (u_{it}^m)^2(x,t)\,dx\,dt \le 2T\|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,\tau})} + 2T\int_{D_\tau} (u_{it}^m)^2\,dx\,dt,
\end{aligned}
$$

from which it follows that

$$
\frac{1}{2}\int_{\omega_\tau} \Big(\sum_{i=1}^n (u_i^m)^2\Big)\,dx \le T\sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + T\int_{D_\tau} \sum_{i=1}^n (u_{it}^m)^2\,dx\,dt. \qquad (2.27)
$$

Combining inequalities (2.24) and (2.27), we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{\omega_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx \\
\le \Big(M_4 + T + \frac{1}{2}\Big)\int_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt \\
+ \frac{1}{2}\int_{D_\tau} \sum_{i=1}^n (F_i^m)^2\,dx\,dt + \Big(\frac{1}{\sqrt{2}}\,\operatorname{mes}\gamma_{1,T} + T\Big)\sum_{i=1}^n \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_3. \qquad (2.28)
\end{aligned}
$$

Assume

$$
w(\tau) = \int_{\omega_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx. \qquad (2.29)
$$

Taking into account that

$$
\int_{D_\tau} \sum_{i=1}^n \big((u_i^m)^2 + (u_{ix}^m)^2 + (u_{it}^m)^2\big)\,dx\,dt = \int_0^\tau w(\sigma)\,d\sigma,
$$

$$
\int_{D_\tau} \sum_{i=1}^n (F_i^m)^2\,dx\,dt \le \operatorname{mes} D_T \sum_{i=1}^n \|F_i^m\|^2_{C(D_T)},
$$

from (2.28), in view of (2.29), we get

$$
w(\tau) \le M_5\int_0^\tau w(\sigma)\,d\sigma + M_6\sum_{i=1}^n \|F_i^m\|^2_{C(D_T)} + M_7\sum_{i=1}^n \|\varphi_i^m\|^2_{C^1(\gamma_{1,T})} + M_8, \qquad (2.30)
$$

where

$$M_5 = 2M_4 + 2T + 1, \quad M_6 = \operatorname{mes} D_T, \quad M_7 = \sqrt{2} \operatorname{mes} \gamma_{1,\tau} + 2T, \quad M_8 = 2M_3. \tag{2.31}$$

According to Gronwall's lemma, it follows from (2.30) that

$$w(\tau) \le \Big[M_6 \sum_{i=1}^{n} \|F_i^m\|^2_{C(D_T)} + M_7 \sum_{i=1}^{n} \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_8 \Big] \exp M_5 T, \ \ 0 \le \tau \le T. \tag{2.32}$$

By virtue of (2.25), (2.29) and (2.32), it is not difficult to see that

$$\begin{aligned}(u_i^m)^2(x,\tau) &\le T \int\limits_{\omega_\tau} \sum_{i=1}^{n} (u_{ix}^m)^2 \, dx \le T w(\tau) \\ &\le T \Big[M_6 \sum_{i=1}^{n} \|F_i^m\|^2_{C(D_T)} + M_7 \sum_{i=1}^{n} \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,T})} + M_8 \Big] \exp M_5 T, \ \ 0 \le \tau \le T. \end{aligned} \tag{2.33}$$

Taking into account the obvious inequality $\big(\sum\limits_{i=1}^{n} a_i^2\big)^{\frac12} \le \sum\limits_{i=1}^{n} |a_i|$, from (2.33) we obtain

$$\begin{aligned}\|u^m\|_{C(\overline{D}_T)} &= \sum_{i=1}^{n} \|u_i^m\|_{C(\overline{D}_T)} \\ &\le n^{\frac12} \Big(\sum_{i=1}^{n} \|u_i^m\|^2_{C(\overline{D}_T)} \Big)^{\frac12} = n^{\frac12} \Big(\sum_{i=1}^{n} \sup_{(x,t)\in D_T} |u_i^m(x,\tau)|^2 \Big)^{\frac12} \\ &\le n^{\frac12} \bigg(nT \Big[M_6 \sum_{i=1}^{n} \|F_i^m\|^2_{C(\overline{D}_T)} + M_7 \sum_{i=1}^{n} \|\varphi_{1i}^m\|^2_{C^1(\gamma_{1,\tau})} + M_8 \Big] \exp M_5 T \bigg)^{\frac12} \\ &\le n^{\frac12} \bigg(n^{\frac12} (TM_6)^{\frac12} \sum_{i=1}^{n} \|F_i^m\|_{C(\overline{D}_T)} + n^{\frac12} (TM_7)^{\frac12} \sum_{i=1}^{n} \|\varphi_{1i}^m\|_{C^1(\gamma_{1,\tau})} + n^{\frac12} (TM_8)^{\frac12} \bigg) \exp \frac12 \, M_5 T \\ &\le n (TM_6)^{\frac12} \exp \frac12 \, M_5 T \sum_{i=1}^{n} \|F_i^m\|_{C(\overline{D}_T)} \\ &\qquad + n (TM_7)^{\frac12} \exp \frac12 \, M_5 T \sum_{i=1}^{n} \|\varphi_{1i}^m\|_{C(\gamma_{1,T})} + n (TM_8)^{\frac12} \exp \frac12 \, M_5 T \\ &= c_1 \|F^m\|_{C(\overline{D}_T)} + c_2 \|\varphi_1^m\|_{C^1(\gamma_{1,\tau})} + c_3. \end{aligned} \tag{2.34}$$

Here

$$c_1 = n(TM_6)^{\frac12} \exp \frac12 \, M_5 T, \quad c_2 = n(TM_7)^{\frac12} \exp \frac12 \, M_5 T, \quad c_3 = n(TM_8)^{\frac12} \exp \frac12 \, M_5 T. \tag{2.35}$$

By (2.3) and (2.4), passing in inequality (2.34) to the limit, as $m \to \infty$, we obtain an a priori estimate (2.2) in which the constants c_1, c_2 and c_3 are given by equalities (2.35), and the constants M_5, M_6, M_7 and M_8 in (2.35) are defined from (2.1), (2.23) and (2.31). In addition, $c_i > 0$, $i = 1, 2$. □

3 Reduction of problem (1.1), (1.2) to a nonlinear system of integral Volterra type equations

As a result of our passage to new independent variables ξ and η:

$$\xi = \frac12 \, (t + x), \quad \eta = \frac12 \, (t - x), \tag{3.1}$$

the domain D_T turns into a triangle $G_T = OP_1P_2$ of the plane $O_{\xi\eta}$, where $O = O(0,0)$, $P_1 = P_1(T,0)$, $P_2 = P_2(\frac{1}{2}T, \frac{1}{2}T)$, and problem (1.1), (1.2) can now be rewritten in the form

$$L_1v := v_{\xi\eta} + A_1(\xi,\eta)v_\xi + B_1(\xi,\eta)v_\eta + c_1(\xi,\eta)v + f_1(\xi,\eta,v) = F_1(\xi,\eta), \ \ (\xi,\eta) \in G_T, \tag{3.2}$$

$$v\big|_{OP_1:\eta=0,\ 0\le\xi\le T} = \psi_1(\xi), \ \ 0 \le \xi \le T, \tag{3.3}$$

$$v\big|_{OP_2:\xi=\eta,\ 0\le\eta\le\frac{1}{2}T} = \psi_2(\eta), \ \ 0 \le \eta \le T, \tag{3.4}$$

with respect to a new unknown vector function $v(\xi,\eta) = u(\xi-\eta,\xi+\eta)$. Here

$$\begin{cases} A_1(\xi,\eta) = \dfrac{1}{2}\big(A(\xi-\eta,\xi+\eta) + B(\xi-\eta,\xi+\eta)\big), \\ B_1(\xi,\eta) = \dfrac{1}{2}\big(B(\xi-\eta,\xi+\eta) - A(\xi-\eta,\xi+\eta)\big), \\ C_1(\xi,\eta) = C(\xi-\eta,\xi+\eta), \\ F_1(\xi,\eta) = F(\xi-\eta,\xi+\eta), \\ f_1(\xi,\eta,v) = f(\xi-\eta,\xi+\eta,v), \end{cases} \tag{3.5}$$

$$\psi_1(\xi) = \varphi_1(\xi), \quad \psi_2(\eta) = \varphi_2(2\eta). \tag{3.6}$$

Below, it will be assumed that $u \in C^2(\overline{D}_T)$ is a classical solution of problem (1.1), (1.2), and according to this fact, $v \in C^2(\overline{D}_T)$ is a classical solution of problem (3.2)–(3.4).

Consider first the case when in equation (3.2)

$$f_1(\xi,\eta,v) = 0, \tag{3.7}$$

and the coefficients A_1, B_1 and C_1 of that equation satisfy the following condition:

$$B_{1\eta} + A_1B_1 - C_1 = 0. \tag{3.8}$$

When conditions (3.7) and (3.8) are fulfilled, equation (3.2) can be rewritten in the form

$$\Big(\frac{\partial}{\partial\eta} + A_1\Big)\Big(\frac{\partial v}{\partial\xi} + B_1v\Big) = F_1, \ \ (\xi,\eta) \in G_T. \tag{3.9}$$

If we adopt the notation

$$w = \frac{\partial v}{\partial\xi} + B_1v, \tag{3.10}$$

then by virtue of (3.3) and (3.9), the vector function $w = w(\xi,\eta)$ for fixed ξ will be a solution of the Cauchy problem

$$w_\eta + A_1(\xi,\eta)w = F_1(\xi,\eta), \tag{3.11}$$

$$w(\xi,0) = \psi_{1\xi}(\xi) + B_1(\xi,0)\psi_1(\xi). \tag{3.12}$$

Since under the above assumptions $A_1 = A_1(\xi,\eta) \in C(\overline{G}_T)$, therefore, as is known, there exists the fundamental matrix $X_1 = X_1(\xi,\eta)$ of the corresponding to (3.11) homogeneous system satisfying both the following matrix equality [14]

$$X_{1\eta} + A_1X_1 = 0 \tag{3.13}$$

and the condition

$$\det X_1(\xi,\eta) \ne 0, \ \ (\xi,\eta) \in G_T. \tag{3.14}$$

Denote by $K = K(\xi,\eta,\zeta)$ the Cauchy matrix of order n of system (3.13) which satisfies the conditions

$$K_\eta + A_1K = 0, \tag{3.15}$$

$$K(\xi,\zeta,\zeta) = I, \tag{3.16}$$

where I is the unit matrix of order n.

As is known, the Cauchy matrix K is given by the equality

$$K(\xi,\eta,\zeta)=X_1(\xi,\eta)X_1^{-1}(\xi,\zeta), \tag{3.17}$$

where $X_1=X_1(\xi,\eta)$ is the fundamental matrix satisfying conditions (3.13), (3.14) [14].

The Cauchy matrix K for the constant matrix A_1 is given by the equality [14]

$$K(\xi,\eta,\zeta)=\exp(A_1(\zeta-\eta)). \tag{3.18}$$

By virtue of (3.15) and (3.16), the unit solution of the Cauchy problem (3.11), (3.12) is defined by the formula [14]

$$w(\xi,\eta)=K(\xi,\eta,0)\big(\psi_{1\xi}(\xi)+B_1(\xi,0)\psi_1(\xi)\big)+\int\limits_0^\eta K(\xi,\eta,\zeta)F_1(\xi,\zeta)\,d\zeta. \tag{3.19}$$

Owing to (3.18), in case the matrix A_1 is constant, formula (3.19) takes the form

$$w(\xi,\eta)=\exp(-A_1\eta)\big(\psi_{1\xi}(\xi)+B_1(\xi,0)\psi_1(\xi)\big)+\int\limits_0^\eta \exp(A_1(\zeta-\eta))F_1(\xi,\zeta)\,d\zeta. \tag{3.20}$$

Taking into account equalities (3.9)–(3.12), it follows from the above reasoning that a solution v of problem (3.2)–(3.4) satisfies the Cauchy problem

$$\frac{\partial v}{\partial\xi}+B_1v=w(\xi,\eta),\quad \eta\le\xi\le T-\eta, \tag{3.21}$$

$$v(\xi,\eta)\big|_{\xi=\eta}=\psi_2(\eta),\quad 0\le\eta\le\frac12\,T, \tag{3.22}$$

where the vector function $w=w(\xi,\eta)$ is given by formula (3.19).

Analogously to the matrix K, we denote by $\Lambda=\Lambda(\eta,\xi,\theta)$ the Cauchy matrix of the corresponding to (3.21) homogeneous system which satisfies the conditions

$$\Lambda_\xi+B_1\Lambda=0, \tag{3.23}$$

$$\Lambda(\eta,\theta,\theta)=1, \tag{3.24}$$

and which is given by the equality

$$\Lambda(\eta,\xi,\theta)=X_2(\eta,\xi)X_2^{-1}(\eta,\theta), \tag{3.25}$$

where $X_2(\eta,\xi)$ is the fundamental matrix for the corresponding to (3.21) homogeneous system.

When the matrix B_1 is constant, the Cauchy matrix Λ is given by the equality

$$\Lambda(\eta,\xi,\theta)=\exp(B_1(\theta-\xi)). \tag{3.26}$$

Owing to (3.23) and (3.24), the unique solution of the Cauchy problem (3.21), (3.22) is defined by the formula [14]

$$v(\xi,\eta)=\Lambda(\eta,\xi,\eta)\psi_2(\eta)+\int\limits_\eta^\xi \Lambda(\eta,\xi,\theta)w(\theta,\eta)\,d\theta. \tag{3.27}$$

By (3.26), when the matrix B_1 is constant, formula (3.27) takes the form

$$v(\xi,\eta)=\exp(B_1(\eta-\xi))\psi_2(\eta)+\int\limits_\eta^\xi \exp(B_1(\theta-\xi))w(\theta,\eta)\,d\theta. \tag{3.28}$$

Substituting (3.19) for the vector function $w(\xi,\eta)$ into the right-hand side of equality (3.27), we obtain

$$\begin{aligned} v(\xi,\eta) &= \Lambda(\eta,\xi,\eta)\psi_2(\eta) \\ &\quad + \int\limits_{\eta}^{\xi} \Lambda(\eta,\xi,\theta)\Big[K(\theta,\eta,0)\big(\psi_{1\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big) + \int\limits_0^{\eta} K(\theta,\eta,\zeta)F_1(\theta,\zeta)\,d\zeta\Big]\,d\theta \\ &= \Lambda(\eta,\xi,\eta)\psi_2(\eta) + \int\limits_{\eta}^{\xi} \Lambda(\eta,\xi,\theta)\big[K(\theta,\eta,0)\big(\psi_{1,\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big)\big]\,d\theta + \\ &\quad + \int\limits_{\eta}^{\xi}\int\limits_0^{\eta} \Lambda(\eta,\xi,\theta)K(\theta,\eta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta, \quad (\xi,\eta)\in G_T. \end{aligned} \tag{3.29}$$

We rewrite equality (3.29) in the form

$$v(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_0^{\eta} R(\xi,\eta;\theta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta + F_2(\xi,\eta), \quad (\xi,\eta)\in G_T. \tag{3.30}$$

where

$$R(\xi,\eta;\theta,\zeta) = \Lambda(\eta,\xi,\theta)K(\theta,\eta,\zeta), \tag{3.31}$$

$$F_2(\xi,\eta) = \Lambda(\eta,\xi,\eta)\psi_2(\eta) + \int\limits_{\eta}^{\xi} \Lambda(\eta,\xi,\theta)\big[K(\theta,\eta,0)\big(\psi_{1\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big)\big]\,d\theta. \tag{3.32}$$

In case the matrices A_1 and B_1 are constant, by virtue of (3.18) and (3.26), equalities (3.31) and (3.32) take the form

$$R(\xi,\eta;\theta,\zeta) = \exp\big(B_1(\theta-\xi)+A_1(\zeta-\eta)\big), \tag{3.33}$$

$$\begin{aligned} F_2(\xi,\eta) &= \exp(B_1(\eta-\xi))\psi_2(\eta) \\ &\quad + \int\limits_{\eta}^{\xi} \exp(B_1(\theta-\xi))\big[\exp(A_1\eta)\big(\psi_{1\xi}(\theta)+B_1(\theta,0)\psi_1(\theta)\big)\big]\,d\theta. \end{aligned} \tag{3.34}$$

Consider now a general case when it is not necessary for conditions (3.7) and (3.8) to be fulfilled. We rewrite system (3.2) in the form

$$\Big(\frac{\partial}{\partial\eta}+A_1\Big)\Big(\frac{\partial v}{\partial\xi}+B_1 v\Big) = (B_{1\eta}+A_1B_1-C_1)v - f_1 + F_1. \tag{3.35}$$

Then, due to representation (3.30), the classical solution of problem (3.2)–(3.4) or, what comes to the same, of problem (3.35), (3.3), (3.4), is given by the formula

$$v(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_0^{\eta} R(\xi,\eta;\theta,\zeta)\big[(B_{1\eta}+A_1B_1-C_1)v(\theta,\zeta)-f_1(\theta,\zeta,v)\big]\,d\zeta\,d\theta + F_3(\xi,\eta), \quad (\xi,\eta)\in G_T, \tag{3.36}$$

where

$$F_3(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_0^{\eta} R(\xi,\eta;\theta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta + F_2(\xi,\eta). \tag{3.37}$$

Remark 3.1. Equality (3.36) can be considered as a nonlinear system of integral Volterra type equations which we rewrite as follows:

$$v = L_2 v + L_3 F_1 + l_0(\psi_0, \psi_2), \tag{3.38}$$

where the operator L_2 acts according to the formula

$$(L_2 v)(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_{0}^{\eta} R(\xi,\eta;\theta,\zeta)\Big[(B_{1\eta} + A_1 B_1 - C_1)v(\theta,\zeta) - f_1(\theta,\zeta,v)\Big]\,d\zeta\,d\theta,\ \ (\xi,\eta)\in G_T, \tag{3.39}$$

and the operators L_3 and l_0, by virtue of (3.32) and (3.37), act by the formulas

$$(L_3 F_1)(\xi,\eta) = \int\limits_{\eta}^{\xi}\int\limits_{0}^{\eta} R(\xi,\eta;\theta,\zeta)F_1(\theta,\zeta)\,d\zeta\,d\theta, \tag{3.40}$$

$$(l_0(\psi_1,\psi_2))(\xi,\eta) = \Lambda(\eta,\xi,\eta)\psi_2(\eta) + \int\limits_{\eta}^{\xi} \Lambda(\eta,\xi,\theta)\big[K(\theta,\eta,0)\big(\psi_{1\xi}(\theta) + B_1(\theta,0)\psi_1(\theta)\big)\big], \tag{3.41}$$

where $(\xi,\eta)\in G_T$.

4 Global solvability of problem (1.1), (1.2) in the class of continuous functions

Remark 4.1. If we impose on the coefficients and on the vector function f appearing in equation (1.1) the requirements of smoothness

$$A, B \in C^2(\overline{D}_T), \quad C \in C^1(\overline{D}_T), \quad f \in C^1(\overline{D}_T \times \mathbb{R}^n), \tag{4.1}$$

and along with equalities (3.17) and (3.25) take into account the properties dealt with the smoothness of solutions of the system of ordinary differential equations, we will have [14]

$$R(\xi,\eta;\theta,\zeta) \in C^2(\overline{G}_T \times \overline{G}_T). \tag{4.2}$$

Remark 4.2. Under conditions (4.1), in view of (4.2) for the operator L_2 acting according to formula (3.39), we have

$$L_2 v \in C^{k+1}(\overline{G}_T), \ \text{ if } \ v \in C^k(\overline{G}_T), \ \ k = 0,1, \tag{4.3}$$

and, hence, the operator $L_2 : C^k(\overline{G}_T) \to C^{k+1}(\overline{G}_T)$ will be continuous.

Arguing as above, we find that

$$L_3 F_1 \in C^{k+1}(\overline{G}_T), \ \text{ if } \ F_1 \in C^k(\overline{G}_T), \ \ k = 0,1, \tag{4.4}$$

and

$$l_0(\psi_1,\psi_2) \in C^{k+1}(\overline{G}_T), \ \text{ if } \ \psi_i \in C^k(OP_i), k = 0,1,2; \ \ i = 1,2. \tag{4.5}$$

In addition, the operators $L_3 : C^k(\overline{G}_T) \to C^{k+1}(\overline{G}_T)$ and $l_0 : C^k(OP_1)\times C^k(OP_2) \to C^k(\overline{G}_T)$ will be continuous.

Remark 4.3. It can be easily verified that if u is a generalized solution of problem (1.1), (1.2) of the class C in the domain D_T, then the vector function $v(\xi,\eta) = u(\xi-\eta,\xi+\eta)$ will be a generalized solution of problem (3.2)–(3.4) of the class C in the domain G_T in the following sense: $v \in C(\overline{G}_T)$, and there exists the sequence of vector functions $v^m \in C^2(\overline{G}_T)$ such that

$$\lim_{m\to\infty} \|v^m - v\|_{C(\overline{G}_T)} = 0, \quad \lim_{m\to\infty} \|L_1 v^m - F_1\|_{C(\overline{G}_T)} = 0, \tag{4.6}$$

$$\lim_{m\to\infty} \big\|v^m\big|_{OP_i} - \psi_i\big\|_{C^1(OP_i)} = 0, \ \ i = 1,2, \tag{4.7}$$

and the converse statement holds, too.

Lemma 4.1. *Let conditions* (4.1) *be fulfilled. Then the vector function will be a generalized solution of problem* (3.2)–(3.4) *of the class C in the domain G_T if and only if v is a solution of the nonlinear system of integral Volterra type equations* (3.38) *of the class $C(\overline{G}_T)$.*

Proof. Let $v \in C(\overline{G}_T)$ be a solution of system (3.38). Since the space $C^k(\overline{G}_T)$, $k = 1, 2$, is the dense in $C(\overline{G}_T)$ and the space $C^2(OP_i)$ is the dense in $C^1(OP_i)$, $i = 1, 2$, [17], there exists the sequence of vector functions $F_{1n} \in C^1(\overline{G}_T)$ ($\psi_{in} \in C^2(OP_i)$, $i = 1, 2$) such that

$$\lim_{n\to\infty} \|F_{1n} - F_1\|_{C(\overline{G}_T)} = 0 \quad \Big(\lim_{n\to\infty} \|\psi_{in} - \psi_i\|_{C^1(OP_i)} = 0, \ \ i = 1, 2\Big). \tag{4.8}$$

Analogously, since $v \in C(\overline{G}_T)$, there exists the sequence of vector functions $w_n \in C^2(\overline{G}_T)$ such that

$$\lim_{n\to\infty} \|w_n - v\|_{C(\overline{G}_T)} = 0. \tag{4.9}$$

Let us now introduce the following sequence of vector functions:

$$v_n = L_2 w_n + L_3 F_{1n} + l_0(\psi_{1n}, \psi_{2n}). \tag{4.10}$$

By virtue of (4.1)–(4.5), the vector function $v_n \in C^2(\overline{G}_T)$, and owing to its construction, we will have

$$v_n\big|_{OP_i} = \psi_{in}, \ \ i = 1, 2. \tag{4.11}$$

Taking into account Remark 4.2 and the limiting equalities (4.8), (4.9), we find that

$$v_n \longrightarrow \big[L_2 v + L_3 F_1 + l_0(\psi_1, \psi_2)\big] \tag{4.12}$$

in the space $C(\overline{G}_T)$, as $n \to \infty$. At the same time, by equality (3.38), we have

$$L_2 v + L_3 F_1 + l_0(\psi_1, \psi_2) = v. \tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\lim_{n\to\infty} \|v_n - v\|_{C(\overline{G}_T)} = 0. \tag{4.14}$$

In view of equality (4.10) and Remark 4.2, as well as of the fact how we have obtained equality (3.30), from the representation (3.9) we get

$$\Big(\frac{\partial}{\partial\eta} + A_1\Big)\Big(\frac{\partial v_n}{\partial\xi} + B_1 v_n\Big) = (B_{1\eta} + A_1 B_1 - C_1) w_n - (f_1(\,\cdot\,, w_n)) + F_{1n}, \tag{4.15}$$

$$v_n\big|_{OP_i} = \psi_{in}, \ \ i = 1, 2. \tag{4.16}$$

By virtue of the representation of equation (3.2) by equality (3.35), from (4.15) we obtain

$$L_1 v_n = (B_{1\eta} + A_1 B_1 - C_1)(w_n - v_n) + \big(f_1(\,\cdot\,, v_n) - f_1(\,\cdot\,, w_n)\big) + F_{1n},$$

whence, in view of (4.9) and (4.14), we get

$$\lim_{n\to\infty} \|L_1 v_n - F_1\|_{C(\overline{G}_T)} = 0.$$

It follows from (4.16) and (4.8) that

$$\lim_{n\to\infty} \big\|v_n\big|_{OP_i} - \psi_i\big\|_{C^1(OP_i)} = 0, \ \ i = 1, 2.$$

The last two limiting equalities show that if $v \in C(\overline{G}_T)$ is a solution of system (3.38), then the vector function v will be a generalized solution of problem (3.2)–(3.4) of the class C in the domain G_T. Thus Lemma 4.1 is proved, since the converse statement can be easily verified. $\square$

As is known, the space $C^1(\overline{G}_T)$ is compactly imbedded in the space $C(\overline{G}_T)$. Therefore, taking into account Remark 4.2 and considering L_2 as the operator acting in the space $C(\overline{G}_T)$ by formula (3.39), the operator

$$L_2 : C(\overline{G}_T) \longrightarrow C(\overline{G}_T)$$

will be compact. In addition, for the fixed ψ_1, ψ_2 and F_1, the operators L_3 and l_0 acting by formulas (3.40) and (3.41) are constant, and hence their sum

$$L_0 := (L_2 + L_3 F_1 + l_0(\psi_1, \psi_2)) : C(\overline{G}_T) \longrightarrow C(\overline{G}_T) \tag{4.17}$$

will likewise be compact. By (4.17), system (3.38) can be rewritten in the form

$$v = L_0 v. \tag{4.18}$$

Let $v \in C(\overline{D}_T)$ be a solution of equation (4.18), and $\psi_2 = 0$. Then, since v is connected with $u \in C(\overline{G}_T)$ by the equality $v(\xi, \eta) = u(\xi - \eta, \xi + \eta)$, and u satisfies a priori estimate (2.2), in view of Lemma 4.1 and Lemma 2.1, an a priori estimate of the same type will take place likewise for the vector function v,

$$\|v\|_{C(\overline{G}_T)} \leq c_1 \|F\|_{C(\overline{G}_T)} + c_2 \|\varphi_1\|_{C^1(\gamma_{1,T})} + c_3, \tag{4.19}$$

where the constants c_i, $i = 1, 2, 3$, are defined from equalities (2.35). It should now be noted that owing to Remark 4.3 and Lemma 4.1, if $v \in C(\overline{G}_T)$ is a solution of equation $v = \tau L_0 v$, where $\tau \in [0, 1]$, then the same a priori estimate (4.19) with the constants c_1, c_2 and c_3, independent in view of (2.1), (2.23), (2.31) and (2.35) of v, F, φ_1 and τ, will be valid. Therefore, taking into account that the operator $L_0 : C(\overline{G}_T) \to C(\overline{G}_T)$ is continuous and compact, it follows from the Lere–Schauder theorem [18] that equation (4.18) has at least one solution in the space $C(\overline{G}_T)$. This, in its turn, in view of the above remarks, implies that problem (1.1), (1.2) has at least one generalized solution of the class C in the domain D_T. Thus, the following theorem is valid.

Theorem 4.1. *Let conditions* (2.1), (4.1) *and* $F \in C(\overline{D}_T)$, $\varphi_1 \in C^1(\gamma_{1,T})$, $\varphi_2 = 0$, *be fulfilled. Then problem* (1.1), (1.2) *has at least one generalized solution of the class* C *in the domain* D_T.

5 The smoothness and uniqueness of a solution of problem (1.1), (1.2). The existence of a global solution in the domain D_∞

By virtue of (4.3), (4.4) and (4.5), from Remark 4.3 and Lemma 4.1 follows

Lemma 5.1. *Let the vector function* u *be a generalized solution of problem* (1.1), (1.2) *of the class* C *in the domain* D_T *in a sense of Definition* 1.1, *and in addition, the conditions of smoothness* (4.1) *and* $F \in C^1(\overline{D}_T)$, $\varphi_1 \in C^2(\gamma_{1,T})$, $i = 1, 2$, *hold. Then* u *belongs to the class* $C^2(\overline{D}_T)$ *and is a classical solution of problem* (1.1), (1.2).

We say that the vector function $f = f(x, t, u)$ satisfies the local Lipschitz condition on the set $\overline{D}_T \times \mathbb{R}$ if

$$\big\|f(x, t, u_2) - f(x, t, u_1)\big\| \leq M(T, R)\|u_2 - u_1\|, \ \ (x, t) \in \overline{D}_T, \ \ \|u_i\| \leq R, \ \ i = 1, 2, \tag{5.1}$$

where $M = M(T, R) = const \geq 0$. Note that if $f \in C^1(\overline{D}_T \times \mathbb{R}^n)$, then condition (5.1) will automatically be fulfilled.

Lemma 5.2. *If the vector function* $f \in C(\overline{D}_T \times \mathbb{R}^n)$ *satisfies condition* (5.1), *then problem* (1.1), (1.2) *fails to have more than one generalized solution of the class* C *in the domain* D_T.

Proof. Assume that problem (1.1), (1.2) has two generalized solutions u_1 and u_2 of the class C in the domain D_T. According to Remark 1.1 and Definition 1.1, there exists a sequence of vector functions $u_j^m \in C_0^2(\overline{D}_T, \gamma_{2,T})$ such that

$$\lim_{m\to\infty} \|u_j^m - u_j\|_{C(\overline{D}_T)} = \lim_{m\to\infty} \|Lu_j^m - F\|_{C(D_T)} = \lim_{m\to\infty} \big\|u^m\big|_{\gamma_{1,T}} - \varphi_1\big\|_{C^1(\gamma_{1,T})} = 0. \tag{5.2}$$

We introduce the notation $w^m = u_2^m - u_1^m$. It is easy to verify that $w^m \in C^2(\overline{D}_T)$ is a solution of the following problem:

$$w_{tt}^m - w_{xx}^m + A(x,t)w_x^m + B(x,t)w_t^m + C(x,t)w^m + g^m = F^m, \tag{5.3}$$

$$w^m\big|_{\gamma_{1,T}} = \varphi_1^m, \quad w^m\big|_{\gamma_{2,T}} = 0. \tag{5.4}$$

Here

$$g^m = f(x,t,u_2^m) - f(x,t,u_1^m), \tag{5.5}$$

$$F^m = Lu_2^m - Lu_1^m, \tag{5.6}$$

$$\varphi_1^m = (u_2^m - u_1^m)\big|_{\gamma_{1,T}}. \tag{5.7}$$

It follows from (5.2) that there exists a number $d = const > 0$ such that it does not depend on the indices j and m, and $\|u_j^m\|_{C(\overline{D}_T)} \le d$. Hence, by virtue of (5.1) and (5.5), we have

$$\|g^m\| \le M(T,d)\|u_2^m - u_1^m\| = M(T,d)\|w^m\|. \tag{5.8}$$

Reasoning now for the solution w^m of problem (5.3), (5.4) in the same way as for the solution u^m of problem (2.5), (2.6), owing to (5.8), we have to take in inequalities (2.1), (2.23), (2.28), (2.30) and (2.34) the constants, corresponding to M_1, M_3, M_8 and c_3, equal to zero. Consequently, instead of inequality (2.34) we will have

$$\|w_j^m\|_{C(\overline{D}_T)} \le \widetilde{c}_1\|F^m\|_{C(\overline{D}_T)} + \widetilde{c}_2\|\varphi_1^m\|_{C^1(\gamma_{1,T})}. \tag{5.9}$$

Here, unlike (2.35), for the constants $\widetilde{c}_1$ and $\widetilde{c}_2$ we have

$$\widetilde{c}_1 = n(TM_6)^{\frac{1}{8}} \exp\frac{1}{2}\,\widetilde{M}_5T, \quad \widetilde{c}_2 = n(TM_7)^{\frac{1}{2}} \exp\frac{1}{2}\,\widetilde{M}_5T,$$

where M_6 and M_7 are defined from (2.31) and, in view of (2.23),

$$\widetilde{M}_5 = 2\widetilde{M}_4 + 2T + 1, \quad \widetilde{M}_4 = M^2(T,d)n + 2nM_0 + \frac{n}{2}\,.$$

It follows from (5.2), (5.6) and (5.7) that

$$\begin{gathered}\lim_{m\to\infty}\|w^m\|_{C(\overline{D}_T)} = \|u_2 - u_1\|_{C(\overline{D}_T)}, \quad \lim_{m\to\infty}\|F^m\|_{C(\overline{D}_T)} = 0,\\ \lim_{m\to\infty}\|\varphi_1^m\|_{C^1(\gamma_{1,T})} = 0.\end{gathered} \tag{5.10}$$

If now we pass in inequality (5.9) to the limit, as $m \to \infty$, then due to the limiting equalities (5.10) we get $\|u_2 - u_1\|_{C(\overline{D}_T)} \le 0$, which implies that $u_2 = u_1$. □

The consequence of Theorem 4.1 and Lemmas 5.1 and 5.2 is the following

Theorem 5.1. *Let for any positive T conditions* (2.1), (4.1) *and $F \in C^1(\overline{D}_\infty)$, $\varphi_1 \in C^2(\gamma_{1,\infty})$, $\varphi_2 = 0$ be fulfilled. Then problem* (1.1), (1.2) *has the unique classical solution $u \in C^2(\overline{D}_\infty)$ in the domain D_∞.*

Proof. It follows from Theorem 4.1 and Lemmas 5.1 and 5.2 that in the domain D_T, where $T = k \in N$, there exists the unique classical solution $u_k \in C^2(\overline{D}_k$ of problem (1.1), (1.2). In addition, $u_{k+1}|_{D_k}$ is likewise the classical solution of problem (1.1), (1.2) in the domain D_k. Therefore, by Lemma 5.2, the equality $u_{k+1}|_{D_k} = u_k$ holds. This implies that the vector function u constructed in the domain D_∞ by the rule: $u(x,t) = u_k(x,t)$, where $k = [t]+1$, $[t]$ is an integer part of the number, and $(x,t) \in D_\infty$, is the unique classical solution of problem (1.1), (1.2) in the domain D_∞. □

References

[1] G. K. Berikelashvili, O. M. Dzhokhadze, B. G. Midodashvili and S. S. Kharibegashvili, On the existence and nonexistence of global solutions of the first Darboux problem for nonlinear wave equations. (Russian) *Differ. Uravn.* **44** (2008), no. 3, 359–372; translation in *Differ. Equ.* 44 (2008), no. 3, 374–389.

[2] G. Berikelashvili, O. Jokhadze, S. Kharibegashvili and B. Midodashvili, Finite difference solution of a nonlinear Klein-Gordon equation with an external source. *Math. Comp.* **80** (2011), no. 274, 847–862.

[3] A. V. Bitsadze, Influence of the lower terms on the correctness of the formulation of characteristic problems for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* **225** (1975), no. 1, 31–34.

[4] A. V. Bitsadze, Some classes of partial differential equations. (Russian) *"Nauka", Moscow*, 1981.

[5] A. V. Bitsadze, On the question of formulating the characteristic problem for second order hyperbolic systems. (Russian) *Dokl. Akad. Nauk SSSR* **223** (1975), no. 6, 1289–1292; translation in *Soviet Math. Dokl.* **16** (1975), 1062–1066.

[6] G. Darboux, Leçons sur la théorie générale des surfaces, troisiéme partie. (French) *Gauthier-Villars et Fils, Paris*, 1894.

[7] E. Goursat, Cours d'analyse mathématique, I–III. 5th ed. *Gauthieir-Villars, Paris*, 1956.

[8] S. S. Kharibegashvili, The characteristic problem for second-order hyperbolic systems with constant coefficients. (Russian) *Differentsial nye Uravneniya* **14** (1978), no. 1, 123–135.

[9] S. S. Kharibegashvili, A boundary value problem for second-order normally hyperbolic systems with variable coefficients. (Russian) *Differentsial nye Uravneniya* **21** (1985), no. 1, 149–155.

[10] S. Kharibegashvili, Goursat and Darboux type problems for linear hyperbolic partial differential equations and systems. *Mem. Differential Equations Math. Phys.* **4** (1995), 1–127.

[11] S. Kharibegashvili, Boundary value problems for some classes of nonlinear wave equations. *Mem. Differential Equations Math. Phys.* **46** (2009), 1–114.

[12] S. S. Kharibegashvili and O. M. Dzhokhadze, The Cauchy–Darboux problem for a one-dimensional wave equation with power nonlinearity. (Russian) *Sibirsk. Mat. Zh.* **54** (2013), no. 6, 1407–1426; translation in *Sib. Math. J.* **54** (2013), no. 6, 1120–1136.

[13] S. S. Kharibegashvili and O. M. Jokhadze, On the solvability of a boundary value problem for nonlinear wave equations in angular domains. (russian) *Differ. Uravn.* **52** (2016), no. 5, 665–686; translation in *Differ. Equ.* **52** (2016), no. 5, 644–666.

[14] I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I Linear theory. (Russian) *"Metsniereba", Tbilisi*, 1997.

[15] T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differ. Equ. Math. Phys.* **1** (1994), 1–144.

[16] R. Kurant, Partial differential equations. (Russian) *Izdat. "Mir", Moscow*, 1964.

[17] R. Narasimhan, Analysis on real and complex manifolds. *Advanced Studies in Pure Mathematics*, Vol. 1. Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam, 1968; Russian translation: *Mir, Mocow*, 1971.

[18] V. A. Trenogin, Functional analysis. (Russian) *"Nauka", Moscow*, 1993.

(Received 30.01.2017)

Authors' addresses:

Giorgi Dekanoidze

Georgian Technical University, 77 M. Kostava St., Tbilisi 0175, Georgia.

E-mail: `ggdekanoidze@gmail.com`

Sergo Kharibagashvili

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili St., Tbilisi 0177, Georgia.

2. Georgian Technical University, 77 M. Kostava St., Tbilisi 0175, Georgia.

E-mail: kharibegashvili@yahoo.com