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RELIABLE SOLUTIONS OF PROBLEMS WITH UNCERTAIN HYSTERESIS OPERATORS

Abstract. Problems in technology lead to initial boundary value problems for partial differential equations. Material properties which appear in constitutive relations are obtained by measurements. These data are uncertain and thus are known to some extent only. Using their mean values in numerical modelling cause several serious failures in technology.

The problem of finding a reliable solution by uncertain data is solved by the so-called worst scenario method introduced by Ivo Babuška and Ivan Hlaváček. The method consists in looking for the worst scenario that may appear in the case of any admissible data, the badness of situation is estimated by means of a criterion-functional evaluating critical parts of the body.

In the contribution, the worst scenario method is applied to boundary value problems for nonlinear equation with a scalar hysteresis operator $\mathcal{F}$ or its inverse $\mathcal{G}$ of Prandtl–Ishlinskii type. The method demands special construction of admissible data and estimates the hysteresis operators. The existence of a reliable solution for the initial boundary value problem for the heat conduction or the diffusion equation $c\,u_t = (\mathcal{F}_\eta[u_x])_x + f$ with various types of criterion-functionals is proved.[*]

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Key words and phrases. Prandtl–Ishlinskii hysteresis operator, reliable solution, uncertain data, worst scenario method, heat conduction equation, diffusion equation.

რეზიუმე. ტექნოლოგიის ამოცანებს მივყავართ კერძოწარმოებულიანი დიფერენციალური განტოლებებისთვის დასმულ საწყის-სასაზღვრო ამოცანებამდე. მატერიალური მახასიათებლები, რომლებიც მონაწილეობს კონსტიტუციურ თანაფარდობებში, მიღებულია გაზომვის საშუალებით. ეს მონაცემები მხოლოდ გარკვეული სიზუსტით არის ცნობილი. მათი საშუალო მნიშვნელობების გამოყენებამ რიცხვით მოდელირებაში გამოიწვია რამდენიმე ტექნოლოგიური მარცხი.

განუსაზღვრელ მონაცემის საშუალებით საიმედო ამონახსნის მოძებნის ამოცანა გადაწყვეტილია ივო ბაბუშკას და ივან ჰლავაჩეკის მიერ შემოტანილი ე.წ. უარესი სცენარის მეთოდით. მეთოდი მდგომარეობს იმ უარესი სცენარის მოძებნაში, რომელიც შეიძლება გამოჩნდეს ნებისმიერი დასაშვები მონაცემების შემთხვევაში. სიტუაციის უარესობა შეფასებულია სხეულის კრიტიკული ნაწილების კრიტერიუმ-ფუნქციონალური შეფასების საშუალებით.

უარესი სცენარის მეთოდი გამოყენებულია სასაზღვრო ამოცანებში არაწრფივი განტოლებისთვის სკალარული ჰისტერეზისის $\mathcal{F}$ ოპერატორით ან მისი პრანდტლ-იშლინსკის ტიპის $\mathcal{G}$ შებრუნებულით. მეთოდი მდგომარეობს შესაძლო მონაცემების სპეციალურ კონსტრუქციაში და ჰისტერეზისის ოპერატორების შეფასებაში. დამტკიცებულია სითბოგამტარებლობის ან დიფუზიის $c\,u_t = (\mathcal{F}_\eta[u_x])_x + f$ განტოლებისთვის საწყისი ამოცანის საიმედო ამონახსნის არსებობა სხვადასხვა ტიპის კრიტერიუმ-ფუნქციონალისთვის.

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1 Introduction

Many problems in technology can be modelled by the initial boundary value problems for partial differential equations with a hysteresis operator. Among them let us consider a scalar one-dimensional equation

$$c\,u_t = q_x + f\,, \quad q = k\,u_x,\;\; x \in (0,\ell),\;\; t \in (0,T),$$

which can be physically interpreted as the heat conduction in a one-dimensional body, particularly in a bar $(0,\ell)$. The unknown $u(x,t)$ is the temperature, $q(x,t)$ is a negative heat flow, c is specific heat capacity and k is thermal conductivity. We take a negative heat flow q in order to obtain the linear Fourier law $q = k\,u_x$ with positive $k > 0$ instead of the usual Fourier law $q = -k\,u_x$ with (positive) heat flow q. We replace this Fourier law by the relation $q = \mathcal{F}[u_x]$ with a hysteresis operator $\mathcal{F}$ which describes behavior of a rate-independent material with memory or phase transition. In this way we obtain the equation

$$c\,u_t = q_x + f\,, \quad q = \mathcal{F}[u_x],\;\; x \in (0,\ell),\;\; t \in (0,T).$$

The equation contains material parameters, which are not known exactly, since they are obtained by measurements. They are uncertain, i.e., they are known to some extent only. In the past, using mean values of the data in the process of mathematical modelling caused several serious failures in technology. This problem with uncertain data has been solved by I. Babuška and I. Hlaváček in a series of papers, see [6, 7]. They proposed the so-called *worst scenario method*.

The method takes into account all data, i.e., all material parameters from their range of uncertainty. Using a criterion-functional which measures the badness of the situation, we seek for the worst scenario that may appear. The method is used in engineering for its simplicity: the model is deterministic (no need to deal with stochastic models), and optimization tools can be used for computing the maximum: theory, numeric analysis and the corresponding software.

The problem of longitudinal vibration of a nonhomogeneous elasto-plastic rod including homogenization problem was solved in [2]. The one-dimensional diffusion equation with a scalar hysteresis operator was solved in [3] and a higher space dimensional heat equation with a scalar hysteresis operator including homogenization problem was studied in [4]. Reliable solutions of the problem of periodic oscillations of an elasto-plastic beam was studied in [9]. Reliable solutions of a homogenization problem with monotone operators was studied in [5].

In the contribution, we study the initial boundary value problem for a nonlinear heat conduction equation (or diffusion equation) with a hysteresis operator of Prandtl–Ishlinskii type. These hysteresis operators are described and studied in e.g. [1, 8, 10]. The aim of the contribution is to propose sets for admissible data, criterion-functionals and to prove the existence of the worst scenario solution.

The paper is organized as follows. Section 2 contains the survey of hysteresis operators and their properties, in Section 3, the existence of a solution of the initial boundary value problem is proved, and the worst scenario method applied to the problem is considered in Section 4 including the setting of a set of admissible data and proposals of various criterion-functionals.

2 Hysteresis operators

In this section we deal with the one-dimensional hysteresis operators. These operators acting in a space of real functions on an interval $I = \langle 0,T\rangle$ representing time can be simply characterized by the following properties. The hysteresis operators $\mathcal{T}$ are:

- *rate independent* – the output $\mathcal{T}[v]$ is independent of speed of the input v: $\mathcal{T}[v\circ\varphi](t) = \mathcal{T}[v](\varphi(t))$ for any increasing mapping φ from I onto I,
- *causal* – the output is independent of future input, i.e., if $u(s) = v(s)$ for all $s \le t$, then $\mathcal{T}[u](t) = \mathcal{T}[v](t)$,
- *locally monotone* – a locally non-decreasing input yields a locally non-decreasing output and also a non-increasing input provides a non-increasing output, i.e.,

$$\mathcal{T}[v]'(t) \cdot v'(t) \ge 0 \;\text{ for a.e. }\; t \in I.$$

For more detailed study of hysteresis operators we can recommend [1, 8, 10].

2.1 Stop and play operators

Here we deal with hysteresis operators of Prandtl–Ishlinskii type. These operators are defined by means of operators called as a stop and a play operator with one parameter $r > 0$. Their definition is based on the solution of the following variational inequality. Let $v \in W^{1,1}(I)$ be an input function and $s_r^0 \in \langle -r, r\rangle$ be an initial state. We look for a function $s \in W^{1,1}(I)$ satisfying:

$$\begin{aligned} &|s(t)| \le r \ \ \forall\, t \in I, \quad s(0) = s_r^0, \\ &(s'(t) - v'(t))(\widetilde{s} - s(t)) \ge 0 \ \ \forall\, |\widetilde{s}| \le r, \ \text{ a.e. } \ t \in I. \end{aligned} \tag{2.1}$$

It should be noted that the above inequality yields $s'(t) = v'(t)$ provided $s(t)$ is inside the interval $(-r, r)$. If $s(t) = r$ and v is increasing, then $s'(t) = 0$ and, also, if $s(t) = -r$ and v is decreasing, then likewise $s'(t) = 0$.

The inequality admits a unique solution $s \in W^{1,1}(I)$ which defines the elementary hysteresis operators:

Definition 2.1. The solution $s(t)$ of the variational inequality (2.1) defines two complementary operators: the stop operator $\mathcal{S}_r$ and the play operator $\mathcal{P}_r$:

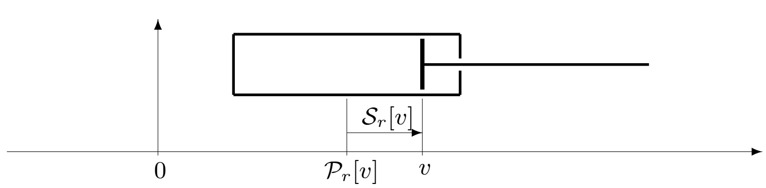
$$\mathcal{S}_r[v](t) := s(t), \quad \mathcal{P}_r[v](t) := v(t) - s(t), \ \ t \in \langle 0, T\rangle. \tag{2.2}$$

To simplify the notation, we have taken for the input $v(t)$ the so-called virgin initial state $s_r^0 = \min\{r, \max\{-r, v(0)\}\}$ and omit s_r^0 in the notation of the operators. We also put the input v into square brackets to indicate that the dependence is not local: the value at time t depends on values on the whole interval $\langle 0, t\rangle$. Let us note that the stop operator can be equivalently introduced on each interval of monotonicity $\langle t_a, t_b\rangle$ of the input $v(t)$ by the relation

$$\mathcal{S}_r[v](t) = \min\Big\{r, \max\big\{-r, \mathcal{S}_r[v](t_a) + v(t) - v(t_a)\big\}\Big\} \ \ \forall\, t \in (t_a, t_b\rangle.$$

Both stop and play operators are rate independent, causal and locally monotone, and in addition, they satisfy $\mathcal{S}_r[v]'(t) \cdot \mathcal{P}_r[v]' = 0$ for a.e. $t \in I$.

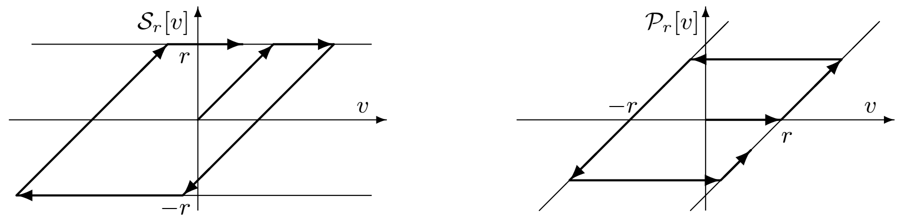
Values of the stop and play operators can be visualized by the so-called "piston in cylinder model". Let us consider a piston freely moving in a cylinder of length $2r$. Position of the piston is the input $v(t)$, position of the cylinder center is the value of the play operator $\mathcal{P}_r[v](t)$, while the position of the piston with respect to the cylinder center is the value of the stop operator $\mathcal{S}_r[v](t)$.



Piston in the cylinder model for the stop and play operators.

In mechanics, the stop operator $\mathcal{S}_r$ can be interpreted as the output stress $\mathcal{S}_r[v](t) = s(t)$ of an elasto-plastic material caused by the input strain (deformation) $v(t)$. Its rheological element consists of an elastic and a friction element combined in series. On the other hand, the play operator $\mathcal{P}_r$ can be interpreted as the output strain $\mathcal{P}_r[v](t) = v(t) - s(t)$ of an elasto-plastic material caused by the input stress $v(t)$. Its rheological element consists of an elastic and a friction elements combined in parallel. In both cases the elasticity modulus is 1 and plasticity limit is r.

Plane diagram $[v, \mathcal{S}_r[v]]$ of the stop operator has straight line segments with slope 0 or 1 with concave increasing branches and convex decreasing branches, while the plane diagram $[v, \mathcal{P}_r[v]]$ of the play operator has also straight line segments with slope 0 or 1, whereas the increasing branches are convex and decreasing branches are concave:



Diagrams of the stop operator $v \mapsto \mathcal{S}_r[v]$ and the play operator $v \mapsto \mathcal{P}_r[v]$.

Let us consider the properties of the stop and play operators (for proofs see, e.g., [2]).

Proposition 2.2. *Let $v_1, v_2 \in W^{1,1}(I)$ and put $s_i(t) = \mathcal{S}_r[v_i](t)$, $p_i(t) = \mathcal{P}_r[v_i](t)$, $i = 1, 2$. Then we have*

$$(p_1'(t) - p_2'(t))(s_1(t) - s_2(t)) \geq 0, \quad \textit{for a.e. } t \in I, \tag{2.3}$$

$$|p_1(t) - p_2(t)| \leq \max\left\{|p_1(0) - p_2(0)|, \|v_1 - v_2\|_{\langle 0,t\rangle}\right\} \;\; \textit{for } t \in I, \tag{2.4}$$

$$|s_1(t) - s_2(t)| \leq \|v_1 - v_2\|_{\langle 0,t\rangle} \;\; \textit{for } t \in I. \tag{2.5}$$

2.2 Prandtl–Ishlinskii operators

Diagrams of the stop and play operators consist of straight line segments with two slopes. But diagrams of the real elasto-plastic materials have curved changing slope branches. To obtain such diagrams we combine the operators with different parameters r of various weights.

The Prandtl–Ishlinskii operator $\mathcal{F}$ of stop type is defined as a parallel combination of the stop operators with increasing parameters r_i and various weights c_i

$$\mathcal{F}[v] = c_1\mathcal{S}_{r_1}[v] + c_2\mathcal{S}_{r_2}[v] + \cdots + c_n\mathcal{S}_{r_n}[v] + c_\infty v.$$

The combination can be rewritten with Stieltjes integral

$$\mathcal{F}[v] = \eta(\infty)\, v - \int_0^\infty \mathcal{S}_r[v]\,\mathrm{d}\eta(r)$$

by a non-increasing distribution function $\eta(r)$, where $\eta(r) = c_\infty$ for $r \in \langle r_n, \infty)$, $\eta(r) = c_i + c_{i+1} + \cdots + c_n + c_\infty$ for $r \in \langle r_{i-1}, r_i)$, $i = 1, 2, \ldots, n-1$, where $r_0 = 0$.

The Stieltjes integral enables us to cover both the discrete combination of stop operators $\mathcal{S}_{r_i}$, when η is piecewise constant, and a continuous combination of stop operators $\mathcal{S}_r$ if η is a continuous function.

Definition 2.3. Let $\alpha, \beta \in \mathbb{R}$ be positive constants, $\alpha\beta < 1$ and let $\eta : \langle 0, \infty) \to \langle 0, \infty)$ be a non-increasing right continuous function satisfying $\alpha \leq \eta(r) \leq \frac{1}{\beta}$ for all r. Then the Prandtl–Ishlinskii operator of stop type is given by

$$\mathcal{F}_\eta[v](t) := \eta(\infty)\, v(t) - \int_0^\infty \mathcal{S}_r[v](t)\,\mathrm{d}\eta(r)\,. \tag{2.6}$$

In the case of elasto-plastic material, the dependence of the stress q on strain $e = u_x$ can be modelled by this operator as

$$q(t) = \mathcal{F}_\eta[e](t). \tag{2.7}$$

The corresponding diagram of dependence of $q(t)$ on $e(t)$ is an oriented continuous curve with concave increasing and convex decreasing parts.

Similarly, the Prandtl–Ishlinskii operator of play type is defined as a serial combination of the play operators with increasing r_i. Again, we use the Stieltjes integral by a non-decreasing function η which enables us to describe both discrete and continuous combinations:

Definition 2.4. Let $\alpha, \beta \in \mathbb{R}$ be positive constants, $\alpha\beta < 1$ and let $\zeta : \langle 0, \infty) \to \langle 0, \infty)$ be a non-decreasing right continuous function satisfying $\beta \le \eta(r) \le \frac{1}{\alpha}$ for all r. Then the Prandtl–Ishlinskii operator of play type is given by

$$\mathcal{G}_\zeta[v](t) := \zeta(0)\, v(t) + \int_0^\infty \mathcal{P}_r[v](t)\, \mathrm{d}\zeta(r). \tag{2.8}$$

In the case of elasto-plastic material, the dependence of the strain e on the stress q can be modelled by this operator as

$$e(t) = \mathcal{G}_\zeta[q](t). \tag{2.9}$$

The corresponding diagram of the dependence of $e(t)$ on $q(t)$ is an oriented continuous curve with convex increasing and concave decreasing parts.

For the increasing input $v(s) = s$, $s \in \langle 0, \infty)$ and the Prandtl–Ishlinskii operator of stop type we obtain the so-called virgin curve $\varphi(s) = \mathcal{F}[v](s)$ which is a continuous increasing concave unbounded function on $\mathbb{R}^+$. Similarly, for the input $v(t) = t$, $t \in \langle 0, \infty)$ and the Prandtl–Ishlinskii operator of play type we obtain the curve $\psi(t) = \mathcal{G}[v](t)$ which is a continuous increasing convex unbounded function on $\mathbb{R}^+$.

Let these functions φ, ψ be a pair of increasing mutually inverse functions, i.e.,

$$t = \varphi(s) \iff s = \psi(t), \quad s, t \in \langle 0, \infty). \tag{2.10}$$

Moreover, the function φ is concave if and only if ψ is convex.

Definition 2.5. Let $\alpha, \beta > 0$, $\alpha\beta < 1$ be positive constants. We say that the functions $[\eta, \zeta]$ defined on $\mathbb{R}^+$ form a pair of Prandtl–Ishlinskii distribution functions if they are right continuous, η non-increasing, ζ non-decreasing, they satisfy

$$\alpha \le \eta(r) \le \frac{1}{\beta} \ \text{ and } \ \beta \le \zeta(r) \le \frac{1}{\alpha} \tag{2.11}$$

and their primitive functions

$$\varphi(s) = \int_0^s \eta(r)\, \mathrm{d}r, \quad \psi(t) = \int_0^t \zeta(r)\, \mathrm{d}r$$

are mutually inverse, i.e., they satisfy (2.10). The set of all such pairs of distribution functions will be denoted by $PI(\alpha, \beta)$ and the set of ζ by $PI^+(\alpha, \beta)$.

These pairs of distribution functions define mutually inverse operators:

Proposition 2.6. *Let $(\eta, \zeta) \in PI(\alpha, \beta)$. Then the corresponding Prandtl–Ishlinskii operators are mutually inverse, i.e., for each inputs e, q*

$$q(t) = \mathcal{F}_\eta[e](t) \ \textit{ if and only if } \ e(t) = \mathcal{G}_\zeta[q](t). \tag{2.12}$$

2.3 Properties of the operators

Let $\alpha, \beta > 0$, $\alpha\beta < 1$ and $(\eta, \zeta) \in PI(\alpha, \beta)$. Let us consider the properties of the corresponding Prandtl–Ishlinskii operators. They are locally monotone and Lipschitz continuous (for proofs, see [2, Propositions 2.7–2.12].

Proposition 2.7. *Let $\alpha, \beta > 0$, $\alpha\beta < 1$, $(\eta, \zeta) \in LP(\alpha, \beta)$. Then the corresponding operators $\mathcal{F}$ and $\mathcal{G}$ map $W^{1,\infty}(I)$ into $W^{1,\infty}(I)$ and also $W^{1,1}(I)$ into $W^{1,1}(I)$.*

Let further $q \in W^{1,1}(I)$ and put $e = \mathcal{G}_\zeta[q]$ or, equivalently, $q = \mathcal{F}_\eta[e]$. Then for a.e. $t \in I$, the derivatives exist and the following estimates hold:

$$\alpha(e'(t))^2 \le e'(t)\, q'(t) \le \frac{1}{\beta}\, (e'(t))^2, \tag{2.13}$$

$$\beta(q'(t))^2 \le e'(t)\, q'(t) \le \frac{1}{\alpha}\, (q'(t))^2. \tag{2.14}$$

The following estimates ensure the Lipschitz continuity of the operators:

Proposition 2.8. *Let $(\eta, \zeta) \in LP(\alpha, \beta)$ and $q_1, q_2, e_1, e_2 \in W^{1,1}(I)$. Then for $t \in I$, we have*

$$\left|\mathcal{F}_\eta[e_1](t) - \mathcal{F}_\eta[e_2](t)\right| \leq \left(\frac{2}{\beta} - \alpha\right) \cdot \|e_1 - e_2\|_{\langle 0,t\rangle}, \tag{2.15}$$

$$\left|\mathcal{G}_\zeta[q_1](t) - \mathcal{G}_\zeta[q_2](t)\right| \leq \frac{1}{\alpha} \cdot \|q_1 - q_2\|_{\langle 0,t\rangle}. \tag{2.16}$$

The following estimate is a consequence of (2.3) (see also [2,3]):

Proposition 2.9. *Let $\zeta \in LP^+(\alpha, \beta)$ and $q_1, q_2 \in W^{1,1}(I)$. Then for a.e. $t \in I$, we have*

$$\big(\mathcal{G}_\zeta[q_1](t) - \mathcal{G}_\zeta[q_2](t)\big)_t (q_1 - q_2) \geq \frac{\beta}{2} \frac{\mathrm{d}}{\mathrm{d}t}[(q_1 - q_2)^2]. \tag{2.17}$$

Finally, the following estimate yields the dependence of the operator $\mathcal{G}_\zeta$ on the distribution functions ζ (for proof see, e.g., [2, Proposition 2.10]).

Proposition 2.10. *Let $\zeta_1, \zeta_2 \in PI^+(\alpha, \beta)$ be two distribution functions, $\mathcal{G}_{\zeta_1}$, $\mathcal{G}_{\zeta_2}$ be the corresponding operators and $q_1, q_2 \in W^{1,1}(I)$ be arbitrary input functions. Then*

$$\|\mathcal{G}_{\zeta_1}[q_1] - \mathcal{G}_{\zeta_2}[q_2]\|_{[0,t]} \leq \zeta_1(\infty)\|q_1 - q_2\|_{[0,t]} + \int_0^{\|q_2\|_{[0,t]}} |\zeta_1(r) - \zeta_2(r)| \,\mathrm{d}r. \tag{2.18}$$

2.4 Space dependent case

In case of nonhomogeneous materials the material properties depend even on the space variable x. Thus both function η and ζ are not only the functions of r, but in addition, they depend on the space variable x, i.e., $\eta = \eta(x, r)$ and $\zeta = \zeta(x, r)$.

3 Heat conduction and diffusion equation with hysteresis operator

We deal with the following equations:

$$c\, u_t = q_x + f, \quad q = \mathcal{F}_\eta[u_x] \text{ or, equivalently, } u_x = \mathcal{G}_\zeta[q] \tag{3.1}$$

on $x \in \Omega \equiv (0, \ell)$ and $t \in I \equiv (0, T)$ with a pair of mutually inverse hysteresis operators $\mathcal{G}_\zeta$ or $\mathcal{F}_\eta$. The equations are completed with the boundary conditions, e.g.,

$$u(0, t) = 0, \quad q(\ell, t) = 0 \text{ for } t \in I \tag{3.2}$$

and the standard initial condition

$$u(x, 0) = u^0(x) \text{ for } x \in \Omega. \tag{3.3}$$

The problem can be physically interpreted as the heat conduction or the diffusion problem in some materials with a changing phase in a bar $(0, \ell)$ and time $(0, T)$. In the case of heat conduction, the variable u stands for temperature and q for a negative heat flow, and in the case of diffusion problem, u denotes concentration and q negative mass flow.

The boundary condition $u = 0$ prescribes zero temperature or zero concentration on the left end of the bar, while $q = 0$ means thermal or mass insulated right end of the bar. The hysteresis operator describes the relation between the negative heat or mass flow q and the temperature or concentration gradient $e = u_x$.

3.1 Solvability of the problem

Hypotheses 3.1. *We adopt the following hypotheses for the data of the problem:*

- $c \in L^\infty(\Omega)$ *and* $c_m \le c(x) \le c_M$ *for a.e.* $x \in \Omega$ *for some* $0 < c_m < c_M$,
- $f \in W^{1,1}(I, L^2(\Omega))$,
- $\eta, \zeta \in L^\infty(\Omega \times I)$ *such that* $(\eta,\zeta)(x,\cdot) \in PI(\alpha,\beta)$ *for a.e.* $x \in \Omega$ *for some constants* $\alpha, \beta > 0$, $\alpha\beta < 1$,
- $u^0 \in W^{1,2}(\Omega)$ *and it satisfies the compatibility condition with the boundary conditions, i.e.,* $u^0(0) = 0$ *and* $\mathcal{F}[u^0_x](\ell) = 0$.

Theorem 3.2. *Let Hypotheses* 3.1 *hold. Then the problem has a unique solution, namely, there exist the functions* $u, q \in C(\Omega \times I)$ *and* $e = u_x \in L^2(\Omega, C(I))$ *such that*

$$u_t, e_t, q_t, q_x, \in L^\infty(I, L^2(\Omega)),$$

and equalities (3.1)–(3.3) *hold almost everywhere.*

The solution is unique. Moreover, all unknowns and their derivatives are bounded in the corresponding norms by the constants depending on α, β, c_m, c_M *and the norm of* f *in* $W^{1,1}(I, L^2(\Omega))$ *and* u^0 *in* $W^{1,2}(\Omega)$.

Let us briefly sketch the proof of the theorem (the details can be found in [3]). The proof will be done in several steps.

3.2 Semidiscretized problem

First we convert the partial differential equation into a system of ordinary differential equations in t. We divide the interval $\Omega = (0,\ell)$ into n parts $\Omega_k = (x_{k-1}, x_k)$, $k = 1, 2, \dots, n$, of length $h = \ell/n$, where $x_k = kh$. In the semidiscretized problem, the space derivative is replaced by the difference, the unknowns u_k, e_k, q_k are the function of time $t \in I$ approximating the value at $x_k = kh$, $k = 0, 1, \dots, n$. In this way, we obtain the following system of equations, $k = 1, 2, \dots, n-1$ and $t \in I$,

$$c_k\, u_k' = \frac{1}{h}\,(q_{k+1} - q_k) + f_k, \tag{3.4}$$

$$e_k = \frac{1}{h}\,(u_k - u_{k-1}), \tag{3.5}$$

$$e_k = \mathcal{G}_k[q_k] \ \text{ or equivalently } \ q_k = \mathcal{F}_k[e_k], \tag{3.6}$$

$$u_k(0) = u_k^0\,, \tag{3.7}$$

where c_k, u_k^0, $f_k(t)$, $\zeta_k(r)$ are the integral means of the corresponding functions over the space interval Ω_k, e.g., $f_k(t) = \frac{1}{h} \int_{\Omega_k} f(x,t)\,\mathrm{d}x$. The operator $\mathcal{G}_k$ is determined by the averaged distribution function $\zeta_k(r)$ and $\mathcal{F}_k$ is the operator, inverse to the operator $\mathcal{G}_k$.

We have obtained a system of ordinary differential equations (3.4) with the initial conditions (3.7) with additional equations (3.5), (3.6). Taking $q_k = \mathcal{F}_k[\frac{1}{h}(u_k - u_{k-1})]$ and the properties of the Prandtl–Ishlinskii operator of stop type, the right-hand side of the ODE (3.4) are Lipschitz continuous in u_k, and thus by the Piccard theorem, the system admits unique solutions $u_k \in W^{2,1}(I)$, $e_k \in W^{2,1}(I)$ and $q_k \in W^{1,\infty}(I)$.

3.3 Estimates

We use the following estimates.

Lemma 3.3. *The solutions* $\{u_k, e_k, q_k\}$ *to system* (3.4)–(3.7) *satisfies*

$$h\sum_{k=1}^{n-1}\Big[(u_k')^2+\Big(\frac{q_{k+1}-q_k}{h}\Big)^2\Big](t)\le C\ \ \forall\, t\in I, \tag{3.8}$$

$$\int_I h\sum_{k=1}^{n-1}\Big[(q_k')^2+(e_k')^2+\Big(\frac{u_k-u_{k-1}}{h}\Big)^2\Big](\tau)\,\mathrm{d}\tau\le C, \tag{3.9}$$

where the constant C *is independent of* n, h.

Proof. To derive the estimate, we differentiate equation (3.4), multiply it by u_k' and sum it with equation (3.5) differentiated and multiplied by q_k':

$$c_k\,u_k''u_k'+e_k'q_k'=\frac{1}{h}\,(q_{k+1}'\,u_k'-q_k'\,u_{k-1}')+f_k'\,u_k'.$$

Further, summing up the equation for $k=1,2,\dots,n-1$, we obtain for a.e. $t\in I$

$$\sum_k c_k\,u_k''u_k'+\sum_k e_k'q_k'=\frac{1}{h}\,(q_n'\,u_{n-1}'-q_1'\,u_0')+\sum_k f_k'\,u_k'.$$

Owing to the boundary conditions, we have $q_n'=0$ and $u_0'=0$. We multiply the equality by h. Since $u_k''\,u_k'=\frac12[(u_k')^2]'$, integration of the last equality from 0 to a fixed $t\le T$ yields

$$h\sum_k\frac{c_k}{2}\,(u_k'(t))^2+\int_0^t h\sum_k e_k'\,q_k'\,\mathrm{d}\tau=h\sum_k\frac{c_k}{2}\,(u_k'(0))^2+\int_0^t h\sum_k f_k'\,u_k'\,\mathrm{d}\tau. \tag{3.10}$$

Using equation (3.4), initial condition (3.7) and properties of the operator $\mathcal{G}_k$, the first term with $u_k'(0)$ can be estimated by a constant. Using the inequalities

$$\int_I|f(t)g(t)|\,\mathrm{d}t\le\max_I|f(t)|\int_I g(t)\,\mathrm{d}t,\qquad\Big(\int_I f(t)\,\mathrm{d}t\Big)^2\le|I|\int_I f^2(t)\,\mathrm{d}t$$

and $|ab|\le\varepsilon a^2+\frac{1}{4\varepsilon}b^2$, we estimate the term with f_k'

$$\Big|\int_0^t h\sum_k f_k'u_k'\,\mathrm{d}\tau\Big|\le\varepsilon h\max_{t\in\langle0,t\rangle}\sum_k(u_k')^2+\int_0^t h\sum_k(f_k')^2\,\mathrm{d}\tau.$$

Since f_k is the integral mean of $f(x)$ over the interval Ω_k, we have $h\sum_k f_k=\int_\Omega f(x)\,\mathrm{d}x$. Thus, by (2.8) $e_k'q_k'\ge0$, for sufficiently small $\varepsilon>0$, we obtain the estimate of the terms on the left-hand side of (3.10). Using inequalities (2.14), (2.15) and equations (3.4), (3.5), we obtain the remaining estimates of Lemma 3.3. □

3.4 Approximate solutions and passage to the limit

For a fixed n, using the solutions u_k, e_k, q_k and the data c_k, f_k, ζ_k, we construct approximated solutions

- $\overline{c}^{(n)}$, $\overline{f}^{(n)}$, $\overline{u}^{(n)}$ are "forward" piecewise constant approximate solutions defined by the relation $\overline{\varphi}^{(n)}=\varphi_{k-1}$ for $x\in\Omega_k$,
- $\overline{e}^{(n)}$, $\overline{q}^{(n)}$, $\overline{\zeta}^{(n)}$ are "backward" piecewise constant approximate solutions defined by the relation $\overline{\varphi}^{(n)}=\varphi_k$ for $x\in\Omega_k$,
- $\widehat{u}^{(n)}$, $\widehat{q}^{(n)}$ are continuous piecewise linear on each Ω_k approximation satisfying $\widehat{\varphi}^{(n)}(x_k)=\varphi_k$.

The above approximations satisfy the system of equations for all t and a.e. $x \in \Omega$:

$$\overline{c}^{(n)}\overline{u}_t^{(n)} = \widehat{q}_x^{(n)} + \overline{f}^{(n)}, \quad \overline{e}_t^{(n)} = \widehat{u}_x^{(n)}, \quad \overline{e}^{(n)} = \overline{\mathcal{G}}^{(n)}[\overline{q}^{(n)}]. \tag{3.11}$$

Estimates (3.8), (3.9) yield the estimates of the corresponding approximate solutions $\overline{u}^{(n)}$, $\overline{e}^{(n)}$, $\overline{q}^{(n)}$, $\widehat{u}^{(n)}$ and $\widehat{q}^{(n)}$. By the compactness, these sequences contain converging subsequences which converge to the functions u, e, q satisfying the problem. Thus the solution to the problem exists.

The proof of the uniqueness of a solution can be found in [3]. Since the uniqueness for the worst scenario method is not necessary, we omit the proof. We have also proved that the unknowns in the corresponding spaces are bounded by the constants depending on the constants ℓ, T, c_m, c_M, α, β and the norms of f and u^0 only.

4 Problems with uncertain data and reliable solutions

Mathematical models of particular problems in engineering contain data, mainly material constants or constitutive relation dependence. These data are obtained by measurements and thus are not known exactly, they are uncertain, their values are known to some extent only. Since using the mean value of the data by modelling already caused several failures of a construction in engineering practice, Ivo Babuška has proposed the so-called worst scenario method.

The method consists in considering the problems with all data admissible by the measurements and the corresponding solutions. According to the character of the problem, a criterion-functional on data and solutions is chosen. This functional should evaluate a rate of danger of the situation. The method thus looks for the data yielding the worst situation, i.e., what the worst can happen within the given uncertain data, although the probability may be very low. The worst scenario method for obtaining reliable solutions was further developed by Ivan Hlaváček and others (see, e.g., [7]) and also applied to many particular problems. The survey paper [6] can be recommended for a brief introduction.

The advantage of the worst scenario method consists in the possibility to use numerical methods, algorithms and software developed for optimization problems. For its deterministic character the method is much more simpler and effective than probabilistic approaches.

4.1 Worst Scenario Method

Here we describe the method. Let us denote by P_a the state problem with data a and the corresponding solution by u_a. The data a may contain coefficients of the equation, right-hand side, values in the boundary conditions, etc. The set of all admissible values of data a will be denoted by $\mathcal{U}_{\rm ad}$. It should be chosen such that for each $a \in \mathcal{U}_{\rm ad}$ the problem P_a admits a solution u_a.

The criterion-functional $\Phi = \Phi(a, u_a)$ "evaluating" danger of the situation will be defined on the data $a \in \mathcal{U}_{\rm ad}$ and the corresponding solutions u_a of the problem P_a. Then the worst scenario problem reads:

Problem. Find the data $a^* \in \mathcal{U}_{\rm ad}$ which maximize the functional Φ, i.e.,

$$\text{Look for } a^* \in \mathcal{U}_{\rm ad} \text{ s.t. } \Phi(a^*, u_{a^*}) \geq \Phi(a, u_a) \text{ for all } a \in \mathcal{U}_{\rm ad}. \tag{4.1}$$

In the case if the solution of the problem P_a exists but is not unique, the problem is modified to:

$$\text{Find } a^* \in \mathcal{U}_{\rm ad} \text{ and } u^* \in U_{a^*} \text{ s.t. } \Phi(a^*, u^*) \geq \Phi(a, u) \ \forall\, a \in \mathcal{U}_{\rm ad} \ \forall\, u \in U_a,$$

where U_a is the set of all solutions u_a to the problem P_a with data a.

The aim of the contribution is to prove that the problem admits a solution, i.e., the functional Φ is bounded and attains its maximum. Let us note that this maximum can be attained for more than one data a^* and in the case of a nonlinear problem the maximum can be reached for the data a^* in the interior of the set $\mathcal{U}_{\rm ad}$ of admissible data, i.e., not on the boundary of $\mathcal{U}_{\rm ad}$ as in the case of a linear problem.

Knowing that for each $a \in \mathcal{U}_{\rm ad}$ the problem P_a admits a solution, the procedure continues with the following steps:

- we choose the set $\mathcal{U}_{\rm ad}$ which is compact, i.e., each sequence $\{a_n\} \subset \mathcal{U}_{\rm ad}$ contains a subsequence converging to an element $a^* \in \mathcal{U}_{\rm ad}$,
- we prove that the mapping $a \mapsto u_a$ is continuous, i.e.,

$$a_n \to a^* \quad \Longrightarrow \quad u_{a_n} \to u_{a^*}$$

- and we verify the continuous dependence of the functional Φ on the data a and the solution u_a, i.e.,

$$a_n \to a^*, \;\; u_{a_n} \to u_{a^*} \quad \Longrightarrow \quad \Phi(a_n, u_{a_n}) \to \Phi(a^*, u_{a^*}).$$

Then the worst scenario problem admits a reliable solution. Indeed, let the sequence of data $\{a_n\}_{n=1}^{\infty}$ maximize the functional Φ on $\mathcal{U}_{\rm ad}$, i.e., $\Phi(a_n, u_{a_n})$ tends to a supremum of Φ on $\mathcal{U}_{\rm ad}$. Since the set $\mathcal{U}_{\rm ad}$ is compact, the sequence $\{a_n\}$ contains a subsequence $\{a_{n'}\}$ converging to $a^* \in \mathcal{U}_{\rm ad}$. The continuity of the mapping $a \to u_a$ yields $u_{a_{n'}} \to u_{a^*}$ and the continuity of Φ yields $\Phi(a_{n'}, u_{a_{n'}})$ tends to $\Phi(a^*, u_{a^*})$. Thus the supremum is a real number and Φ admits a maximum on $\mathcal{U}_{\rm ad}$.

4.2 Admissible data

The data in our problem are of several types: the material constant c, the pair of distribution functions (η, ζ) of the hysteresis operator, the right-hand side f, the initial condition u^0 and the boundary conditions for the unknown at $x = 0$ and $x = \ell$; for the sake of simplicity, they were chosen to be zero. For simplicity, we also take the initial condition u^0 and right-hand side to be certain.

We consider a nonhomogeneous medium composed of two or more homogeneous materials occupying the parts $\Omega_1, \dots, \Omega_k$. For the sake of simplicity, we assume that the parts Ω_i are known, only the constant c on each Ω_i is uncertain, i.e., its real values are within the intervals $\langle c_{im}, c_{iM}\rangle$. Thus $c(x)$ will be piecewise constant functions from the admissible set $\mathcal{C}_{\rm ad}$

$$\mathcal{C}_{\rm ad} = \big\{c:\ \Omega \to \mathbb{R},\ c(x) = c_i \in \langle c_{im}, c_{iM}\rangle \ \text{ for } \ x \in \Omega_i,\ i = 1, \dots, k\big\}, \tag{4.2}$$

where the constants $0 < c_{im} \le c_{iM}$ are given. Since the set $\mathcal{C}_{\rm ad}$ is "equivalent" to the cartesian product of compact intervals $\langle c_{im}, c_{iM}\rangle$, we have arrived to

Lemma 4.1. *The set $\mathcal{C}_{\rm ad}$ is compact in the maximum norm.*

4.3 Admissible data for hysteresis operators

Mutually inverse Prandtl–Ishlinskii operators $\mathcal{F}$ and G are fully determined by their distribution function $\eta \in PI(\alpha, \beta)^-$ or $\zeta \in PI(\alpha, \beta)^+$. Since the use is made of the operator $\mathcal{G}$ of play type, we define the set of admissible functions for ζ as a subset of $PI(\alpha, \beta)^+$ which are constant outside of the interval $\langle 0, R\rangle$ with some $R > 0$. Thus for $\alpha, \beta > 0$, $\alpha\beta < 1$ and $R > 0$ let $\mathcal{Z}(\alpha, \beta, R)$ be the set of all functions $\zeta = \zeta(r)$ satisfying:

(a) ζ is right-continuous nondecreasing function $\langle 0, \infty) \to \langle 0, \infty)$,

(b) $\beta \le \zeta(r) \le 1/\alpha \ \forall\, r \in \langle 0, \infty)$ (i.e., $\zeta \in PI(\alpha, \beta)^+$),

(c) $\zeta(r)$ is constant on $\langle R, \infty)$.

The set $\mathcal{Z}(\alpha, \beta, R)$ is compact in the following sense: Each sequence of $\{\zeta_n\}_{n=1}^{\infty}$ in the set $\mathcal{Z}(\alpha, \beta, R)$ contains a subsequence $\{\zeta_{n'}\}$ and there exists a function ζ^* in $\mathcal{Z}(\alpha, \beta, R)$ such that $\int_0^R |\zeta_{n'}(r) - \zeta^*(r)|\, \mathrm{d}r$ tends to zero as $n' \to \infty$.

This compactness can be proved by constructing finite ε-nets of piecewise constant functions. Let us divide the interval $\langle 0, R\rangle$ by the points $r_i = i\, R/n_r$ into n_r parts. Further, let us divide the interval of values $\langle \beta, \frac{1}{\alpha}\rangle$ by $z_j = \beta + (\frac{1}{\alpha} - \beta)j/n_z$ into n_z equal parts. Then the functions $\zeta(r)$ which are nondecreasing and take the values z_j on each part $\langle r_{j-1}, r_j)$ make for sufficiently large n_r and n_z a finite ε-net in $\mathcal{Z}(\alpha, \beta, R)$ which proves the compactness of $\mathcal{Z}(\alpha, \beta, R)$.

Let $\{q_n(t)\}$ be a bounded sequence $\|q_n\|_{\langle 0,T\rangle} \le R$ of functions uniformly converging to $q^*(t)$, i.e., $\|q_n(t) - q^*(t)\|_{\langle 0,T\rangle} \to 0$. By (2.18), the convergence of $\zeta_n(r)$ ensures that of the corresponding $e_n = \mathcal{G}_{\zeta_n}[q_n]$.

Since the medium consists of k homogeneous materials, we take the functions $\zeta(r,x)$ constant in x on each $x \in \Omega_i$. For $\alpha_i, \beta_i > 0$, $\alpha_i\beta_i < 1$ and $R_i > 0$, $i = 1, 2, \ldots, k$, we put

$$\mathcal{Z}_{\mathrm{ad}} = \big\{\zeta(r,x),\ \text{ s.t. } \zeta(x,r) \in \mathcal{Z}(\alpha_i, \beta_i, R_i) \text{ for } x \in \Omega_i,\ i = 1, 2, \ldots, k\big\}. \tag{4.3}$$

Thus we have arrived to

Lemma 4.2. *Each sequence of $\{\zeta_n\}_{n=1}^{\infty}$ in the set $\mathcal{Z}_{\mathrm{ad}}$ of admissible distribution functions contains a subsequence $\{\zeta_{n'}\}$, and there exists a function ζ^* in $\mathcal{Z}_{\mathrm{ad}}$ such that for any $q_n \in C(\Omega \times I)$ converging uniformly to a $q^* \in C(\Omega \times I)$ the sequence of $e_n = \mathcal{G}_{\zeta_n}[q_n]$ converges uniformly to $e^* = \mathcal{G}_{\zeta^*}[q^*]$.*

Thus the set for admissible data for our problem will be

$$\mathcal{U}_{\mathrm{ad}} = \mathcal{C}_{\mathrm{ad}} \times \mathcal{Z}_{\mathrm{ad}}.$$

4.4 Continuity of the mapping $a \mapsto u_a$.

It remains to prove that the convergence of data $a_n \to a^*$ implies that of solutions $u_{a_n} \to u_{a^*}$, where $u_a = (u, e, q)$. Let us denote by $u_n = (u^n, e^n, q^n)$ the solution of problem P_{a_n} with the data $a_n = (c^n, \zeta^n)$ and by $u^* = (u^*, e^*, q^*)$ the solution of problem P_{a^*} with the data $a^* = (c^*, \zeta^*)$, i.e.,

$$\begin{aligned} c^n u^n_t &= q^n_x + f, \quad e^n = u^n_x, \quad e^n = \mathcal{G}_{\zeta^n}[q^n], \\ c^* u^*_t &= q^*_x + f, \quad e^* = u^*_x, \quad e^* = \mathcal{G}_{\zeta^*}[q^*]. \end{aligned}$$

Comparing the first pair of equations and splitting the left-hand side, we obtain

$$c^n u^n_t - c^* u^*_t \equiv (c^n - c^*)u^n_t + c^*(u^n_t - u^*_t) = q^n_x - q^*_x.$$

Multiplying the equation with $(u^n_t - u^*_t)$, we obtain

$$(c^n - c^*)u^n_t(u^n_t - u^*_t) + c^*(u^n_t - u^*_t)^2 = (q^n_x - q^*_x)(u^n_t - u^*_t). \tag{4.4}$$

The second pair of equalities yields $e^n - e^* = u^n_x - u^*_x$. Differentiating it by t and multiplying it by $(q^n - q^*)$, we obtain

$$(e^n_t - e^*_t)(q^n - q^*) = (u^n_{xt} - u^*_{xt})(q^n - q^*). \tag{4.5}$$

Summing up (4.4) and (4.5) and using formula $f_x g + f g_x = (fg)_x$, we obtain

$$(c^n - c^*)u^n_t(u^n_t - u^*_t) + c^*(u^n_t - u^*_t)^2 + (e^n_t - e^*_t)(q^n - q^*) = ((u^n_t - u^*_t)(q^n - q^*))_x.$$

We integrate the equation over G. Formula $\int_0^{\ell} f_x \,\mathrm{d}x = f(\ell) - f(0)$ and the zero boundary conditions for $x = 0$ and $x = \ell$ give zero in the right-hand side. Finally, integrating the equality over $(0,t)$, we obtain

$$\int_{\Omega \times (0,t)} \big[(c^n - c^*)u^n_t(u^n_t - u^*_t) + c^*(u^n_t - u^*_t)^2 + (e^n_t - e^*_t)(q^n - q^*)\big] \,\mathrm{d}x \,\mathrm{d}\tau = 0. \tag{4.6}$$

Splitting $e^n - e^* = \mathcal{G}_{\zeta^n}[q^n] - \mathcal{G}_{\zeta^n}[q^*] + \mathcal{G}_{\zeta^n}[q^*] - \mathcal{G}_{\zeta^*}[q^*]$ and using the inequality

$$\big(\mathcal{G}_{\zeta^n}[q^n] - \mathcal{G}_{\zeta_n}[q^*]\big)_t (q^n - q^*) \ge \frac{\beta}{2} \frac{\mathrm{d}}{\mathrm{d}t} [q^n - q^*]^2$$

(see Proposition 2.9), for sufficiently large R, we obtain

$$\begin{aligned} &\int_{\Omega \times (0,t)} c^*(u^n_t - u^*_t)^2 \,\mathrm{d}x \,\mathrm{d}\tau + \frac{\beta}{2} \int_{\Omega} (q^n(t) - q^*(t))^2 \,\mathrm{d}x \\ &\qquad \le \int_{\Omega \times (0,t)} \Big[\big|(c^n - c^*)u^n_t(u^n_t - u^*_t)\big| + \int_0^R |\zeta^n - \zeta^*| \,\mathrm{d}r\Big] |q^n - q^*| \,\mathrm{d}x \,\mathrm{d}\tau. \end{aligned} \tag{4.7}$$

Since by the compactness of $\mathcal{U}_{\mathrm{ad}}$ we have $c^n \rightrightarrows c^*$ and $\zeta^n - \zeta^* \to 0$, the right-hand side tends to zero which proves the convergence of the solutions. In this way even the uniqueness of the solution in Theorem 3.2 is proved. □

4.5 Criterion-functional

In mechanics, the dangerous situations are extremes of the deformation or stress. In our heat conduction problem or the diffusion problem the extremes of the temperature or concentration can be critical, the extremes of the temperature or concentration gradient can be critical, as well.

From the mathematical point of view, a continuous function on a compact (i.e., closed bounded) set attains its maximum. If the function is integrable only, say in the L^p space, then its values are determined except for measure zero sets and thus the value of the function at a point x has no sense. Instead of it we have to take integral mean of a small part G of Ω

$$\Phi(a, u_a) = \frac{1}{|G|} \int_G u_a(x)\,\mathrm{d}x,$$

where G is the small part of Ω, where the failure of the construction may be expected.

Following the existence Theorem 3.2, the solutions u, q are the continuous functions on $C(\Omega \times I)$. Thus for any point $x_0 \in \Omega$ and any time t_0 the criterion-functional for the data $a \in \mathcal{U}_{\mathrm{ad}}$ and the corresponding solution $u_a = (u, q, e)$ can be defined as the value of u or q for (x_0, t_0) or its maximum at the point x_0 or time t_0, for example,

$$\begin{aligned} \Phi_1(a, u_a) &= u(x_0, t_0), & \Phi_2(a, u_a) &= q(x_0, t_0), \\ \Phi_3(a, u_a) &= \max_{x \in G} u(x, t_0), & \Phi_4(a, u_a) &= \max_{t \in J} q(x_0, t), \\ \Phi_5(a, u_a) &= \max_{(x,t) \in G \times J} u(x, t), & \Phi_6(a, u_a) &= \max_{(x,t) \in G \times J} q(x, t), \end{aligned}$$

where G is a closed subset of Ω and J is a closed subinterval of I.

Following Theorem 3.2, the unknown $e = u_x \in L^2(\Omega, C(I))$. Then the criterion-functional may be be the integral mean of u_x e.g.

$$\Phi_7(a, u_a) = \frac{1}{|G|} \int_G |u_x(x, t_0)|\,\mathrm{d}x, \quad \Phi_8(a, u_a) = \frac{1}{|G|} \int_G \max_{t \in J} |u_x(x, t)|\,\mathrm{d}x,$$

where G is an open subset of Ω and J a closed subinterval of I.

Finally, following Theorem 3.2, the gradients u_t, e_t, q_t, q_x are in $L^\infty(I, L^2(\Omega)$. Thus the criterion-functional may be the integral mean over a closed $G \subset \Omega$ and J a subinterval of I, e.g.,

$$\Phi_9(a, u_a) = \frac{1}{|G| \cdot |J|} \int_{G \times J} |v(x, t)|\,\mathrm{d}t\,\mathrm{d}x,$$

where v stands for any of u_t, e_t, q_t, q_x.

4.6 Main result

Since the set of admissible data $\mathcal{U}_{\mathrm{ad}}$ is compact with respect to the corresponding norms, the mapping $a \mapsto u_a$ is continuous and also each functional of type $\Phi_1, \dots, \Phi_9$ is continuous, we have arrived at the main result:

Theorem 4.3. *Let Hypothesis* 3.1 *be satisfied, the set of admissible data* $\mathcal{U}_{\mathrm{ad}} = \mathcal{C}_{\mathrm{ad}} \times \mathcal{Z}_{\mathrm{ad}}$ *be defined by* (4.2)*,* (4.3)*.*

Then the worst scenario problem for the problem (3.1)–(3.3) *with any criterion-functionals of type* $\Phi_1, \dots, \Phi_9$ *or their combination admits the solution.*

5 Concluding remarks

For the sake of simplicity, we have assumed certain zero boundary conditions for $u(0,t) = 0$, and $q(\ell,t) = 0$, certain initial condition and certain right-hand side $f(x,t)$. The result can be extended even to uncertain both boundary conditions $u(0,t) = u_0(t)$, or uncertain $q(0,t) = q_0(t)$ and similar uncertain data on the right end $x = \ell$. Also, the initial condition $u^0(t)$ and right-hand side $f(x,t)$ may be uncertain. One should define the corresponding convenient compact sets for these uncertain functions, and their difference will appear in the right-hand side of (4.7).

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