Memoirs on Differential Equations and Mathematical Physics

Volume 75, 2018, 93–104

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EXACT CONDITIONS FOR THE EXISTENCE OF HOMOCLINIC ORBITS IN THE LIÉNARD SYSTEMS

Abstract. We consider the Liénard system $\dot{x} = y - F(x)$ and $\dot{y} = -g(x)$. Under the assumptions that the origin is a unique equilibrium, we investigate the existence of homoclinic orbits of this system which is closely related to the stability of the zero solution, center problem, global attractively of the origin, and oscillation of solutions of the system. We present the necessary and sufficient conditions for this system to have a positive orbit which starts at a point on the vertical isocline $y = F(x)$ and approaches the origin without intersecting the x-axis. Our results solve the problem completely in some sense.

2010 Mathematics Subject Classification. Primary 37C29; Secondary 34A12.

Key words and phrases. Homoclinic orbit, Liénard system, oscillation.

რეზიუმე. განვიხილავთ ლიენარდის სისტემას $\dot{x} = y - F(x)$ და $\dot{y} = -g(x)$. იმ დაშვებით, რომ სათავე არის ერთადერთი წონასწორობის წერტილი, ვსწავლობთ ამ სისტემის ჰომოკლინური ორბიტების არსებობას, რაც მჭიდროდ არის დაკავშირებული ნულოვანი ამოცანის მდგრადობასთან, ცენტრის პრობლემასთან, სათავის გლობალურ მიზიდულობასთან და სისტემის ამონახსნთა რხევადობასთან. მოყვანილია აუცილებელი და საკმარისი პირობები, რათა ამ სისტემას გააჩნდეს დადებითი ორბიტები, რომლებიც იწყება $y = F(x)$ ვერტიკალური იზოკლინის წერტილში და უახლოვდება სათავეს ისე, რომ არ გადაკვეთს x ღერძს. შედეგები გარკვეული თვალსაზრისით სრულად ხსნის დასმულ ამოცანას.

1 Introduction

It is well known that the Liénard system

$$\begin{aligned}\frac{dx}{dt} &= y - F(x),\\ \frac{dy}{dt} &= -g(x),\end{aligned} \tag{1.1}$$

is of great importance in various applications. Hence, asymptotic and qualitative behavior of this system and some of its extensions have been widely studied by many authors; results can be found in many books and papers [1–22]. In system (1.1), a trajectory is said to be a homoclinic orbit if its α- and ω-limit sets are the origin. The existence of homoclinic orbits in the Liénard-type systems (see [5]) is closely connected with the stability of the zero solution and the center problem. If system (1.1) has a homoclinic orbit, then the zero solution is no longer stable. A homoclinic orbit and a center cannot exist together in system (1.1). Our subject also has a near relation to the global attractivity of the origin and oscillation of solutions (see [9, 11]).

Taking the vector field of (1.1) into account, we see that every homoclinic orbit is in the upper or in the lower half-plane. In other words, no homoclinic orbit crosses the x-axis. When a homoclinic orbit appears in the upper (resp. lower) half-plane, all other homoclinic orbits exist in the same half-plane.

We say that system (1.1) has property (Z_1^+) (resp. (Z_3^+)) if there exists a point $P(x_0, y_0)$ with $y_0 = F(x_0)$ and $x_0 > 0$ (resp. $x_0 < 0$) such that the positive semitrajectory of (1.1) starting at P approaches the origin through only the first (resp. third) quadrant. We also say that system (1.1) has property (Z_2^-) (resp. (Z_4^-)) if there exists a point $P(x_0, y_0)$ with $y_0 = F(x_0)$ and $x_0 < 0$ (resp. $x_0 > 0$) such that the negative semitrajectory of (1.1) starting at P approaches the origin through only the second (resp. fourth) quadrant. If system (1.1) has both properties (Z_1^+) and (Z_2^-), then a homoclinic orbit exists in the upper half-plane. Similarly, if system (1.1) has both properties (Z_3^+) and (Z_4^-), then a homoclinic orbit exists in the lower half-plane. Notice that by the transformation $x \to -x$ and $t \to -t$, we can transfer any result for property (Z_1^+) to an analogous result with respect to property (Z_2^-). Also, by the transformation $x \to -x$ and $y \to -y$, we can transfer any result for property (Z_1^+) (resp. (Z_2^-)) to an analogous result with respect to property (Z_3^+) (resp. (Z_4^-)).

In this paper, we intend to give some conditions on $F(x)$ and $g(x)$ under which system (1.1) has properties (Z_1^+), (Z_2^-), (Z_3^+), or (Z_4^-). We assume that F and g are continuous on an open interval I which contains 0 and satisfy smoothness conditions for uniqueness of solutions of the initial value problems. We also assume that $F(0) = 0$ and

$$xg(x) > 0 \ \text{ for } \ x \neq 0,$$

which guarantee that the origin is the unique equilibrium of (1.1). Throughout this paper, in the results related to property (Z_1^+) (resp. (Z_2^-)), we assume that $F(x) > 0$ for $x > 0$ (resp. $x < 0$),$|x|$ sufficiently small. Because if $F(x)$ has an infinite number of positive (resp. negative) zeroes clustering at $x = 0$, then the system (1.1) fails to have property (Z_1^+) (resp. (Z_2^-)). Similarly, in the results related to property (Z_3^+) (resp. (Z_4^-)), we assume that $F(x) < 0$ for $x < 0$ (resp. $x > 0$), $|x|$ sufficiently small.

T. Hara and T. Yoneyama [10] considered system (1.1) and proved that if there exists $\delta > 0$ such that

$$F(x) > 0, \qquad \frac{1}{F(x)} \int_0^x \frac{g(\eta)}{F(\eta)}\, d\eta \leq \frac{1}{4}$$

for $0 < x < \delta$, then system (1.1) has property (Z_1^+). They also proved that if there exist $a > 0$ such that $F(x) > 0$ for $0 < x \leq a$ and some $\alpha > \frac{1}{4}$ such that

$$\frac{1}{F(x)} \int_0^x \frac{g(\eta)}{F(\eta)}\, d\eta \geq \alpha,$$

then system (1.1) fails to have property (Z_1^+) (see also [6, 9, 15, 19]).

In this paper, we present an implicit necessary and sufficient condition for system (1.1) to have property (Z_1^+). Then we drive sharp explicit conditions and solve this problem completely in some sense. We formulate similar results for properties (Z_2^-), (Z_3^+), and (Z_4^-).

The paper is organized as follows. In Section 2, we give implicit conditions for system (1.1) to have property (Z_1^+). In Section 3, we use our results obtained in Section 2 and present sufficient conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-). In Section 4, we present the necessary conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-) and show that the sufficient conditions presented in Section 3 are best possible.

2 Implicit conditions for property (Z_1^+)

In this section we present implicit conditions for system (1.1) to have property (Z_1^+). First, we introduce a system which is equivalent to (1.1). Let the function $\lambda(x)$ be defined by

$$\lambda(x) = \begin{cases} \sqrt{2G(x)} & \text{for } x \geq 0, \\ -\sqrt{2G(x)} & \text{for } x < 0 \end{cases}$$

and the mapping $\Lambda : R^2 \to R^2$ by

$$\Lambda(x, y) = (\lambda(x), y) \equiv (u, v).$$

Consider the canonical form of the Liénard systems

$$\begin{aligned} \frac{du}{d\tau} &= v - F^*(u), \\ \frac{dv}{d\tau} &= -u, \end{aligned} \tag{2.1}$$

in which $d\tau = [g(x)\operatorname{sgn}(x)/\sqrt{2G(x)}]\, dt$ and a continuous function F^* is defined by

$$F^*(u) = \begin{cases} F(G^{-1}\Big(\frac{1}{2}\,u^2\Big) & \text{if } u \geq 0, \\ F(G^{-1}\Big(-\frac{1}{2}\,u^2\Big) & \text{if } u < 0, \end{cases}$$

where $G^{-1}(w)$ is the inverse function to $G(x)\operatorname{sgn}(x)$. Then the mapping Λ is a homeomorphism of the (x, y)-plane onto an open subset of the (u, v)-plane which contains zero. It is obvious that Λ maps the x-axis into the u-axis. Consequently, we have only to determine whether system (2.1), instead of (1.1), has property (Z_1^+) or not. Hereafter we denote τ by t again.

Theorem 2.1. *Let $F^* \in C^1([0, \alpha])$ for some $\alpha > 0$. Then system (2.1) has property (Z_1^+) if and only if there exist a constant $b \leq \alpha$ and a function $\varphi \in C^1([0, b])$ such that $\varphi(0) = 0$,*

$$\varphi(u) > 0, \quad (F^*)'(u) \geq \frac{u}{\varphi(u)} + \varphi'(u) \;\; \textit{for} \;\; 0 < u \leq b. \tag{2.2}$$

Proof. Sufficiency. Consider the positive semitrajectory of (2.1) starting at a point $(b, F^*(b))$. This trajectory is considered as a solution $v(u)$ of

$$\frac{dv}{du} = -\frac{u}{v - F^*(u)} \tag{2.3}$$

with $v(b) = F^*(b)$. Suppose that the positive semitrajectory $v(u)$ crosses the negative y-axis. Then it also meets the curve $v = F^*(u) - \varphi(u)$ at a point $(s, F^*(s) - \varphi(s))$ with $s < b$ such that

$$\frac{dv}{du}(s) = \frac{-s}{(F^*(s) - \varphi(s)) - F^*(s)} > (F^*)'(s) - \varphi'(s).$$

Thus

$$(F^*)'(s) < \frac{s}{\varphi(s)} + \varphi'(s).$$

This is a contradiction. Hence, the trajectory $v(u)$ does not cross the negative y-axis, and, therefore, system (2.1) has property (Z_1^+).

Necessity. Suppose that system (2.1) has property (Z_1^+). Then there exists a positive semitrajectory of (2.1) starting at a point $(b, F^*(b))$ with $b > 0$, which does not meet the negative y-axis. This trajectory can be regarded as the graph of a continuously differentiable function $\psi(u)$ which is a solution of (2.3). Let $\varphi(u) = F^*(u) - \psi(u)$. Then it is clear that $\varphi(0) = 0$,

$$\varphi(u) > 0, \quad (F^*)'(u) = \frac{u}{\varphi(u)} + \varphi'(u) \ \text{ for } \ 0 < u \leq b.$$

Hence, the condition (2.2) is verified. □

Theorem 2.2. *Suppose that system* (2.1) *with* F_1 *has property* (Z_1^+). *If*

$$F_2(u) \geq F_1(u) \tag{2.4}$$

for $u > 0$ *sufficiently small, then system* (2.1) *corresponding to* F_2 *has property* (Z_1^+).

Proof. Since system (2.1) with $F_1(u)$ has property (Z_1^+), there exists a positive semitrajectory of (2.1) starting at a point (u_0, v_0) with $u_0 > 0$, which approaches the origin through only the first quadrant. This trajectory can be regarded as the graph of a function $v = \psi_1(u)$ which is a solution of (2.3). Let $v = \psi_2(u)$ be the graph of the solution of system (2.3) corresponding to F_2 such that $(u(0), v(0)) = (u_0, v_0)$. We can assume that u_0 is sufficiently small, thus from (2.4) we have

$$\psi_2'(u) = \frac{-u}{v - F_2(u)} \leq \frac{-u}{v - F_1(u)} = \psi_1'(u) \ \text{ for } \ 0 < u \leq u_0.$$

Hence, $\psi_2(u) \geq \psi_1(u) > 0$ for $0 < u \leq u_0$. Therefore, system (2.1) corresponding to F_2 has property (Z_1^+). □

3 Explicit sufficient conditions for property (Z_1^+)

In this section we use our implicit conditions to drive explicit sufficient conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-). To this end, for $u > 0$ sufficiently small we define

$$L_1(u) = \log ku$$

and

$$L_n(u) = \log ku \times \log(b|\log ku|) \times \cdots \times \underbrace{\log\log\cdots\log}_{(n-1)\text{-times}}\big(b|\log ku|\big) \ \text{ for } \ n \geq 2,$$

where $k, b > 0$. Notice that $L_n(u) < 0$ for $u > 0$ sufficiently small.

Theorem 3.1. *Let* $k, b > 0$. *If*

$$F^*(u) \geq 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2}$$

for some $n \geq 2$ *and* $u > 0$ *sufficiently small, then system* (2.1) *has property* (Z_1^+).

Proof. By Theorem 2.2, it suffices to prove the theorem when

$$F^*(u) = 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2}\,.$$

Let

$$M_n(u)=\sum_{j=1}^{n-1}\Big(\frac{1}{L_j(u)}\sum_{i=1}^{j}\frac{1}{L_i(u)}\Big), \tag{3.1}$$

$$N_n(u)=\sum_{j=1}^{n-1}\frac{1}{L_j(u)}\,,\quad \varphi_n(u)=u+\frac{1}{2}\,uN_{n+1}(u). \tag{3.2}$$

We have

$$u\,\frac{d}{du}\,(L_n(u))=N_n(u)L_n(u)+1,\quad 2M_n(u)-(N_n(u))^2=\sum_{j=1}^{n-1}\frac{1}{(L_j(u))^2}$$

and

$$\frac{d}{du}\,(N_n(u))=-\frac{M_n(u)}{u}\,.$$

Thus

$$\frac{u}{\varphi_n(u)}+\varphi_n'(u)=2-\frac{1}{4(1+\frac{1}{2}N_{n+1}(u))}\Big(\sum_{j=1}^{n}\frac{1}{(L_j(u))^2}+N_{n+1}(u)M_{n+1}(u)\Big),$$

or

$$\frac{u}{\varphi_n(u)}+\varphi_n'(u)=2-\frac{1}{4}\sum_{j=1}^{n}\frac{1}{(L_j(u))^2}-\frac{(N_{n+1}(u))^3}{8(1-\frac{1}{2}N_{n+1}(u))} \tag{3.3}$$

for $u>0$ sufficiently small. On the other hand,

$$(F^*)'(u)=2-\frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{(L_j(u))^2}+\frac{1}{2}\sum_{j=1}^{n-1}\frac{N_j(u)L_j(u)+1}{(L_j(u))^3}\,. \tag{3.4}$$

It is easy to check that

$$(F^*)'(u)>\frac{u}{\varphi_n(u)}+\varphi_n'(u)$$

for $u>0$ sufficiently small. Hence, (2.2) holds and, by Theorem 2.1, system (2.1) has property (Z_1^+). □

Recall defining the function $F^*(u)$ as follows:

$$F^*(u)=F\Big(G^{-1}\Big(\frac{1}{2}u^2\Big)\Big)\ \text{ for }\ u\ge 0.$$

Put $x=G^{-1}(\frac{1}{2}u^2)$. Then for system (1.1) to have property (Z_1^+) we have the following sufficient condition.

Theorem 3.2. *Assume $k,b>0$. If*

$$F(x)\ge\sqrt{8G(x)}-\frac{1}{4}\sum_{j=1}^{n-1}\frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2}$$

for some $n\ge 2$ and $x>0$ sufficiently small, then system (1.1) has property (Z_1^+).

Similarly, for system (1.1) to have properties (Z_2^-), (Z_3^+), and (Z_4^-), we have the following sufficient conditions.

Theorem 3.3. *Assume $k,b>0$. If*

$$F(x)\ge\sqrt{8G(x)}-\frac{1}{4}\sum_{j=1}^{n-1}\frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2}$$

for some $n\ge 2$ and $x<0$, $|x|$ sufficiently small, then system (1.1) has property (Z_2^-).

Theorem 3.4. *Assume $k, b > 0$. If*

$$F(x) \le -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2}\,,$$

for some $n \ge 2$ and $x < 0$, $|x|$ sufficiently small, then system (1.1) *has property* (Z_3^+).

Theorem 3.5. *Assume $k, b > 0$. If*

$$F(x) \le -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2},$$

for some $n \ge 2$ and $x > 0$ sufficiently small, then system (1.1) *has property* (Z_4^-).

4 Explicit necessary conditions for property (Z_1^+)

In this section we drive explicit necessary conditions for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-) and show that the sufficient conditions presented in Section 2 are best possible.

Definition 4.1. Let $f_1(u)$ and $f_2(u)$ be real-valued functions. By $f_1(u) \preceq f_2(u)$ we mean that there exists $b > 0$ such that $f_1(u) \le f_2(u)$ for $0 < u \le b$.

In proving Theorem 4.1 we will need the following

Lemma 4.1. *Suppose that $\varphi \in C^1([0, \alpha])$ for some $\alpha > 0$, $\varphi(0) = 0$, and $\varphi(u) > 0$ for $u > 0$ sufficiently small. If*

$$\frac{d}{du}\left(2u - \frac{1}{4}\sum_{j=1}^{n-1}\frac{u}{(L_j(u))^2} - \frac{\lambda u}{(L_n(u))^2}\right) \ge \frac{u}{\varphi(u)} + \varphi'(u), \quad \lambda \ge \frac{1}{4}\,, \tag{4.1}$$

for some $n \ge 2$, $k > 0$, $b > 0$, and $u > 0$ sufficiently small, then

(i) $\lim\limits_{u\to 0^+} \frac{\varphi(u)}{u} = 1,$

(ii) $|\frac{\varphi(u)-u}{u}| \le \frac{1}{|\log ku|}$ *for every $k > 0$ and $u > 0$ sufficiently small.*

Proof. It is easy to check that the left-hand side of inequality (4.1) tends to 2 as $u \to 0^+$. Thus, from (4.1) we get

$$\lim_{u\to 0^+}\Big(\frac{u}{\varphi(u)} + \varphi'(u)\Big) = \frac{1}{\varphi'(0^+)} + \varphi'(0^+) \le 2.$$

Hence,

$$\lim_{u\to 0^+}\frac{\varphi(u)}{u} = \varphi'(0^+) = 1.$$

This completes the proof of (i). Now let $\varphi(u) = u + h(u)$. Then we have

$$-\Big(\frac{u}{\varphi(u)} + \varphi'(u)\Big) = -2 + \frac{h(u)}{u + h(u)} - h'(u). \tag{4.2}$$

From (4.1) and (4.2) we conclude that

$$\frac{h(u)}{u+h(u)} - h'(u) > 0 \tag{4.3}$$

for u sufficiently small. Suppose that $\{u_n\}$ tends to zero and $h(u_n) = 0$, then there exists a sequence $\{c_n\}$ such that c_n tends to zero as $n \to \infty$, $h'(c_n) = 0$, and $h(c_n) \le 0$. This contradicts (4.3). Hence,

$h(u)$ is positive or negative for $u > 0$ sufficiently small, and we can let $h(u) = \frac{u}{f(u)}$ for $0 < u \le c$ with c sufficiently small. Notice that, by (i), $|f(u)| \to \infty$ as $u \to 0$. Since $\varphi(u) > 0$ for u sufficiently small,

$$\frac{f(u)+1}{f(u)} = \frac{\varphi(u)}{u} > 0. \tag{4.4}$$

Thus, from (4.3) and (4.4) we have

$$f'(u)\Big(\frac{f(u)+1}{f(u)}\Big) > \frac{1}{u}$$

for $0 < u \le b$ with b sufficiently small. Integration of the above leads to

$$f(u) + \log(|f(u)|) - f(b) - \log(|f(b)|) \le \log(u) - \log(b)$$

for $0 < u \le b$. Hence, $f(u) \to -\infty$ as $u \to 0^+$, and $|f(u)| > |\log ku|$ for every $k > 0$ and $u > 0$ sufficiently small. □

Theorem 4.1. *Suppose that there exist $\lambda > 1/4$, $n \ge 2$, and $k, b > 0$ such that*

$$F^*(u) \le 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2} - \frac{\lambda u}{(L_n(u))^2}$$

for $u > 0$ sufficiently small. Then system (2.1) fails to have property (Z_1^+).

Proof. By Theorem 2.2, it suffices to prove the theorem when

$$F^*(u) = 2u - \frac{1}{4}\sum_{j=1}^{n-1} \frac{u}{(L_j(u))^2} - \frac{\lambda u}{(L_n(u))^2}\,, \quad \lambda > \frac{1}{4}\,,$$

for $u > 0$ sufficiently small. We prove the theorem by contradiction. Suppose that there exists a continuously differentiable function φ such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ sufficiently small, and

$$(F^*)'(u) \succeq \frac{u}{\varphi(u)} + \varphi'(u). \tag{4.5}$$

Let

$$h(u) = \varphi(u) - \varphi_{n-1}(u) = \varphi(u) - u\Big(1 + \frac{1}{2}\,N_n(u)\Big).$$

From (4.5), (3.3), and (3.4) we have

$$\begin{aligned} \frac{u}{\varphi_{n-1}(u)} - \frac{u}{\varphi_{n-1}(u)+h(u)} - h'(u) &\succeq \frac{u}{\varphi_{n-1}(u)} + \varphi'_{n-1}(u) - (F^*)'(u) \\ &= \frac{\lambda}{(L_n(u))^2} - \Big(2\lambda + \frac{1}{2}\Big)\sum_{j=1}^{n-1} \frac{N_j(u)L_j(u)+1}{(L_j(u))^3} - \frac{(N_{n+1}(u))^3}{8(1-\frac{1}{2}N_{n+1}(u))}\,. \end{aligned}$$

Then

$$\frac{\lambda'}{(L_n(u))^2} \preceq \frac{u}{\varphi_{n-1}(u)} - \frac{u}{\varphi_{n-1}(u)+h(u)} - h'(u), \tag{4.6}$$

where $1/4 < \lambda' < \lambda$. Suppose that $\{u_n\}$ tends to zero and $h(u_n) = 0$, then there exists a sequence $\{c_n\}$ such that c_n tends to zero as $n \to \infty$, $h'(c_n) = 0$, and $h(c_n) \le 0$. This contradicts (4.6). Hence, $h(u) \ne 0$ for $x > 0$ sufficiently small, and we can let $f(u) = \frac{u}{h(u)}$ for $0 < u \le c$ with c sufficiently small. From (4.5), Lemma 4.1, and the fact that $|N_n(u)| \preceq \frac{2}{|\log ku|}$, we conclude that

$$\frac{1}{|f(u)|} = |\frac{\varphi(u)-u}{u} - \frac{N_n(u)}{2}| \le \frac{2}{|\log ku|} \tag{4.7}$$

for $u>0$ sufficiently small.

Let

$$T_n(u)=\Big(1+\frac{N_n(u)}{2}\Big)\Big(1+\frac{N_n(u)}{2}+\frac{1}{f(u)}\Big)$$

and

$$g(u)=\frac{f(u)}{L_n(u)}\,.$$

Then from (3.2) and (4.6) we have

$$\frac{\lambda'}{(L_n(u))^2}\preceq\frac{1}{1+\frac12 N_n(u)}-\frac{1}{1+\frac12\,N_n(u)+\frac{1}{f(u)}}-\frac{f(u)-f'(u)u}{f^2(u)}=\frac{1}{f(u)T_n(u)}-\frac{1}{f(u)}+\frac{f'(u)u}{f^2(u)}\,.$$

Hence,

$$\lambda'\preceq\frac{L_n(u)}{g(u)T_n(u)}-\frac{L_n(u)}{g(u)}+\frac{(g(u)L_n(u))'u}{g^2(u)}\,. \tag{4.8}$$

Notice that $u(L_n(u))'=N_n(u)L_n(u)+1$, thus, from (4.8),

$$\lambda' g^2(u)\preceq g'(u)uL_n(u)+g(u)L_n(u)\Big(\frac{1-T_n(u)+N_n(u)T_n(u)}{T_n(u)}\Big)+g(u),$$

or

$$\begin{aligned}&\Big(\lambda'-\frac14\Big)g^2(u)+\Big(\frac{g(u)}{2}-1\Big)^2\\ &\quad\preceq g'(u)uL_n(u)+\Big(1-\frac{1}{T_n(u)}\Big)-\frac{N_n(u)}{2T_n(u)}-\frac{g(u)\big(N_n(u)L_n(u)(1-T_n(u))+\frac{(N_n(u))^2}{4}L_n(u)\big)}{T_n(u)}\,.\end{aligned}$$

Now, let

$$A(u)=-\frac{(N_n(u)L_n(u)(1-T_n(u))+\frac{(N_n(u))^2}{4}\,L_n(u))}{T_n(u)}$$

and

$$B(u)=1-\frac{1}{T_n(u)}-\frac{N_n(u)}{2T_n(u)}\,.$$

It is easy to check that

$$\lim_{u\to0^+}(1-T_n(u))=\lim_{u\to0^+}(N_n(u))^2L_n(u)=0.$$

Also, by (4.7), we conclude that

$$\lim_{u\to0^+}N_n(u)L_n(u)(1-T_n(u))=0,$$

thus, $A(u)$ and $B(u)$ tend to 0 as $u\to0^+$, and we have

$$\Big(\lambda'-\frac14\Big)g^2(u)+\Big(\frac{g(u)}{2}-1\Big)^2\preceq g'(u)uL_n(u)+A(u)g(u)+B(u),\quad \lambda'>\frac14\,, \tag{4.9}$$

and

$$\Big(\frac{g(u)}{2}-1\Big)^2\preceq g'(u)uL_n(u)+A(u)g(u)+B(u). \tag{4.10}$$

We now prove that if (4.10) holds, then

$$\lim_{u\to0^+}g(u)=2. \tag{4.11}$$

Suppose $u_n>0$ tends to zero and $g'(u_n)=0$. Then from (4.10) we conclude that

$$\lim_{n\to\infty}g(u_n)=2.$$

Since g' vanishes at the extremum points, if $g(u)$ is not increasing or decreasing for $u>0$ sufficiently small, then

$$\liminf_{u\to 0^+} g(u) = \limsup_{u\to 0^+} g(u) = 2,$$

and (4.11) holds. Suppose now that $g(u)$ is increasing or decreasing for $u>0$ sufficiently small. If $\lim\limits_{u\to 0^+} g(u)\neq 2$, then from (4.10) we conclude that there exists $c>0$ such that

$$\frac{c}{uL_n(u)} > \frac{g'(u)}{(\frac{g(u)}{2}-1)^2}$$

for $0<u\le l$ with l sufficiently small. Integration of the above leads to

$$c\Big(\underbrace{\log\log\cdots\log}_{(n-1)\text{-times}}\big(b|\log kl|\big) - \underbrace{\log\log\cdots\log}_{(n-1)\text{-times}}\big(b|\log ku|\big)\Big) > \frac{-2}{\frac{g(l)}{2}-1} + \frac{2}{\frac{g(u)}{2}-1}$$

and, therefore, $\lim\limits_{u\to 0^+} g(u) = 2$. This is a contradiction, thus $\lim\limits_{u\to 0^+} g(u) = 2$. But if $\lim\limits_{u\to 0^+} g(u) = 2$, then from (4.9) we conclude that there exists $d>0$ such that

$$g'(u) \le \frac{d}{uL_n(u)}$$

for $u>0$ sufficiently small. Hence, $\lim\limits_{u\to 0^+} g(u) = -\infty$. This is a contradiction and condition (2.2) does not hold. Thus, by Theorem 2.1, system (2.1) fails to have property (Z_1^+). □

The following theorem gives a necessary condition for system (1.1) to have property (Z_1^+).

Theorem 4.2. *If there exist $\lambda > 1/4$, $n\ge 2$, and $k,b>0$ such that*

$$F(x) \le \sqrt{8G(x)} - \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} - \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x>0$ sufficiently small, then system (1.1) *fails to have property (Z_1^+).*

Similarly, we have the following necessary conditions for the properties (Z_2^-), (Z_3^+), and (Z_4^-).

Theorem 4.3. *If there exist $\lambda > 1/4$, $n\ge 2$, and $k,b>0$ such that*

$$F(x) \le \sqrt{8G(x)} - \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} - \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x<0$, $|x|$ sufficiently small, then system (1.1) *fails to have property (Z_2^-).*

Theorem 4.4. *If there exist $\lambda > 1/4$, $n\ge 2$, and $k,b>0$ such that*

$$F(x) \ge -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} + \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x<0$, $|x|$ sufficiently small, then system (1.1) *fails to have property (Z_3^+).*

Theorem 4.5. *If there exist $\lambda > 1/4$, $n\ge 2$, and $k,b>0$ such that*

$$F(x) \ge -\sqrt{8G(x)} + \frac{1}{4}\sum_{j=1}^{n-1} \frac{\sqrt{2G(x)}}{(L_j(\sqrt{2G(x)}))^2} + \frac{\lambda\sqrt{2G(x)}}{(L_n)(\sqrt{2G(x)})^2}$$

for $x>0$ sufficiently small, then system (1.1) *fails to have property (Z_4^-).*

Remark 4.1. Paying attention to the explicit sufficient and necessary conditions presented for properties (Z_1^+), (Z_2^-), (Z_3^+), and (Z_4^-), it seems that these results have solved the problem of the existence of homoclinic orbits in system (1.1) completely in some sense.

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(Received 02.06.2017)

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