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LOCALIZED LOCAL MAXIMA FOR NON-NEGATIVE GROUND STATE SOLUTION OF NONLINEAR SCHRÖDINGER EQUATION WITH NON-MONOTONE EXTERNAL POTENTIAL

Abstract. A non-negative ground state solution $u(x)$ of the nonlinear Schrödinger equation with non-monotone potential is studied. The existence of local maxima of $u(x)$ which are attained on the given intervals in one-dimensional space variable x is shown. Next, it is proved that the stationary point of $u(x)$ per one interval is unique. The co-existence of the local extrema of ground state solution and external potential on the same interval is considered, too.[1]

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Key words and phrases. Schrödinger equation, ground state solution, extrema, non-monotonic behaviour, particle density, Bose–Einstein condensates.

რეზიუმე. შროდინგერის არაწრფივი დიფერენციალური განტოლებისთვის არამონოტონური პოტენციალით შესწავლილია არაუარყოფითი ძირითადი მდგომარეობის $u(x)$ ამონახსნი. ნაჩვენებია $u(x)$-ის ლოკალური მაქსიმუმების არსებობა, რომლებიც მიიღწევა ერთგანზომილებიანი სივრცითი x ცვლადის მოცემულ ინტერვალებზე. დამტკიცებულია, რომ $u(x)$-ის სტაციონარული წერტილი თითოეული ინტერვალისთვის არის ერთადერთი. განხილულია აგრეთვე ძირითადი მდგომარეობის ამონახსნის ლოკალური ექსტრემუმებისა და იმავე ინტერვალზე გარე პოტენციალის თანაარსებობის საკითხი.

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1 Introduction and mathematical setting

1.1 Localized local maxima

Let $[a,b] \subset \mathbb{R}$ be a bounded interval and $u : \mathbb{R} \to \mathbb{R}$, $u = u(x)$, be a C^1-function. Recall that $u(x)$ *attains a local maximum in a prescribed interval* $[a,b]$ if there exists a point $x_{\mathrm{s}} \in [a,b]$ such that $u'(x_{\mathrm{s}}) = 0$ (stationary point of $u(x)$) and $u'(x)$ changes sign at x_{s} such that $u'(x) > 0$ in $(x_{\mathrm{s}} - \varepsilon, x_{\mathrm{s}})$ and $u'(x) < 0$ in $(x_{\mathrm{s}}, x_{\mathrm{s}} + \varepsilon)$ for some $\varepsilon > 0$. One can say that x_{s} is *localized* on $[a,b]$.

For instance, if $[a,b] = [0,\pi]$ and $u(x) = \exp(\sin(x))$, then the differential equation $u'' + (\sin(x) - \cos^2(x))u = 0$ possesses a positive solution $u(x)$ having a local maximum at $x_{\mathrm{s}} = \pi/2$, which is localized and unique in $[a,b]$.

1.2 Time-independent nonlinear Schrödinger equation (NLSE)

In the paper, we consider C^2-solutions $u(x)$ of the following one-dimensional time-independent nonlinear Schrödinger equation:

$$u'' + \Big(\mu - \frac{2m}{\hbar^2}\, V(x)\Big)u + \frac{2m}{\hbar^2}\, f\big(x, |u|^2\big)u = 0, \tag{1.1}$$

where $\mu \in \mathbb{R}$ is the chemical potential, $\hbar$ is the Planck constant, m is the particle mass, $V(x)$ is a continuous the so-called *linear,* or *external,* or *trapping potential* and the *nonlinear potential* f satisfies:

$$f(x,s^2) \geq -g(x), \quad (x,s) \in \mathbb{R}^2, \tag{1.2}$$

where $g(x)$ is a continuous function. In the accordance with (1.2), the following two cases occur:

(1) if $g(x) \leq 0$, then $f(x,s^2)$ is an *attractive potential*: $f(x,s^2) \geq 0$, $(x,s) \in \mathbb{R}^2$; especially for $g(x) \equiv 0$, assumption (1.2) allows $f(x,s^2)$ to be a *classic attractive potential*: $f(x,s^2) = f_0(x)s^2$ with $f_0(x) \geq 0$; hence, in this case, our result can be interperted as the non-monotonic behaviour of particle density in the Bose–Einstein condensate (BEC);

(2) if $g(x) \geq 0$ and $g(x) \not\equiv 0$, then assumption (1.2) allows $f(x,s^2)$ to be a *repulsive potential*: $f(x,s^2) \leq 0$, $(x,s) \in \mathbb{R}^2$, but not a *classic repulsive potential*: $f(x,s^2) = f_0(x)s^2$ with $f_0(x) \leq 0$; an example of a repulsive potential satisfying (1.2) is $f(x,s^2) = -g_0(x)\arctan(s^2)$, where $g(x) = \frac{\pi}{2}\, g_0(x)$ with $g_0(x) \geq 0$.

1.3 Motivation for mathematical treatment of localized local maxima of ground state solution of NLSE

The so-called solitary wave $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(x,t) = e^{-\mathrm{i}\, \frac{\hbar\mu}{2m} t}\, u(x) \tag{1.3}$$

satisfies the time-dependent nonlinear Schrödinger equation

$$\mathrm{i}\hbar\, \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\, \frac{\partial^2\psi}{\partial x^2} + V(x)\psi - f\big(x, |\psi|^2\big)\psi, \tag{1.4}$$

provided $u(x)$ is a solution of our main equation (1.1). In such a situation, $u(x)$ is called as the *ground state solution* of NLSE (1.1). If $f(x,s^2) = f_0(x)s^2$, equation (1.4) is known as the Gross–Pitaevski equation (GPE), which is a model for a wave function of the particles in an atomic cloud in BEC. The quantity $|\psi(x,t)|^2$ represents the particle density in BEC, which has the common stationary points in the variable x with a non-negative ground state solution, since

$$|\psi(x,t)|^2 = u^2(x) \ \text{ and } \ \frac{\partial}{\partial x}\, |\psi(x,t)|^2 = (u^2(x))' = 2u(x)u'(x). \tag{1.5}$$

Hence, the non-monotonic behaviour of particle density $|\psi(x,t)|^2$ is strictly related with the extrema of the ground state solution $u(x)$. Among all known numerical simulations in which we can see the non-monotonic behaviour of particle density in BEC (see [1–4] and [7–11]), we point out the next three:

- **BEC with spatially modulated parameters – Figure 1.** The exact ground state solution $u(x) = \rho(x)\Phi(\theta(x))$ of the main equation (1.1) especially for $f(x, s^2) = f_0(x)s^2$, where $\Phi(t)$ is a solution of the corresponding Duffing equation. The potential $V(x)$, the spatially modulation $f_0(x)$ and the frequency $\theta(x)$ are generated by the amplitude function $\rho(x)$ via certain differential relations derived by the similarity transformations (for details see [4]).

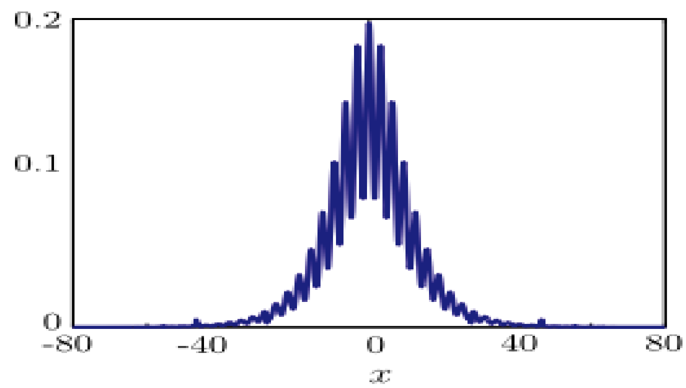

Figure 1. [4, Figure 2 – case (a)]

- **A spin-orbit coupled BEC – Figure 2.** The numerical simulation realized by a split-step Crank–Nicolson method for the stationary states $|\psi_1|$ and $|\psi_2|$ of an integrable system of coupled GPEs (1.4) solved by combining the Lax pair method and gauge transformation approach (for details see [11]).

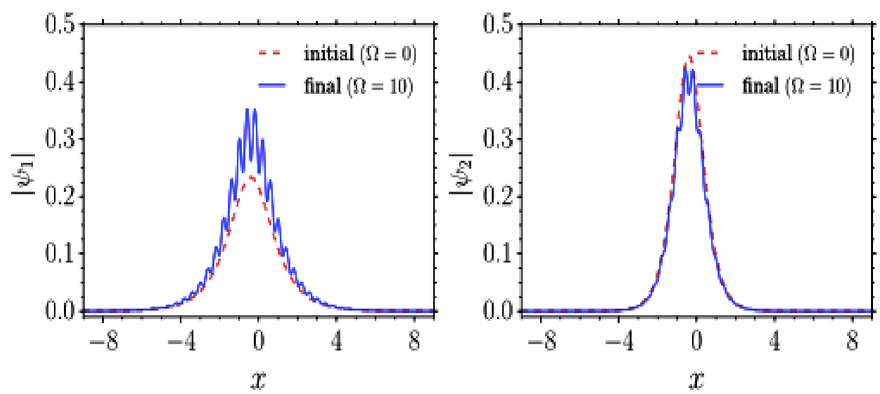

Figure 2. [11, Figure 7]

- **The ground and first excited states in BEC – Figure 3.** The numerically ground state solution $u(x)$ of the main equation (1.1), which is computed by the gradient flow with discrete normalization, where the discretizing has been made in two ways (the backward Euler sine-pseudospectral and backward/forward Euler sine-pseudospectral methods) (for details see [3]).

This numerical simulation is the most interesting for our consideration in the paper, because it visualizes the next two issues:

- *relation between non-monotonic behaviours of $u(x)$ and $V(x)$:* when $V(x)$ is non-monotonic, then $u(x)$ is non-monotonic too, although it is very well known that the classic theory for the

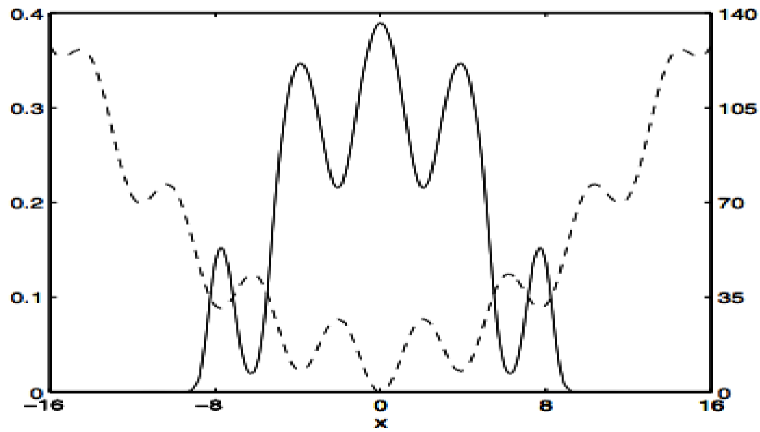

Figure 3. [3, Figure 1(b), $u(x)$ – solid line, $V(x)$ – dashed lines]

linear Schrödinger equation says that when $V(x)$ is a harmonic potential: $V(x) = A|x|^2$, $A > 0$, which is increasing on $(0, \infty)$, then $u(x)$ is of Gaussian type: $u(x) = Be^{-|x|^2}$, $B > 0$, which is decreasing on $(0, \infty)$, see in [8, Section 2.3: Density profile and velocity distribution];

- *the co-existence of local extrema on the same interval:* $u(x)$ attains the local maxima (resp., minima) in the intervals where the $V(x)$ attains its minima (resp., maxima).

In Section 2, we state and describe our main assumptions and results, which are proved in Section 3. The essential advantages of our method with respect to the method presented in the recently published paper [5] are: the assumption for strictly positivity of $u(x)$ is relaxed so that $u(x)$ is now a non-negative ground state solution having the most finite number of zeros per one interval; here, the nonlinear potential $f(x, s^2)$ is not only of attractive type but it can also be of a repulsive type, which is described above just after (1.2); our conditions on the external potential $V(x)$ is more general than related one considered in [6], which is shown below in Subsection 2.2.

2 Statement of the basic assumptions and main results

2.1 Basic assumptions

Let $[a, b] \subset \mathbb{R}$ be a bounded interval on which the ground state solution $u(x)$ satisfies:

$$u(x) \text{ possesses at most finite number of zeros in } [a, b], \tag{H_0}$$

and the potential difference between μ and $(V(x) + g(x))2m/\hbar^2$ satisfies:

$$\mu - \frac{2m}{\hbar^2}\,(V(x) + g(x)) > 0 \ \text{ in } \ [a, b]. \tag{H-basic}$$

The next consequence of the assumptions (H_0) and (H-basic) is worth to be pointed out.

Proposition 2.1. *Let* (1.2) *and* (H-basic) *hold. If the ground state solution* $u(x)$ *of* (1.1) *satisfies* (H_0) *and* $u(x) \geq 0$ *in* $[a, b]$*, then* $u(x)$ *has at most one stationary point in* $[a, b]$.

Indeed, if the ground state solution $u(x)$ is non-negative in $[a, b]$ and has two stationary points $x_1, x_2 \in [a, b]$, $x_1 \neq x_2$, then integrating (1.1) over $[x_1, x_2]$ together with assumptions (1.2), (H_0) and (H-basic), we have

$$0 = u'(x_2) - u'(x_1) \leq -\int\limits_{x_1}^{x_2} \Big[\mu - \frac{2m}{\hbar^2}\,(V(x) + g(x))\Big] u(x)\, dx < 0,$$

which is not possible. Thus, the stationary point of $u(x)$ in $[a,b]$ is unique if it exists of course.

Next, the assumption (H_0) is more general than the next one,

$$u(x) \neq 0, \quad x \in [a,b]. \tag{$H_{\neq 0}$}$$

Although $(H_{\neq 0})$ is involved in all preceding Figures 1–3, the general assumption (H_0) is also appearing in the context of particle density in BEC (see, for instance, [2]).

Remark 2.1. Especially for $g(x) \equiv 0$ (attractive case) or $g(x) \geq 0$ (repulsive case), the assumption (H-basic) implies

$$\mu - \frac{2m}{\hbar^2} V(x) > 0 \ \text{ in } \ [a,b]. \tag{2.1}$$

Since the chemical potential μ is a constant and $V(x)$ is a continuous potential in $\mathbb{R}$, thanks to (2.1) it is possible to take for $[a,b]$ such an interval in which $V(x)$ attains its minimum. This is in the accordance with the numerical simulation given in Figure 3 above. More accurate relation between the non-monotonic behaviours of $u(x)$ and $V(x)$ is considered in Subsection 2.3 below about the co-existence of local extrema of $u(x)$ and $V(x)$.

2.2 The existence of localized local extrema of $u(x)$

On a given interval $[a,b]$, we involve on the potentials μ, $V(x)$ and $g(x)$ the following additional assumption: for some $\varphi \in C^1(a,b)$, $\varphi(a) = \varphi(b) = 0$, $\varphi(x) \neq 0$ in (a,b), we have

$$\int_a^b |\varphi(x)|^2 \, dx > \int_a^b \frac{|\varphi'(x)|^2}{\mu - \frac{2m}{\hbar^2}\left(V(x) + g(x)\right)} \, dx. \tag{H-general}$$

The condition (H-general) is particularly related with the eigenvalue problem for the one-dimensional Laplacian operator in (a,b) with respect to the first eigenvalue $\lambda_1 > 0$ and the corresponding eigenvalue vector $\varphi \in C^2(a,b)$ (let us remark that $\lambda_1 = (\pi/(b-a))^2$ and $\varphi(x) = \sin(\sqrt{\lambda_1}(x-a))$):

$$\varphi'' + \lambda_1 \varphi = 0 \ \text{ in } \ (a,b), \quad \varphi(a) = \varphi(b) = 0. \tag{2.2}$$

Indeed, if we suppose

$$\mu - \frac{2m}{\hbar^2}\left(V(x) + g(x)\right) > \lambda_1 \ \text{ in } \ [a,b], \tag{2.3}$$

which is a more concrete condition than (H-general), from (2.2) and (2.3) we get

$$\int_a^b |\varphi(x)|^2 \, dx = \frac{1}{\lambda_1} \int_a^b |\varphi'(x)|^2 \, dx > \int_a^b \frac{|\varphi'(x)|^2}{\mu - \frac{2m}{\hbar^2}\left(V(x) + g(x)\right)} \, dx.$$

Thus, condition (2.3) is a particular case of (H-general) taking for $\varphi(x)$ the eigenfunction from (2.2).

The first main result is

Theorem 2.1. *Suppose that* (1.2) *is satisfied and let* $[a,b]$ *be an interval such that* (H-basic) *and* (H-general) *hold. Then every solution* $u(x)$ *of the nonlinear Schrödinger equation* (1.1) *has a stationary point in* $[a,b]$. *Furthermore, if* $u(x) \geq 0$ *in* $[a,b]$ *and satisfies* (H_0), *then the stationary point of* $u(x)$ *is unique in* $[a,b]$. *Moreover,* $u(x)$ *attains its local maximum in* $[a,b]$.

Since (2.3) is a particular case of (H-general), we have also derived the next interesting consequence of the main result.

Theorem 2.2. *Suppose that* (1.2) *holds and let* $[a,b]$ *be an interval such that the potentials* μ, $V(x)$ *and* $g(x)$ *satisfy* (2.3). *Then every solution* $u(x)$ *of the nonlinear Schrödinger equation* (1.1) *has a stationary point on* $[a,b]$. *Furthermore, if* $u(x) \geq 0$ *in* $[a,b]$ *and satisfies* (H_0), *then the stationary point of* $u(x)$ *is unique in* $[a,b]$. *Moreover,* $u(x)$ *attains its local maximum in* $[a,b]$.

Thus, Theorem 2.2 is a particular case of Theorem 2.1, and Theorem 2.1 is more general than [6, Theorem 3.1] even in the case $g(x) \equiv 0$, because the condition $(H_{\neq 0})$ is relaxed here with (H_0).

2.3 The co-existence of local extrema of ground state solution $u(x)$ and potential $V(x) + g(x)$

According to Theorem 2.1, we are able now to explain the case in which the ground state solution $u(x)$ attains a local minimum on an interval where the potential $V(x) + g(x)$ attains its local maximum. This is also visualized in the next figure:

Figure 4. $u(x)$ - solid line, $V(x) + g(x)$ – dashed lines.

For this purpose, we need to work with two disjoint intervals $[a_1, b_1]$ and $[a_2, b_2]$ such that

$$a_1 < b_1 < a_2 < b_2. \tag{2.4}$$

In order to simplify the notation, let

$$W(x) = \mu - \frac{2m}{\hbar^2} (V(x) + g(x)).$$

Let the assumptions (H-basic), (H-general) and $u(x) \geq 0$ with (H_0) be satisfied on both intervals $[a_k, b_k]$, $k \in \{1, 2\}$. Firstly, it implies that $W(x) > 0$ on $[a_1, b_1] \cup [a_2, b_2]$. Since $W(x)$ is a continuous potential on $\mathbb{R}$, we have $W(x) > 0$ on $[a_1, b_1 + \varepsilon) \cup (a_2 - \varepsilon, b_2]$ for some small enough $\varepsilon > 0$. Secondly, from Theorem 2.1 applied to $[a_1, b_1]$ and $[a_2, b_2]$ simultaneously, we obtain that $u(x)$ has two points of local maximum $x_1 \in [a_1, b_1]$ and $x_2 \in [a_2, b_2]$ as well as x_1 (resp., x_2) is a unique stationary point on $[a_1, b_1]$ (resp., $[a_2, b_2]$). Hence, $u(x)$ attains its local minimum on $[b_1, a_2]$. On the other hand, we claim that

$$\text{there exists } x_0 \in (b_1 + \varepsilon, a_2 - \varepsilon) \text{ such that } W(x_0) < 0. \tag{2.5}$$

Indeed, if we suppose the contrary, then $W(x) \geq 0$ in $(b_1 + \varepsilon, a_2 - \varepsilon)$ and hence, $W(x) > 0$ on $J_\varepsilon := [x_1, b_1 + \varepsilon) \cup (a_2 - \varepsilon, x_2]$. Next, since $u'(x_1) = u'(x_2) = 0$, integrating equation (1.1) over $[x_1, x_2] \subset [a_1, b_2]$, as in the proof of Proposition 2.1, we obtain

$$0 \leq -\int_{x_1}^{x_2} W(x)u(x)\, dx. \tag{2.6}$$

Since $W(x) > 0$ on J_ε and $u(x) \geq 0$, from (H_0) and (2.6) it follows that $0 < 0$. Hence, $W(x)$ has to satisfy (2.5). Since $W(x)$ is supposed to be strictly positive on $[a_k, b_k]$, $k \in \{1, 2\}$, this implies that $W(x)$ has a negative minimum on $[b_1, a_2]$ and hence, $V(x) + g(x)$ attains a local maximum on $[b_1, a_2]$. Thus, we have shown the next result.

Theorem 2.3. *Suppose that* (1.2) *is satisfied and let* $[a_k, b_k]$, $k \in \{1, 2\}$ *be two disjoint intervals such that* (2.4) *hold. If* (H-basic) *and* (H-general) *are satisfied on* $[a_k, b_k]$, $k \in \{1, 2\}$, *then on the interval* $[b_1, a_2]$ *the ground state solution* $u(x)$ *has a local minimum and the potential* $V(x) + g(x)$ *attains a local maximum.*

In particular, for $g(x) \equiv 0$, Theorem 2.3 shows that $V(x)$ has to be necessarily a non-monotonic potential on $[b_1, a_2]$.

3 Proofs of main results

3.1 Some propositions

Before stating two propositions used in the proof of Theorem 2.1, we first state and prove the next

Proposition 3.1. *Every solution $u(x)$ of NLSE* (1.1) *which satisfies* $(H_{\neq 0})$ *has a stationary point in* $[a,b]$ *if and only if there is no any solution (v,R) of the first-order system*

$$\begin{cases} R' = 1 + R^2\Big[\Big(\mu - \dfrac{2m}{\hbar^2}\,V(x)\Big) + \dfrac{2m}{\hbar^2}\,f\big(x,|v(x)|^2\big)\Big] & \text{in } (a,b),\\ v' = \dfrac{1}{R(x)}\,v & \text{in } (a,b), \end{cases} \tag{3.1}$$

such that $v, R \in C([a,b]) \cap C^1(a,b)$, $v(x) \neq 0$ and $R(x) \neq 0$, $\forall x \in [a,b]$.

Proof. (Direction $\Longrightarrow$) Arguing by contradiction, let there exist a function $v \in C([a,b]) \cap C^1(a,b)$, $v(x) \neq 0$ on $[a,b]$ and a function $R \in C([a,b]) \cap C^1(a,b)$, $R(x) \neq 0$ on $[a,b]$ which satisfy the first-order system (3.1). Then

$$\begin{aligned} v''(x) &= \frac{v'(x)}{R(x)} - \frac{v(x)}{R^2(x)}\,R'(x)\\ &= \frac{v(x)}{R^2(x)}\,(1 - R'(x)) = -\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|v(x)|^2\big)\Big]v(x) \end{aligned}$$

and thus, $v(x)$ is a solution of NLSE (1.1) such that $v'(x) = v(x)/R(x) \neq 0$ on $[a,b]$. It contradicts the assumption that every solution of NLSE (1.1) has a stationary point in $[a,b]$.

(Direction $\Longleftarrow$) On the contrary, if $u(x)$ is a solution of NLSE (1.1) such that $u'(x) \neq 0$ on $[a,b]$, then the pair of functions $R(x) := u(x)/u'(x)$ and $v(x) := u(x)$ is the solution of system (3.1) such that $R(x) \neq 0$ and $u(x) \neq 0$ on $[a,b]$, because of $(H_{\neq 0})$ and

$$\begin{aligned} R'(x) &= 1 - \frac{u(x)}{u'^2(x)}\,u''(x)\\ &= 1 + \frac{u^2(x)}{u'^2(x)}\,\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|u(x)|^2\big)\Big]\\ &= 1 + R^2(x)\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|u(x)|^2\big)\Big]. \end{aligned}$$

This contradicts the assumption that (3.1) has no such a solution. It completes the proof of this proposition. □

In the absence of the strong assumption $(H_{\neq 0})$, we have the following essential proposition, which is weaker than Proposition 3.1, but it is used in the proof of the main result.

Proposition 3.2. *If for a function $v(x)$ there is no any solution $R \in C([a,b]) \cap C^1(a,b)$, $R = R(x)$ of the first-order differential equation*

$$R' = 1 + R^2\Big[\Big(\mu - \frac{2m}{\hbar^2}\,V(x)\Big) + \frac{2m}{\hbar^2}\,f\big(x,|v(x)|^2\big)\Big] \quad \text{in } (a,b), \tag{3.2}$$

then every solution $u(x)$ of NLSE (1.1) *has a stationary point in* $[a,b]$.

Proof. By contradiction, let $u(x)$ be a solution of (1.1) such that $u'(x) \neq 0$ for all $x \in [a,b]$. Then the function $R(x) = u(x)/u'(x)$ is well defined on $[a,b]$, $R \in C([a,b]) \cap C^1(a,b)$ and satisfies equation (3.2) with $v(x) = u(x)$ (because we can use the similar computation as in the proof of Proposition 3.1). This contradicts the main assumption of this lemma and hence, there exists $x_{\mathrm{s}} \in [a,b]$ such that $u'(x_{\mathrm{s}}) = 0$, which proves the proposition. □

Now we give a condition ensuring that $u(x)$ attains its local maximum at a stationary point.

Proposition 3.3. *Suppose that* (1.2) *holds and let* $x_{\mathrm{s}} \in [a,b]$ *be a stationary point of a solution* $u(x)$ *of NLSE* (1.1). *If* $u(x) \geq 0$ *on* $[a,b]$ *and satisfies* (H_0)*, and the potentials* μ*,* $V(x)$ *and* $g(x)$ *satisfy* (H-basic)*, then* x_{s} *is a unique stationary point of* $u(x)$*. Moreover,* $u(x)$ *attains a local maximum at* x_{s}*.*

Proof. Let $u(x) \geq 0$ and satisfy (H_0). Since all potentials in (H-basic) are continuous, there exists $\varepsilon > 0$ such that

$$\mu - \frac{2m}{\hbar^2}\left(V(x) + g(x)\right) > 0 \ \text{ in } \ (a-\varepsilon, b+\varepsilon). \tag{3.3}$$

Integrating (1.1) over $[x, x_{\mathrm{s}}]$, where $x \in (a-\varepsilon, x_{\mathrm{s}})$, and using (1.2), (H_0) and (3.3), we obtain

$$\begin{aligned} -u'(x) &= -\int_x^{x_{\mathrm{s}}} \Big(\mu - \frac{2m}{\hbar^2} V(\sigma)\Big) u(\sigma)\, d\sigma - \frac{2m}{\hbar^2} \int_x^{x_{\mathrm{s}}} f\big(\sigma, |u(\sigma)|^2\big) u(\sigma)\, d\sigma \\ &\leq -\int_x^{x_{\mathrm{s}}} \Big[\mu - \frac{2m}{\hbar^2}\left(V(\sigma) + g(\sigma)\right)\Big] u(\sigma)\, d\sigma < 0, \end{aligned}$$

which shows that $u'(x) > 0$ for all $x \in (a-\varepsilon, x_{\mathrm{s}})$. Analogously, integrating (1.1) over $[x_{\mathrm{s}}, x]$, where $x \in (x_{\mathrm{s}}, b+\varepsilon)$, we obtain

$$u'(x) \leq -\int_{x_{\mathrm{s}}}^{x} \Big[\mu - \frac{2m}{\hbar^2}\left(V(\sigma) + g(\sigma)\right)\Big] u(\sigma)\, d\sigma < 0,$$

which shows that $u'(x) < 0$ for all $x \in (x_{\mathrm{s}}, b+\varepsilon)$. Thus, $u(x)$ has a local maximum at the given stationary point x_{s}. The uniqueness of x_{s} immediately follows from Proposition 2.1. $\square$

3.2 Proof of Theorem 2.1

By Proposition 3.2 it is enough to show that the assumption (H-general) ensures that for any $v(x)$ there is no any solution $R(x)$, $R \in C([a,b]) \cap C^1(a,b)$ of equation (3.2). Indeed, if there exists such a solution, then multiplying (3.2) by $\varphi^2(x)$, where $\varphi \in C([a,b]) \cap C^1(a,b)$, $\varphi(x) \neq 0$ in (a,b), $\varphi(a) = \varphi(b) = 0$ and using (1.2), we obtain

$$\int_a^b \varphi^2(x)\, dx \leq -\int_a^b \left[\sqrt{Q(x)}\,\varphi(x) R(x) + \frac{\varphi'(x)}{\sqrt{Q(x)}}\right]^2 dx + \int_a^b \frac{\varphi'^2(x)}{Q(x)}\, dx,$$

where $Q(x) := \mu - \frac{2m}{\hbar^2}\left(V(x) + g(x)\right)$ and $Q(x) > 0$ on $[a,b]$ due to the assumption (H-basic). Previous inequality contradicts the main assumption of this theorem and hence, there is no any solution $R(x)$, $R \in C([a,b]) \cap C^1(a,b)$ of equation (3.2). Therefore, Proposition 3.2 gives the existence of a stationary point of $u(x)$ in $[a,b]$. Now, the rest of this proof immediately follows from Proposition 3.3.

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