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ON A NUMERICAL SOLUTION OF TWO-DIMENSIONAL NONLINEAR MITCHISON MODEL

Abstract. In the paper, for the construction of a numerical solution of two-dimensional Mitchison nonlinear partial differential system, the variable directions difference scheme and the difference scheme corresponding to the average method are used. Practical realization of those algorithms and comparative analysis of the obtained results are carried out. Numerical experiments are in accordance with theoretical findings. On the basis of experiments the corresponding tables of data are given.

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Key words and phrases. Nonlinear partial differential equations, numerical methods, economic algorithms, finite difference scheme, variable directions method, average method.

რეზიუმე. ნაშრომში მიჩისონის ორგანზომილებიან არაწრფივ კერძოწარმოებულებიან დიფერენციალურ განტოლებათა სისტემის რიცხვითი ამონახსნის ასაგებად გამოყენებულია ცვალებადი მიმართულებისა და გასაშუალებული მეთოდის შესაბამისი სხვაობიანი სქემები. განხორციელებულია ამ ალგორითმების პრაქტიკული რეალიზაცია და ჩატარებულია მიღებული შედეგების შედარებითი ანალიზი. რიცხვითი ექსპერიმენტების შედეგები შესაბამისობაშია თეორიულ კვლევებთან. ექსპერიმენტებზე დაყრდნობით მოცემულია შესაბამისი მონაცემების ცხრილები.

1 Introduction

Using the nonlinear partial differential equations, a lot of natural processes are described. Among them there is one of the important mathematical model that describes vein formation in the leaves of higher plants. This model was proposed by J. Michison [15].

The model proposed by Michison has the form:

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{\partial}{\partial x_1}\Big(D_1 \frac{\partial S}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(D_2 \frac{\partial S}{\partial x_2}\Big), \\ \frac{\partial D_i}{\partial t} &= f_i\Big(D_i, D_i \frac{\partial S}{\partial x_i}\Big), \quad i = 1, 2, \end{aligned} \tag{1.1}$$

where $S(t, x_1, x_2)$ is concentration of signal, D_1 and D_2 are diffusion coefficients to the Ox_1- and Ox_2-axis, respectively.

Some qualitative and structural properties of solutions of system (1.1) are established in [15]. Investigations for one-dimensional analogue of system (1.1) with two unknown functions S and D_1 are carried out in [2]. In [2,15] and [16], the authors pointed out on theoretical and practical importance of the investigation and construction of approximate solutions of the initial boundary value problems for systems (1.1). In biological modeling there are many other works where this and many models of similar processes are also presented and discussed (see, e.g., [3,6,7,16,19,20] and the references therein).

The complexity of model (1.1), besides of the nonlinearity, is caused by its two-dimensionality. In general, a numerical solution of multi-dimensional problems is often carried out by applying decomposition methods.

Investigations for one-dimensional analogue of system (1.1) were carried out in [2].

Starting from the basic works [4,18], the methods of constructing the effective algorithms for the numerical solution of the multi-dimensional problems of mathematical physics and the class of problems solvable with the help of those algorithms were essentially extended [8,14,21]. Those algorithms belong mainly to the methods of splitting-up or sum approximation. Some schemes of the variable directions are constructed and studied in [1], too.

Some questions of construction and investigation of the schemes of variable directions and the average model of sum approximation as well as the difference schemes for one-dimensional case for the system of type (1.1) are discussed in [5,9–13,17].

The paper is organized as follows. In Section 2, the statement of the problem is given. In Section 3, two economic difference schemes are constructed and the theorem of stability and convergence for the variable direction scheme is stated. Section 4 contains some results of numerical experiments. The brief conclusion in Section 5 ends the paper.

2 Statement of the problem

In the domain $Q = \Omega \times [0, T]$, where $\Omega = (0, 1) \times (0, 1)$, let us consider the certain function f and pose the following initial boundary value problem for the special case of two-dimensional system (1.1):

$$\begin{aligned} \frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\Big(V_1 \frac{\partial U}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(V_2 \frac{\partial U}{\partial y}\Big) &= 0, \\ \frac{\partial V_1}{\partial t} + V_1 - g_1\Big(V_1 \frac{\partial U}{\partial x}\Big) &= 0, \\ \frac{\partial V_2}{\partial t} + V_2 - g_2\Big(V_2 \frac{\partial U}{\partial y}\Big) &= 0 \end{aligned} \tag{2.1}$$

with initial

$$\begin{aligned} U(x, y, 0) &= U_0(x, y), \quad (x, y) \in \overline{\Omega}, \\ V_1(x, y, 0) &= V_{10}(x, y), \quad (x, y) \in \overline{\Omega}, \\ V_2(x, y, 0) &= V_{20}(x, y), \quad (x, y) \in \overline{\Omega}, \end{aligned} \tag{2.2}$$

and boundary conditions

$$U(x,y,t)=0,\ \ (x,y,t)\in\partial\Omega\times[0,T]. \tag{2.3}$$

Here g_α, U_0, $V_{\alpha 0}$, $\alpha=1,2$, are the given sufficiently smooth functions such that

$$\begin{gathered} V_{\alpha 0}\ge\delta_0,\ \ \delta_0=const>0,\ \ (x,y)\in\overline{\Omega},\\ g_0\le g_\alpha(\xi_\alpha)\le G_0,\ \ |g'_\alpha(\xi_\alpha)|\le G_1,\ \ \xi_\alpha\in R, \end{gathered} \tag{2.4}$$

where δ_0, g_0, G_0, G_1 are some positive constants.

3 Economic schemes

In the sequel, for the construction of the grid on the domain $\overline{Q}$ we follow the known notation:

$$\begin{gathered} \overline{\omega}_h=\big\{(x_i,y_j)=(ih,jh)\big\},\quad \overline{\omega}_{1h}=\Big\{(x_i,y_j)=\Big(\frac{i-1}{2}\,h,jh\Big)\Big\},\\ \overline{\omega}_{2h}=\Big\{(x_i,y_j)=\Big(ih,\Big(j-\frac{1}{2}\Big)h\Big)\Big\},\quad i,j=0,\dots,M,\ \ Mh=1,\\ \omega_h=\Omega\cap\overline{\omega}_h,\quad \gamma_h=\frac{\overline{\omega}_h}{\omega_h}\,,\quad \overline{\omega}_h=\omega_h\cup\gamma_h,\\ \omega_\tau=\big\{t_k=k\tau,\ k=0,\dots,N,\ N\tau=T\big\}. \end{gathered} \tag{3.1}$$

Following the known notation [21], let us correspond to problem (2.1)–(2.3) the following difference scheme of variable directions:

$$\begin{gathered} u_{1t}-(\widehat{v}_1\widehat{u}_{1\overline{x}})_x-(v_2u_{2\overline{y}})_y=0,\quad u_{2t}-(\widehat{v}_1\widehat{u}_{1\overline{x}})_x-(\widehat{v}_2\widehat{u}_{2\overline{y}})_y=0,\\ v_{1t}+\widehat{v}_1-g_1(v_1u_{1\overline{x}})=0,\quad v_{2t}+\widehat{v}_2-g_2(v_2u_{2\overline{y}})=0,\\ u_1(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ u_2(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ v_1(x,y,0)=V_{10},\ \ (x,y)\in\overline{\omega}_{1h},\\ v_2(x,y,0)=V_{20},\ \ (x,y)\in\overline{\omega}_{2h},\\ u_1(x,y,t)=u_2(x,y,t)=0,\\ (x,y,t)\in\gamma_h\times\omega_\tau. \end{gathered} \tag{3.2}$$

Using the continuous variant of the averaged model of sum approximation [5], we correspond to problem (2.1)–(2.3) the following decomposition finite difference scheme:

$$\begin{gathered} u_{1t}-(\widehat{v}_1\widehat{u}_{1\overline{x}})_x=0,\quad u_{2t}-(\widehat{v}_2\widehat{u}_{2\overline{y}})_y=0,\\ v_{1t}+\widehat{v}_1-g_1(v_1u_{1\overline{x}})=0,\quad v_{2t}+\widehat{v}_2-g_2(v_2u_{2\overline{y}})=0,\\ u_1(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ u_2(x,y,0)=U_0(x,y),\ \ (x,y)\in\overline{\omega}_h,\\ v_1(x,y,0)=V_{10},\ \ (x,y)\in\overline{\omega}_{1h},\\ v_2(x,y,0)=V_{20},\ \ (x,y)\in\overline{\omega}_{2h},\\ u_1(x,y,t)=u_2(x,y,t)=0,\\ (x,y,t)\in\gamma_h\times\omega_\tau,\\ u=\eta_1u_1+\eta_2u_2,\ \ \eta_1>0,\ \ \eta_2>0,\ \ \eta_1+\eta_2=1. \end{gathered} \tag{3.3}$$

Let us introduce the following notation for the errors: $Z_1=u_1-U$, $Z_2=u_2-U$, $S_1=v_1-V_1$, $S_2=v_2-V_2$.

Theorem. *If problem* (2.1)–(2.3) *has a sufficiently smooth solution, then the finite difference scheme* (3.2) *is stable, its solution converges to the exact solution of problem* (2.1)–(2.3) *as $\tau\to0$, $h\to0$, and the inequality*

$$\|Z_1\|_{\overline{\omega}_h}+\|Z_2\|_{\overline{\omega}_h}+\|S_1\|_{\overline{\omega}_{1h}}+\|S_2\|_{\overline{\omega}_{2h}}\le C(\tau+h^2)$$

holds.

Table 1. CPU time and error for solution u, v_1, v_2 applying scheme of variable directions (3.2).

t	CPU time	Error u	Error v_1	Error v_2
0.2	0.074	0.00013912790131447	0.00000712766408961	0.00002916084998672
0.4	0.148	0.00022425859907783	0.00001730244454379	0.00009005618525060
0.6	0.224	0.00031286373416026	0.00004804529821700	0.00017715471240609
0.8	0.301	0.00040788793632886	0.00009668298990784	0.00028758192640277
1.0	0.378	0.00051151056363487	0.00016425091499817	0.00041715052893787

Table 2. CPU time and error for solution u, v_1, v_2 applying difference scheme (3.3) corresponding to averaged method.

t	CPU time	Error u	Error v_1	Error v_2
0.2	0.072	0.00006973950435170	0.00001634140038553	0.00001662571352523
0.4	0.146	0.00007422011594080	0.00003786305693865	0.00003781271060488
0.6	0.221	0.00007890208614024	0.00006202878270467	0.00005790906416947
0.8	0.295	0.00008480943243865	0.00008875495749039	0.00007978157763566
1.0	0.369	0.00009205402490850	0.00011818090303972	0.00010625023389577

Here C is a positive constant independent of τ and h, the norms are discrete analogous of the norm of space L_2.

4 Numerical experiments

Using the algorithms proposed in (3.2) and (3.3), let us carry out comparative analysis of the numerical results for the above schemes.

Let us take

$$g_1(\xi) = g_2(\xi) = \frac{1}{1+(1+\xi)^2}$$

and choose the right-hand sides of the corresponding nonhomogeneous system (2.1) so that the solution of problem (2.1)–(2.3) is:

$$\begin{aligned} U &= xy(1-x)(1-y)(1+t), \\ V_1 &= 1 + xy(1-x)(1-y)(1+t+t^2), \\ V_2 &= 1 + xy(1-x)(1-y)(1+t+t^3). \end{aligned}$$

CPU time and errors for the variable directions difference scheme (3.2) are given in Table 1 and the CPU time and errors for scheme (3.3) are given in Table 2.

The approximation error for the variable direction difference scheme (3.2) is smaller compared with the scheme (3.3). However, CPU time is better for scheme (3.3) than for scheme (3.2).

Table 3. Absolute value of maximum errors and rate of convergence with respect to τ and h for the function u.

τ	h	Error	Rate of τ	Rate of h
0.00125	0.05	0.00024074087939129	0.99175505389520200	1.98351010779040000
0.0008	0.04	0.00015728407178949	0.98629676971885200	1.97259353943770000
0.0003125	0.025	0.00006418736860213	0.99204420486615900	1.98408840973232000
0.0002	0.02	0.00004172715815061	0.99421791633935300	1.98843583267871000
0.00005	0.01	0.00001084525005050		

Table 4. Absolute value of maximum errors and rate of convergence with respect to τ and h for the function v_1.

τ	h	Error	Rate of τ	Rate of h
0.00125	0.05	0.00015579938599405	0.99768312053704900	1.99536624107410000
0.0008	0.04	0.00009981476150336	0.99879576821464700	1.99759153642929000
0.0003125	0.025	0.00003903430252067	0.99941764475219500	1.99883528950439000
0.0002	0.02	0.00002498844720772	0.99974995294653000	1.99949990589306000
0.00005	0.01	0.00002498844720772		

Table 5. Absolute value of maximum errors and rate of convergence with respect to τ and h for the function v_2.

τ	h	Error	Rate of τ	Rate of h
0.00125	0.05	0.00015579938599405	0.99732714185756800	1.99465428371514000
0.0008	0.04	0.00009981476150336	0.99873489122313100	1.99746978244626000
0.0003125	0.025	0.00003903430252067	0.99935953799193100	1.99871907598386000
0.0002	0.02	0.00002498844720772	0.99972504350513300	1.99945008701027000
0.00005	0.01	0.00002498844720772		

In Tables 3–5 we also computed errors for different values of time and space steps applying scheme (3.2) for $T = 1$ and obtained the rates of convergence confirming the theoretical result in theorem from the previous section.

5 Conclusion

Numerous numerical experiments are performed for problem (2.1)–(2.3) by using schemes (3.2) and (3.3). The approximation errors for the variable direction difference scheme (3.2) are smaller compared with scheme (3.3), but CPU time is better for scheme (3.3) than for scheme (3.2). We have carried out various numerical experiments and calculated the absolute value of maximum errors for different time and space steps and obtained the rate of convergence of scheme (3.2). In all cases, the numerical results fully agree with the theoretical ones.

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