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ON A RESOLVENT APPROACH IN A MIXED PROBLEM FOR THE WAVE EQUATION ON A GRAPH

Abstract. We study a mixed problem for the wave equation with integrable potential on the simplest geometric graph consisting of two ring edges that touch at a point. We use a new resolvent approach in the Fourier method. We do not use refined asymptotic formulas for the eigenvalues and any information on the eigenfunctions.*

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Key words and phrases. Wave equation, geometric graph, Fourier method, resolvent approach.

რეზიუმე. ტალღის განტოლებისათვის ინტეგრებადი პოტენციალით შესწავლილია შერეული ამოცანა უმარტივეს გრაფზე, რომელიც შედგება ერთ წერტილში მხები რგოლის ორი კიდესგან. გამოყენებულია ახალი რეზოლვენტური მიდგომა ფურიეს მეთოდში. ამასთან არ არის გამოყენებული დაზუსტებული ასიმპტოტური ფორმულები საკუთრივი მნიშვნელობებისთვის და რაიმე ინფორმაცია საკუთრივი ფუნქციების მიმართ.

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We consider the simplest geometric graph consisting of two ring edges that touch at a point (at the node of the graph). Parametrizing each edge by the interval $[0,1]$, we study the following mixed problem for the wave equation on this graph:

$$\frac{\partial^2 u_j(x,t)}{\partial t^2} = \frac{\partial^2 u_j(x,t)}{\partial x^2} - q_j(x)u_j(x,t), \quad x \in [0,1], \quad t \in (-\infty,+\infty) \ \ (j=1,2), \tag{1}$$

$$u_1(0,t) = u_1(1,t) = u_2(0,t) = u_2(1,t), \tag{2}$$

$$u'_{1x}(0,t) - u'_{1x}(1,t) + u'_{2x}(0,t) - u'_{2x}(1,t) = 0, \tag{3}$$

$$u_1(x,0) = \varphi_1(x), \quad u_2(x,0) = \varphi_2(x), \quad u'_{1t}(x,0) = u'_{2t}(x,0) = 0. \tag{4}$$

Conditions (2), (3) are generated by the structure of the graph.

In this problem the application of the Fourier method causes difficulties associated with the fact that the eigenvalues of the corresponding spectral problem might be multiple. These difficulties can be coped with by applying the resolvent approach [1]. Note that we do not use refined asymptotic formulas for the eigenvalues and any information on the eigenfunctions. Besides, we use Krylov's idea [2, Chapter VI] concerning the convergence acceleration of Fourier-like series.

The following result was obtained in [3]:

Theorem 1. *If $q_j(x) \in C[0,1]$ are complex-valued, $\varphi_j(x) \in C^2[0,1]$ and are complex-valued, $\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1)$, $\varphi'_1(0) - \varphi'_1(1) + \varphi'_2(0) - \varphi'_2(1) = 0$, $\varphi''_1(0) = \varphi''_1(1) = \varphi''_2(0) = \varphi''_2(1)$, then the formal solution by Fourier method is a classical solution of problem* (1)–(4).

Now, we assume that $q_j(x) \in L[0,1]$ are complex-valued. Then a classical solution is defined as a function $u(x,t)$ such that $u(x,t)$ and its first derivatives with respect to x and t are absolutely continuous, and satisfies the boundary and initial conditions (2)–(4) and the differential equation (1) almost everywhere. Here we use the scheme of analysis given in [4–6].

We assume that the vector functions $\varphi(x)$ and $\varphi'(x)$ are absolutely continuous and such that satisfy the following conditions:

$$\varphi_1(0) = \varphi_1(1) = \varphi_2(0) = \varphi_2(1), \quad \varphi'_1(0) - \varphi'_1(1) + \varphi'_2(0) - \varphi'_2(1) = 0, \ \ L\varphi \in L_2^2[0,1]. \tag{5}$$

Everywhere, by $L_2^2[0,1]$ we denote the space of vector functions $f(x) = (f_1(x), f_2(x))^T$ such that $f_k(x) \in L_2[0,1]$ $(k=1,2)$, T denotes the transpose.

1 The transformation of a formal solution

The Fourier method is related to the spectral problem $Ly = \lambda y$ for the operator

$$Ly = \left(-y''_1(x) - q_1(x)y_1(x), -y''_2(x) - q_2(x)y_2(x)\right)^T, \quad y = y(x) = (y_1(x), y_2(x))^T$$

with the boundary conditions

$$y_1(0) = y_1(1) = y_2(0) = y_2(1), \quad y'_1(0) - y'_1(1) + y'_2(0) - y'_2(1) = 0.$$

By $R_\lambda = (L - \lambda E)^{-1}$, $R_\lambda^0 = (L^0 - \lambda E)^{-1}$ are denoted the resolvents of the operators L and L^0, where L^0 is L with $q_j(x) \equiv 0$ (E is the identity operator, and λ is the spectral parameter). In the sequel the notation corresponding to L^0 is marked with a zero index.

The formal solution $u(x,t) = (u_1(x,t), u_2(x,t))^T$ of problem (1)–(4) produced by the Fourier method can be represented as

$$u(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n \ge n_0}\int\limits_{\gamma_n}\right)(R_\lambda\varphi)(x)\cos\rho t\, d\lambda,$$

where $r > 0$ is fixed and such that all the eigenvalues λ_n, with $n < n_0$, belong to the disk $|\lambda| < r$, and there are no eigenvalues of L on the contour $|\lambda| = r$; γ_n are the contours of sufficiently small radius in λ-plane such that all the eigenvalues of operators L and L^0 with $n \ge n_0$ are only inside γ_n.

Proceeding as in [1], we obtain the following result.

Theorem 2. *The formal solution can be represented as*

$$u(x,t) = u_0(x,t) + u_1(x,t),$$

where

$$u_0(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\geq n_0}\int\limits_{\gamma_n}\right)\frac{R_\lambda^0 g}{\lambda-\mu_0}\cos\rho t\, d\lambda,$$

$$u_1(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\geq n_0}\int\limits_{\gamma_n}\right)\frac{1}{\lambda-\mu_0}\left[R_\lambda g - R_\lambda^0 g\right]\cos\rho t\, d\lambda,$$

$g = (L-\mu_0 E)\varphi$*,* μ_0 *is not an eigenvalue of* L *or* L^0*,* $|\mu_0| > r$*, and* μ_0 *lies outside* γ_n *for* $n \geq n_0$*.*

2 Spectral problem and resolvent

Let $\lambda = \rho^2$, where $\operatorname{Re}\rho \geq 0$. Denote by $\{y_{j1}(x), y_{j2}(x)\}$ $(j = 1,2)$, the fundamental systems of solutions of the equations

$$y_j''(x) - q_j(x)y_j(x) + \rho^2 y_j(x) = 0, \quad (j = 1,2)$$

with initial conditions

$$y_{j1}(0) = 1, \quad y_{j1}'(0) = 0,$$
$$y_{j2}(0) = 0, \quad y_{j2}'(0) = 1.$$

Then $y_{ij}(x)$ are entire functions of ρ and λ. If $q_j(x) \equiv 0$, then

$$y_{j1}^0(x) = \cos\rho x, \quad (y_{j1}^0(x))' = -\rho\sin\rho x,$$
$$y_{j2}^0(x) = \frac{\sin\rho x}{\rho}, \quad (y_{j2}^0(x))' = \cos\rho x.$$

From [7] it follows that all ρ for which $\lambda = \rho^2$ are the eigenvalues of the operator L belong to the semi-infinite strip $S = \{\rho | \operatorname{Re}\rho \geq 0, |\operatorname{Im}\rho| \leq h\}$, where $h > 0$ is sufficiently large.

Just as in [6, Lemma 7] we obtain

Lemma 1. *If* $|\operatorname{Im}\rho| \leq h$*, then*

$$y_{j1}(x,\rho) = \cos\rho x + \frac{1}{2\rho}\sin\rho x\int\limits_0^x q_j(\tau)\,d\tau + \frac{1}{4\rho}\int\limits_0^x\left[q_j\left(\frac{x-\tau}{2}\right) + q_j\left(\frac{x+\tau}{2}\right)\right]\sin\rho\tau\,d\tau + O(\rho^{-2}),$$

$$y_{j2}(x,\rho) = \frac{\sin\rho x}{\rho} + \frac{1}{2\rho^2}\cos\rho x\int\limits_0^x q_j(\tau)\,d\tau + \frac{1}{4\rho^2}\int\limits_0^x\left[q_j\left(\frac{x-\tau}{2}\right) + q_j\left(\frac{x+\tau}{2}\right)\right]\cos\rho\tau\,d\tau + O(\rho^{-3}),$$

where the $O(\dots)$ *estimates are uniform with respect to* $x \in [0,1]$*.*

The eigenvalues of operator L are the zeros of the determinant

$$\Delta(\rho) = \begin{vmatrix} 1-y_{11}(1) & -y_{12}(1) & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1-y_{21}(1) & -y_{22}(1) \\ -y_{11}'(1) & 1-y_{12}'(1) & -y_{21}'(1) & 1-y_{22}'(1) \end{vmatrix}$$

The eigenvalues of L^0 (the zeros of $\Delta^0(\rho)$) are $\lambda_n^0 = (\rho_n^0)^2$, where $\rho_n^0 = n\pi$ $(n = 0, 1, 2, \dots)$. If n is even, then eigenvalues are multiple. The eigenvalues λ_n of the operator L asymptotically approach λ_n^0 for large n.

Theorem 3. *For the resolvent $R_\lambda = (R_{1\lambda}, R_{2\lambda})^T$, the formula*

$$R_{j\lambda} f(x) = (M_{j\rho} f_j)(x) + \Omega_{j\lambda}(x, f), \quad f = (f_1, f_2)^T \ \ (j = 1, 2) \tag{6}$$

holds, where

$$(M_{j\rho} f_j)(x) = \int_0^x M_j(x, \xi, \rho) f_j(\xi)\, d\xi, \quad M_j(x, \xi, \rho) = \begin{vmatrix} y_{j1}(\xi) & y_{j2}(\xi) \\ y_{j1}(x) & y_{j2}(x) \end{vmatrix},$$

$$\Omega_{j\lambda}(x, f) = v_{j1}(x)(f_1, y_{11}) + v_{j2}(x)(f_1, y_{12}) + v_{j3}(x)(f_2, y_{21}) + v_{j4}(x)(f_2, y_{22}) \ \ (j = 1, 2), \tag{7}$$

$$v_{11}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[\Delta_{1k}(\rho) y_{12}(1) + \Delta_{4k}(\rho) y'_{12}(1)\big],$$

$$v_{12}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[-\Delta_{1k}(\rho) y_{11}(1) - \Delta_{4k}(\rho) y'_{11}(1)\big],$$

$$v_{13}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[\Delta_{3k}(\rho) y_{22}(1) + \Delta_{4k}(\rho) y'_{22}(1)\big],$$

$$v_{14}(x) = \sum_{k=1}^{2} \frac{y_{1k}(x)}{\Delta(\rho)} \big[-\Delta_{3k}(\rho) y_{21}(1) - \Delta_{4k}(\rho) y'_{21}(1)\big],$$

$\Delta_{k,s}(\rho)$ are algebraic adjuncts of $\Delta(\rho)$, and $v_{2j}(x)$ are obtained by replacing Δ_{k1}, Δ_{k2} by Δ_{k3}, Δ_{k4}, and $y_{11}(x)$, $y_{12}(x)$ by $y_{21}(x)$, $y_{22}(x)$; $(f, g) = \int_0^1 f(x) g(x)\, dx$.

Proof. For $y = (y_1, y_2)^T = R_\lambda f$, we have

$$y''_j(x) - q_j(x) y_j(x) + \rho^2 y_j(x) = f_j(x), \ \ j = 1, 2,$$

whence

$$y_k(x) = c_{k1} y_{k1}(x) + c_{k2} y_{k2}(x) + (M_{k\rho} f_k)(x), \ \ k = 1, 2.$$

From the boundary conditions for operator L follows (6), where

$$\Omega_{1\lambda}(x, f) = \frac{y_{11}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,1}(\rho) + \frac{y_{12}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,2}(\rho),$$

$$\Omega_{2\lambda}(x, f) = \frac{y_{21}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,3}(\rho) + \frac{y_{22}(x)}{\Delta(\rho)} \sum_{k=1}^{4} d_j \Delta_{k,4}(\rho),$$

$$d_1 = (M_{1\rho} f_1)\Big|_{x=1}, \quad d_2 = 0, \quad d_3 = (M_{2\rho} f_2)\Big|_{x=1},$$

$$d_4 = \int_0^1 \frac{d}{dx} M_1(x, \xi, \rho)\Big|_{x=1} f_1(\xi)\, d\xi + \int_0^1 \frac{d}{dx} M_2(x, \xi, \rho)\Big|_{x=1} f_2(\xi)\, d\xi.$$

Calculating the coefficients d_k in an explicit form, we get (7). □

Define $\widetilde{\gamma}_n = \{\rho \,|\, |\rho - \pi n| = \delta\}$, where $\delta > 0$ is sufficiently small, $n \geq n_0$, and n_0 is chosen so that all λ_n with $n \geq n_0$ lie inside $\widetilde{\gamma}_n$. Let γ_n be the image of $\widetilde{\gamma}_n$ in the λ-plane ($\lambda = \rho^2$).

Lemma 2. *If $\rho \in \widetilde{\gamma}_n$, then*

$$v_{k1}^{(j)}(x,\rho) = {v_{k1}^0}^{(j)}(x,\rho) + O(\rho^{j-2}) \quad (j=0,1),$$
$$v_{k2}^{(j)}(x,\rho) = {v_{k2}^0}^{(j)}(x,\rho) + O(\rho^{j-1}) \quad (j=0,1),$$
$$v_{k1}''(x,\rho) - q_1(x)v_{k1}(x,\rho) - {v_{k1}^0}''(x,\rho) = O(1),$$
$$v_{k2}''(x,\rho) - q_2(x)v_{k2}(x,\rho) - {v_{k2}^0}''(x,\rho) = O(\rho)$$

$(k=1,2)$, where the derivatives are taken with respect to x and the $O(\dots)$ estimates are uniform with respect to $x \in [0,1]$ (in the last two relations $O(\dots)$ stands for $\|O(\omega)\|_\infty \le c|\omega|$).

Proof. Since $v_j''(x,\rho) - q(x)v_j(x,\rho) = -\rho^2 v_j(x,\rho)$, this lemma follows from Lemma 2 in [4]. □

Just as in [6], we can prove the following assertions.

Lemma 3. *By $p(x)$ denote the functions $\int_x^1 m(\xi)q((\xi-x)/2)\,d\xi$ or $\int_x^1 m(\xi)q((\xi+x)/2)\,d\xi$, where $m(\xi)$ is $g_1(\xi)$ or $g_2(\xi)$ $(g=(g_1,g_2)^T = (L-\mu_0 E)\varphi)$, and $q(x)$ is $q_1(x)$ or $q_2(x)$. Then*

$$\|p\|_{L_2} \le 2\|m\|_{L_2} \cdot \|q\|_{L_1},$$

where $\|\cdot\|_{L_s}$ is the norm on $L_s[0,1]$.

Lemma 4. *Let $\psi(x)$ denote the function $\cos x$ or $\sin x$. Let $m(x) \in L_2[0,1]$ and $m(x,\mu) = m(x)\psi(\mu x)$, for $\mu \in \gamma_0$, and $\beta_n(\mu) = (m(x,\mu), \psi(\pi n x))$. Further, by $\widetilde{\beta}_n(\mu)$ we denote the sum of all $|\beta_n(\mu)|$, where $m(x)$ is one of the functions $g_j(x)$, $g_j(x)\int_0^x q_s(\xi)\,d\xi$, $p(x)$ ($p(x)$ is one of the functions from Lemma 3). Then*

$$\sum_{n=n_1}^{n_2} \frac{1}{n}\,\widetilde{\beta}_n(\mu) \le c\sqrt{\sum_{n=n_1}^{n_2} \frac{1}{n^2}}\,\|g\|_2,$$

where $c>0$ is a constant independent of n_1, n_2, and $\mu \in \gamma_0$, and by $\|g\|_2$ is denoted the norm of vector function $g(x) = (g_1(x), g_2(x))^T$ on $L_2^2[0,1]$.

Lemma 5. *If $g(x) = (g_1(x), g_2(x))^T \in L_2^2[0,1]$, $\rho \in \widetilde{\gamma}_n$, and $\rho = \pi n + \mu$, then*

$$(g_s, y_{j1}) = O(\widetilde{\beta}_n(\mu)) + O(\rho^{-1}\widetilde{\beta}_n(\mu)) + O(\rho^{-2}\|g\|_2),$$
$$(g_s, y_{j1} - y_{j1}^0) = O(\rho^{-1}\widetilde{\beta}_n(\mu)) + O(\rho^{-2}\|g\|_2),$$
$$(g_s, y_{j2}) = O(\rho^{-1}\widetilde{\beta}_n(\mu)) + O(\rho^{-2}\widetilde{\beta}_n(\mu)) + O(\rho^{-3}\|g\|_2),$$
$$(g_s, y_{j2} - y_{j2}^0) = O(\rho^{-2}\widetilde{\beta}_n(\mu)) + O(\rho^{-3}\|g\|_2),$$

where $j=1,2$, $s=1,2$.

From Lemmas 2–5 follows

Lemma 6. *If $\rho = \pi n + \mu$, $\mu \in \widetilde{\gamma}_0$, $\Omega_\lambda(x,g) = (\Omega_{1\lambda}(x,g), \Omega_{2\lambda}(x,g))^T$, then*

$$\frac{d^j}{dx^j}\,(\Omega_\lambda(x,g)) = O(\rho^{j-1}\widetilde{\beta}_n(\mu)) + O(\rho^{j-2}\|g\|_2) \quad (j=0,1),$$
$$\frac{d^j}{dx^j}\left(\Omega_\lambda(x,g) - \Omega_\lambda^0(x,g)\right) = O(\rho^{j-2}\widetilde{\beta}_n(\mu)) + O(\rho^{j-3}\|g\|_2) \quad (j=0,1).$$

3 Investigation of the function $u_0(x,t)$

Since $(M_{j\rho}g_j)(x)$, $(M^0_{j\rho}g_j)(x)$ are entire functions, it follows that

$$u_0(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\geq n_0}\int\limits_{\gamma_n}\right)\frac{\Omega^0_\lambda(x,g)}{\lambda-\mu_0}\cos\rho t\,d\lambda.$$

From [3, Lemmas 3, 4] we have

Lemma 7. *It is true that*

$$u_0(x,t) = \frac{1}{2}\left(F(x+t)+F(x-t)\right),$$

where

$$F(x) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\geq n_0}\int\limits_{\gamma_n}\right)\frac{1}{\lambda-\mu_0}\Omega^0_\lambda(x,g)\,d\lambda.$$

Lemma 8. *For $F(x) = (F_1(x),F_2(x))^T$, the relations*

$$\begin{aligned}
F_1(-x) &= \frac{1}{2}\big[F_1(1-x)+F_2(1-x)-F_1(x)+F_2(x)\big],\\
F_2(-x) &= \frac{1}{2}\big[F_1(1-x)+F_2(1-x)+F_1(x)-F_2(x)\big],\\
F_1(1+x) &= \frac{1}{2}\big[F_1(x)-F_1(1-x)+F_2(x)+F_2(1-x)\big],\\
F_2(1+x) &= \frac{1}{2}\big[F_1(x)+F_1(1-x)+F_2(x)-F_2(1-x)\big]
\end{aligned}$$

hold, and $F(x) = \widetilde{\varphi}(x) = R^0_{\mu_0}g$ for $x\in[0,1]$.

Therefore, as in [6], we get

Lemma 9. *The vector functions $F(x)$, $F'(x)$ are absolutely continuous, $F''(x)\in L^2_2[-A,A]$ for all $A>0$, and $F(x)=F(x+2)$.*

Theorem 4. *The function $u_0(x,t)$ is a classical solution of the reference problem obtained from (1)–(4) by setting $q_j(x)\equiv 0$ with initial conditions (4), where $\varphi(x)$ is replaced by $\widetilde{\varphi}(x) = R^0_{\mu_0}g$, and equation (1) is satisfied almost everywhere.*

4 Investigation of the function $u_1(x,t)$

For $u_1(x,t)$ we have

$$u_1(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\geq n_0}\int\limits_{\gamma_n}\right)\frac{1}{\lambda-\mu_0}[\Omega_\lambda(x,g)-\Omega^0_\lambda(x,g)]\cos\rho t d\lambda.$$

By the methods in [6], we obtain the following assertions.

Lemma 10. *The series $u_1(x,t)$ and the series obtained by differentiating $u_1(x,t)$ term by term with respect to x once and with respect to t twice is convergent absolutely and uniformly in $Q_T = [0,1]\times[-T,T]$, where $T>0$ is any fixed number.*

Lemma 11. *The function $u'_{1,x}(x,t)$ is absolutely continuous with respect to x, and the relation*

$$u''_{1,x^2}(x,t) = Q(x)u_1(x,t) + d(x,t)$$

holds for almost all x and t in the rectangle Q_T. Here $Q(x) = \operatorname{diag}(q_1(x),q_2(x))$,

$$d(x,t) = -\frac{1}{2\pi i}\left(\int\limits_{|\lambda|=r} + \sum_{n\geq n_0}\int\limits_{\gamma_n}\right)\frac{\lambda}{\lambda-\mu_0}[\Omega_\lambda(x,g)-\Omega^0_\lambda(x,g)]\cos\rho t d\lambda,$$

and the series $d(x,t)$ is convergent absolutely and uniformly in Q_T.

Using Theorem 4 and Lemmas 10 and 11, we obtain

Theorem 5. *If $q_j(x) \in L[0,1]$, the vector functions $\varphi(x)$ and $\varphi'(x)$ are absolutely continuous and such that they satisfy the conditions* (5)*, then the sum $u(x,t)$ of the formal solution has the following properties: the function $u(x,t)$ is continuously differentiable with respect to x and t; the function $u'_x(x,t)$* (*respectively, $u'_t(x,t)$*) *is absolutely continuous with respect to x* (*respectively, with respect to t*)*; and the function $u(x,t)$ satisfies equation* (1) *almost everywhere and conditions* (2)–(4)*; i.e., $u(x,t)$ is a classical solution of problem* (1)–(4) *with* (1) *satisfied almost everywhere.*

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