Memoirs on Differential Equations and Mathematical Physics

Volume 75, 2018, 115–128

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SOME OPTIMAL CONDITIONS FOR THE SOLVABILITY AND UNIQUE SOLVABILITY OF THE TWO–POINT NEUMANN PROBLEM

Abstract. For second order ordinary differential equations, unimprovable sufficient conditions are established for the solvability and unique solvability of the Neumann boundary value problem.

2010 Mathematics Subject Classification. 34B05, 34B15.

Key words and phrases. Second order ordinary differential equation, linear, nonlinear, the Neumann problem, existence theorem, uniqueness theorem.

რეზიუმე. მეორე რიგის ჩვეულებრივი დიფერენციალური განტოლებებისათვის დადგენილია ნეიმანის სასაზღვრო ამოცანის ამოხსნადობისა და ცალსახად ამოხსნადობის არაგაუმჯობესებადი საკმარისი პირობები.

1 Formulation of the main results

On a finite interval $[a,b]$, we consider the differential equation

$$u'' = f(t,u) \tag{1.1}$$

with the Neumann two-point boundary conditions

$$u'(a) = c_1, \quad u'(b) = c_2, \tag{1.2}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, while c_1 and c_2 are real constants.

A number of interesting and unimprovable in a certain sense results concerning the existence and uniqueness of a solution of problem (1.1), (1.2) are known (see, e.g., [1–3, 5–8, 12] and the references therein). In the present paper, general theorems on the existence and uniqueness of a solution of that problem are proved which are nonlinear analogues of the first Fredholm theorem. Based on these theorems, unimprovable sufficient conditions, different from the above mentioned results, for the solvability and unique solvability of problem (1.1), (1.2) are obtained.

We use the following notation.

$\mathbb{R}$ is the set of real numbers; $\mathbb{R}_+ = [0,+\infty[$; $\mathbb{R}_- =]-\infty,0]$;

$$[x]_- = \frac{|x| - x}{2};$$

$L([a,b])$ is the space of Lebesgue integrable functions.

Definition 1.1. Let $p_i \in L([a,b])$ $(i = 1,2)$ and

$$p_1(t) \le p_2(t) \ \text{ for almost all } \ t \in [a,b]. \tag{1.3}$$

We say that the vector function (p_1,p_2) belongs to the set $\boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$ if for any measurable function $p : [a,b] \to \mathbb{R}$, satisfying the inequality

$$p_1(t) \le p(t) \le p_2(t) \ \text{ for almost all } \ t \in [a,b], \tag{1.4}$$

the homogeneous Neumann problem

$$u'' = p(t)u, \tag{1.5}$$

$$u'(a) = 0, \quad u'(b) = 0 \tag{1.6}$$

has only the trivial solution.

Theorem 1.1. *Let there exist $(p_1,p_2) \in \boldsymbol{\mathcal{N}\mathrm{eum}([a,b])}$ and an integrable in the first and non-decreasing in the second argument function $q : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\lim_{x\to+\infty} \int_a^b \frac{q(t,x)}{x}\,dt = 0, \tag{1.7}$$

and on the set $[a,b] \times \mathbb{R}$ the inequality

$$p_1(t)|x| - q(t,|x|) \le f(t,x)\operatorname{sgn}(x) \le p_2(t)|x| + q(t,|x|) \tag{1.8}$$

holds. Then problem (1.1), (1.2) has at least one solution.

Corollary 1.1. *Let on the set $[a,b] \times \mathbb{R}$ inequality (1.8) be satisfied, where $p_i \in L([a,b])$ $(i = 1,2)$ are the functions satisfying inequality (1.3), and $q : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is an integrable in the first and non-decreasing in the second argument function satisfying condition (1.7). Let, moreover,*

$$\int_a^b p_2(t)\,dt \le 0, \quad \operatorname{mes}\{[t \in [a,b] : \ p_2(t) < 0\} > 0, \tag{1.9}$$

and there exist a number $\lambda \geq 1$ *such that*

$$\int_a^b [p_1(t)]_-^\lambda \, dt \leq \frac{4(b-a)}{\pi^2} \left(\frac{\pi}{b-a}\right)^{2\lambda}. \tag{1.10}$$

Then problem (1.1), (1.2) *has at least one solution.*

Corollary 1.2. *Let on the set* $[a,b] \times \mathbb{R}$ *inequality* (1.8) *be satisfied, where* $p_1 : [a,b] \to \mathbb{R}_-$ *and* $p_2 : [a,b] \to \mathbb{R}$ *are integrable functions satisfying inequalities* (1.3) *and* (1.9), *while* $q : [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is an integrable in the first and non-decreasing in the second argument function satisfying condition* (1.7). *Let, moreover, there exist* $t_0 \in]a,b[$ *such that the function* p_1 *is non-increasing and non-decreasing in the intervals* $]a,t_0[$ *and* $]t_0,b[$, *respectively, and*

$$\int_a^{t_0} \sqrt{|p_1(t)|} \, dt \leq \frac{\pi}{2}, \quad \int_{t_0}^b \sqrt{|p_1(t)|} \, dt \leq \frac{\pi}{2}, \quad \int_a^b \sqrt{|p_1(t)|} \, dt < \pi. \tag{1.11}$$

Then problem (1.1), (1.2) *has at least one solution.*

Theorem 1.2. *Let on the set* $[a,b] \times \mathbb{R}$ *the inequality*

$$p_1(t)|x-y| \leq (f(t,x) - f(t,y)) \operatorname{sgn}(x-y) \leq p_2(t)|x-y| \tag{1.12}$$

be satisfed, where $(p_1,p_2) \in \mathcal{N}\mathbf{eum}([a,b])$. *Then problem* (1.1), (1.2) *has one and only one solution.*

Corollary 1.3. *Let on the set* $[a,b] \times \mathbb{R}$ *condition* (1.12) *hold, where* $p_i \in L([a,b])$ $(i=1,2)$ *are the functions satisfying inequalities* (1.3) *and* (1.9). *If, moreover, for some* $\lambda \geq 1$ *inequality* (1.10) *is satisfied, then problem* (1.1), (1.2) *has one and only one solution.*

Corollary 1.4. *Let on the set* $[a,b] \times \mathbb{R}$ *inequality* (1.12) *hold, where* $p_1 : [a,b] \to \mathbb{R}_-$ *and* $p_2 : [a,b] \to \mathbb{R}$ *are integrable functions satisfying inequalities* (1.3) *and* (1.9). *Let, moreover, there exist* $t_0 \in]a,b[$ *such that the function* p_2 *is non-increasing and non-decreasing in the intervals* $]a,t_0[$ *and* $]t_0,b[$, *respectively, and satisfies inequality* (1.11). *Then problem* (1.1), (1.2) *has one and only one solution.*

The following two corollaries of Theorem 1.2 concern the linear differential equation

$$u'' = p(t)u + q(t), \tag{1.13}$$

where p and $q \in L([a,b])$.

Corollary 1.5. *Let*

$$\int_a^b p(t) \, dt \leq 0, \quad \operatorname{mes}\{t \in [a,b] : \; p(t) < 0\} > 0, \tag{1.14}$$

and let there exist a number $\lambda \geq 1$ *such that*

$$\int_a^b [p(t)]_-^\lambda \, dt \leq \frac{4(b-a)}{\pi^2} \left(\frac{\pi}{b-a}\right)^{2\lambda}. \tag{1.15}$$

Then problem (1.13), (1.2) *has one and only one solution.*

Corollary 1.6. *Let there exist a number* $t_0 \in]a,b[$ *such that the function* p *along with* (1.14) *satisfies the conditions*

$$p_0(t) = \operatorname{ess\,sup}\big\{[p(s)]_- : \; a < s < t\big\} < +\infty \quad \text{for } a < t < t_0, \tag{1.16}$$

$$p_0(t) = \operatorname{ess\,sup}\big\{[p(s)]_- : \; t < s < b\big\} < +\infty \quad \text{for } t_0 < t < b, \tag{1.17}$$

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt \leq \frac{\pi}{2}, \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt \leq \frac{\pi}{2}, \quad \int_a^b \sqrt{p_0(t)} \, dt < \pi. \tag{1.18}$$

Then problem (1.13), (1.2) *has one and only one solution.*

Remark 1.1. In the case, where instead of (1.14) the more hard condition

$$p(t) \le 0 \quad \text{for} \ \ a < t < b, \quad \operatorname{mes}\{t \in [a,b]: \ p(t) < 0\} > 0 \tag{1.19}$$

is satisfied, the results analogous to Corollary 1.5 previously were obtained in [5,6,12]. More precisely, in [12] it is required that along with (1.19) the inequalities

$$\int\limits_a^b |p(t)|\,dt \le \frac{4}{b-a}, \quad \operatorname{ess\,sup}\{|p(t)|: \ a \le t \le b\} < +\infty$$

be satisfied (see [12, Theorem 3]), while in [5] and [6] it is assumed, respectively, that

$$\int\limits_a^b |p(t)|\,dt \le \frac{4}{b-a}$$

(see [5, Corollary 1.2]), and

$$\int\limits_a^b |p(t)|^\lambda\,dt \le \frac{4(b-a)}{\pi^2}\left(\frac{\pi}{b-a}\right)^{2\lambda},$$

where $\lambda \equiv const \ge 1$ (see [6, Corollary 1.3]).

Example 1.1. Suppose

$$p(t) \equiv -\left(\frac{\pi}{b-a}\right)^2,$$

ε is arbitrarily small positive number, while λ is so large that

$$\left(1+\frac{\varepsilon}{\pi}\right)^\lambda > \frac{\pi}{2}.$$

Then instead of (1.15) the inequality

$$\int\limits_a^b [p(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\left(\frac{\pi+\varepsilon}{b-a}\right)^{2\lambda} \tag{1.20}$$

is satisfied. On the other hand, the homogeneous problem (1.5), (1.6) has a nontrivial solution $u_0(t) = \cos\frac{\pi(t-a)}{b-a}$, and the nonhomogeneous problem (1.13), (1.2) has no solution if only

$$c_1 + c_2 + \int\limits_a^b u_0(t)q(t)\,dt \ne 0.$$

Consequently, condition (1.15) in Corollary 1.5 is unimprovable and it cannot be replaced by condition (1.20).

The above example shows also that condition (1.10) in Corollaries 1.1 and 1.3 is unimprovable in the sense that it cannot be replaced by the condition

$$\int\limits_a^b [p_1(t)]_-^\lambda\,dt < \frac{4(b-a)}{\pi^2}\left(\frac{\pi+\varepsilon}{b-a}\right)^{2\lambda},$$

where ε is a positive constant independent of λ.

Note that condition (1.10) in the above mentioned corollaries is unimprovable also in the case where $\lambda = 1$, and it cannot be replaced by the condition

$$\int_a^b [p_1(t)]_- \, dt < \frac{4+\varepsilon}{b-a}$$

no matter how small $\varepsilon > 0$ would be (see [5, p. 357, Remark 1.1]).

Example 1.2. Suppose $t_0 \in]a, b[$ and

$$p(t) = \begin{cases} -\dfrac{\pi^2}{4(t_0-a)^2} & \text{for } a \le t \le t_0, \\[2mm] -\dfrac{\pi^2}{4(b-t_0)^2} & \text{for } t_0 < t \le b. \end{cases}$$

Then inequalities (1.16), (1.17) hold, and instead of (1.18) we have

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt = \frac{\pi}{2}, \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt = \frac{\pi}{2}.$$

On the other hand, the homogeneous problem (1.5), (1.6) has a nontrivial solution

$$u_0(t) = \begin{cases} (t_0-a)\cos\dfrac{\pi(t-a)}{2(t_0-a)} & \text{for } a \le t \le t_0, \\[2mm] (t_0-b)\cos\dfrac{\pi(b-t)}{2(b-t_0)} & \text{for } t_0 < t \le b, \end{cases}$$

while the nonhomogeneous problem (1.13), (1.2) has no solution if only

$$(t_0-a)c_1 + (b-t_0)c_2 + \int_a^b u_0(t)q(t) \, dt \neq 0.$$

Consequently, condition (1.18) in Corollary 1.6 is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^{t_0} \sqrt{p_0(t)} \, dt \le \frac{\pi}{2}, \quad \int_{t_0}^b \sqrt{p_0(t)} \, dt \le \frac{\pi}{2}.$$

From the above said it is also clear that condition (1.11) in both Corollary 1.2 and Corollary 1.4 is unimprovable and it cannot be replaced by the condition

$$\int_a^{t_0} \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2}, \quad \int_{t_0}^b \sqrt{|p_1(t)|} \, dt \le \frac{\pi}{2}\,.$$

2 Auxiliary propositions

2.1. Lemma on a priori estimate. In the segment $[a, b]$, we consider the differential inequality

$$p_1(t)|u(t)| - q(t) \le u''(t)\operatorname{sgn}(u(t)) \le p_2(t)|u(t)| + q(t), \tag{2.1}$$

where

$$(p_1, p_2) \in \mathcal{N}\mathbf{eum}([\boldsymbol{a}, \boldsymbol{b}]), \tag{2.2}$$

and $q \in L([a,b])$ is a non-negative function.

A function $u : [a,b] \to \mathbb{R}$ is said to be a solution of the differential inequality (2.1) if it is continuously differentiable, has an absolutely continuous on $[a,b]$ first derivative, and almost everywhere on this segment satisfies inequality (2.1).

Lemma 2.1. *If condition* (2.2) *holds, then there exists a positive constant r_0 such that for any non-negative function $q \in L([a,b])$ every solution of the differential inequality* (2.1) *admits the estimate*

$$|u(t)| \le r_o\bigg(|u'(a)| + |u'(b)| + \int\limits_a^b q(s)\,ds\bigg) \quad \textit{for} \ \ a \le t \le b. \tag{2.3}$$

Proof. Assume the contrary that the lemma is not true. Then for any natural number k there exist a non-negative function $q_k \in L([a,b])$ and a solution u_k of the differential inequality (2.1) such that

$$\|u_k\| > k^2\bigg(|u_k'(a)| + |u_k'(b)| + \int\limits_a^b q_k(s)\,ds\bigg),$$

where $\|u_k\| = \max\{|u_k(t)| : \ t \in [a,b]\}$.

Let I_k be the set of all $t \in [a,b]$ at which there exists $u_k''(t)$,

$$u_{0k}(t) = u_k(t)/\|u_k\| \ \text{ for } \ t \in [a,b], \quad q_{0k}(t) = kq(t)/\|u_k\| \ \text{ for } \ t \in I_k.$$

Then

$$p_1(t)|u_{0k}(t)| - q_{0k}(t)/k \le u_{0k}''(t)\operatorname{sgn}(u_{0k}(t)) \le p_2(t)|u_{0k}(t)| + q_{0k}(t)/k \ \text{ for } \ t \in I_k, \tag{2.4}$$

$$|u_{0k}'(a)| + |u_{0k}'(b)| < \frac{1}{k}, \quad \|u_{0k}\| = 1, \tag{2.5}$$

$$\int\limits_a^b q_{0k}(s)\,ds < \frac{1}{k}. \tag{2.6}$$

Put

$$I_{1k} = \bigg\{t \in I_k : \ |u_{0k}(t)| \ge \frac{1}{k}\bigg\}, \quad I_{2k} = I_k \setminus I_{1k},$$

$$p_{0k}(t) = \begin{cases} \dfrac{u_{0k}''(t)}{u_{0k}(t)} & \text{for } \ t \in I_{1k}, \\ p_1(t) & \text{for } \ t \in I_{2k}, \end{cases}$$

$$q_{1k}(t) = \begin{cases} 0 & \text{for } \ t \in I_{1k}, \\ u_{0k}''(t) - p_1(t)u_{0k}(t) & \text{for } \ t \in I_{2k}, \end{cases}$$

$$P_k(t) = \int\limits_a^t p_{0k}(s)\,ds.$$

Then

$$u_{0k}''(t) = p_{0k}(t)u_{0k}(t) + q_{1k}(t) \ \text{ for } \ t \in I_k. \tag{2.7}$$

On the other hand, according to conditions (2.4) and (2.5) we have

$$|u_{0k}''(t)| \le \ell(t) + q_{0k}(t) \ \text{ for } \ t \in I_k,$$

$$p_1(t) - q_{0k}(t) \le p_{0k}(t) \le p_2(t) + q_{0k}(t) \ \text{ for } \ t \in I_k,$$

$$|q_{1k}(t)| \le \big(|p_1(t)| + \ell(t) + q_{0k}(t)\big)/k \ \text{ for } \ t \in I_k,$$

where $\ell(t) = |p_1(t)| + |p_2(t)|$.

If along with these estimates we take into account inequality (2.6), then it becomes evident that

$$|u_{0k}'(t) - u_{0k}'(\tau)| \leq \int_\tau^t \ell(s)\,ds + \frac{1}{k} \quad \text{for } a \leq \tau < t \leq b, \tag{2.8}$$

$$P_k(a) = 0, \ \int_\tau^t p_1(s)\,ds - \frac{1}{k} < P_k(t) - P_k(\tau) < \int_\tau^t p_2(s)\,ds + \frac{1}{k} \quad \text{for } a \leq \tau < t \leq b, \tag{2.9}$$

$$\int_a^b |p_{0k}(s)|\,ds < \ell_0, \tag{2.10}$$

$$\int_a^b |q_{1k}(s)|\,ds < \frac{\ell_0}{k}, \tag{2.11}$$

where

$$\ell_0 = 1 + \int_a^b \big(|p_1(s)| + \ell(s)\big)\,ds.$$

By virtue of conditions (2.5), (2.8) and(2.9), the sequences $(u_k)_{k=1}^{+\infty}$, $(u_k')_{k=1}^{+\infty}$, $(P_k)_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on $[a,b]$. By the Arzelà–Ascoli lemma, without loss of generality we can assume that these sequences are uniformly convergent.

Put

$$u(t) = \lim_{k\to+\infty} u_{0k}(t), \quad P(t) = \lim_{k\to+\infty} P_k(t). \tag{2.12}$$

If we pass to the limit in inequality (2.9) as $k \to +\infty$, then we get

$$P(a) = 0, \quad \int_\tau^t p_1(s)\,ds \leq P(t) - P_(\tau) \leq \int_\tau^t p_2(s)\,ds \quad \text{for } a \leq \tau < t \leq b.$$

Hence it is clear that the function P is absolutely continuous and admits the representation

$$P(t) = \int_a^t p(s)\,ds \quad \text{for } a \leq t \leq b, \tag{2.13}$$

where $p \in L([a,b])$ is a function satisfying inequality (1.4).

By Lemma 1.1 from [4], conditions (2.10), (2.12) and (2.13) guarantee the validity of the equality

$$\lim_{k\to+\infty} \int_a^t p_{0k}(s)u_{0k}(s)\,ds = \int_a^t p(s)u(s)\,ds \quad \text{for } a \leq t \leq b. \tag{2.14}$$

In view of (2.7) we have

$$u_{0k}'(t) = u_{0k}'(a) + \int_a^t \big(p_{0k}(s)u_{0k}(s) + q_{1k}(s)\big)\,ds \quad \text{for } a \leq t \leq b.$$

If along with this identity we take into account conditions (2.5), (2.11) and (2.14), then we find

$$u'(t) = \int_a^t p(s)u(s)\,ds \quad \text{for } a \leq t \leq b$$

$$u'(a) = u'(b) = 0, \quad \|u\| = 1.$$

Consequently, u is a nontrivial solution of the homogeneous problem (1.5), (1.6). On the other hand, due to conditions (1.4) and (2.2), this problem has no nontrivial solution. The contradiction obtained proves the lemma. □

2.2. Lemmas on two-point boundary value problems for equation (1.5). Let $p \in L([a,b])$. We consider the differential equation (1.5) with the boundary conditions

$$u'(a) = 0, \quad u(b) = 0, \tag{2.15}$$

or

$$u(a) = 0, \quad u'(b) = 0. \tag{2.16}$$

Lemma 2.2 (T. Kiguradze)**.** *Let*

$$p(t) \geq -p_0(t) \quad \text{for almost all } t \in [a,b], \tag{2.17}$$

where $p_0 \in L([a,b])$ *is a non-negative function. If, moreover, for some* $\lambda \geq 1$ *the inequality*

$$\int_a^b (b-t)p_0^\lambda(t)\,dt \leq \left(\frac{\pi}{2(b-a)}\right)^{2\lambda-2}$$

holds, then problem (1.5), (2.15) *has only the trivial solution. And if*

$$\int_a^b (t-a)p_0^\lambda(t)\,dt \leq \left(\frac{\pi}{2(b-a)}\right)^{2\lambda-2},$$

then problem (1.5), (2.16) *has only the trivial solution.*

This lemma is a corollary of Theorem 1.3 from [10].

Lemma 2.3. *Let inequality* (2.17) *hold where* $p_0 \in L([a,b])$ *is a non-negative non-decreasing* (*non-increasing*) *function such that*

$$\int_a^b \sqrt{p_0(t)}\,dt < \frac{\pi}{2}. \tag{2.18}$$

Then problem (1.5), (2.15) (*problem* (1.5), (2.16)) *has only the trivial solution.*

Proof. We consider only problem (1.5), (2.15) since problem (1.5), (2.16) can be considered analogously.

Assume that problem (1.5), (2.15) has a nontrivial solution u. Without loss of generality we can assume that $u'(b) < 0$. Then there exists $a_0 \in [a,b[$ such that

$$\begin{gathered} u(t) > 0, \quad u'(t) < 0 \quad \text{for } a_0 < t < b, \\ u'(a_0) = 0. \end{gathered} \tag{2.19}$$

By virtue of conditions (2.17) and (2.19), almost everywhere on $[a_0,b]$ the inequality

$$u''(t)u'(t) \leq -p_0(t)u'(t)u(t)$$

is satisfied. If along with this we take into account the fact that p_0 is a non-decreasing function, then we obtain

$$u'^{\,2}(t) \leq -2\int_{a_0}^t p_0(s)u'(s)u(s)\,ds \leq p_0(t)\left(-\int_{a_0}^t u'(s)u(s)\,ds\right) = p_0(t)(u^2(a_0) - u^2(t)) \quad \text{for } a_0 \leq t \leq b.$$

Consequently,

$$\sqrt{p_0(t)} \geq \frac{-u'(t)}{\sqrt{u^2(a_0) - u^2(t)}} \quad \text{for } a_0 < t \leq b.$$

Integrating this inequality from a_0 to b, we get

$$\int_{a_0}^{b} \sqrt{p_0(t)}\,dt \geq -\int_{a_0}^{b} \frac{-u'(t)\,dt}{\sqrt{u^2(a_0) - u^2(t)}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2},$$

which contradicts inequality (2.18). The contradiction obtained provers the lemma. □

Remark 2.1. From Lemma 2.3 it follows, in particular, that if $p : [a,b] \to \mathbb{R}_-$ is a non-decreasing (a non-increasing) function and for some $t_0 \in \,]a,b[$ the inequalities

$$\int_a^{t_0} \sqrt{|p(s)|}\,ds \leq \frac{\pi}{2}, \quad p(t_0) > -\frac{\pi^2}{4(b-t_0)^2} \quad \left(p(t_0) > -\frac{\pi^2}{4(t_0-a)^2}, \quad \int_{t_0}^{b} \sqrt{|p(s)|}\,ds \leq \frac{\pi}{2}\right)$$

hold, then the Dirichlet problem

$$u'' = p(t)u, \quad u(a) = u(b) = 0$$

has only the trivial solution. This result generalizes Z. Nehari's theorem [11, Theorem 1], where it is assumed that

$$\int_a^b \sqrt{|p(s)|}\,ds \leq \frac{\pi}{2}.$$

Along with Lemmas 2.2 and 2.3, below we need Lemma 2.4 as well, concerning problem (1.5), (1.6).

Lemma 2.4. *If condition* (1.14) *holds, then every solution of problem* (1.5), (1.6) *has at least one zero in the interval* $]a,b[$.

Proof. Assume the contrary that problem (1.5), (1.6) has a solution u not having a zero in $]a,b[$. Then by (1.6),

$$u(t) \neq 0 \quad \text{for } a \leq t \leq b,$$

and almost everywhere on $[a,b]$ the equality

$$\frac{u''(t)}{u(t)} = p(t)$$

holds. If we integrate this identity from a to b, then by conditions (1.6) and (1.14) we get

$$0 < \int_a^b \frac{u'^{\,2}(t)}{u^2(t)}\,dt = \int_a^b p(t)\,dt \leq 0.$$

The contradiction obtained provers the lemma. □

2.3. Lemmas on the set $\mathcal{N}$eum$([a,b])$.

Lemma 2.5. *Let* $p_i \in L([a,b])$ $(i = 1,2)$ *be functions satisfying inequalities* (1.3), (1.9) *and* (1.10), *where* $\lambda \geq 1$. *Then*

$$(p_1, p_2) \in \mathcal{N}\mathbf{eum}([a,b]).$$

Proof. Assume the contrary that

$$(p_1, p_2) \notin \mathcal{N}\mathbf{eum}([a,b]).$$

Then there exists a function $p \in L([a,b])$, satisfying condition (1.4), such that problem (1.5), (1.6) has a nontrivial solution u.

Inequalities (1.4) and (1.9) imply inequalities (1.14). Hence by Lemma 2.4 follows the existence of $t_1 \in \,]a,b[$ such that

$$u(t_1) = 0. \tag{2.20}$$

On the other hand, by Lemma 2.2 inequality (1.4) and equalities (1.6) and (2.20) result in

$$\Big(\frac{\pi}{2}\Big)^{2\lambda-2} < (t_1-a)^{2\lambda-2}\int_a^{t_1}(t_1-t)[p_1(t)]_-^\lambda\,dt < (t_1-a)^{2\lambda-1}\int_a^{t_1}[p_1(t)]_-^\lambda\,dt,$$

$$\Big(\frac{\pi}{2}\Big)^{2\lambda-2} < (b-t_1)^{2\lambda-2}\int_{t_1}^{b}(t-t_1)[p_1(t)]_-^\lambda\,dt < (b-t_1)^{2\lambda-1}\int_{t_1}^{b}[p_1(t)]_-^\lambda\,dt.$$

Thus

$$\Big(\frac{\pi}{2}\Big)^{4\lambda-4} < ((t_1-a)(b-t_1))^{2\lambda-1}\Big(\int_a^{t_1}[p_1(t)]_-^\lambda\,dt\Big)\Big(\int_{t_1}^{b}[p_1(t)]_-^\lambda\,dt\Big).$$

Hence, in view of the inequalities

$$(t_1-a)(b-t_1) \le \frac{1}{4}(b-a)^2,$$

$$\Big(\int_a^{t_1}[p_1(t)]_-^\lambda\,dt\Big)\Big(\int_{t_1}^{b}[p_1(t)]_-^\lambda\,dt\Big) \le \frac{1}{4}\Big(\int_a^{b}[p_1(t)]_-^\lambda\,dt\Big)^2,$$

it follows that

$$\Big(\frac{\pi}{2}\Big)^{4\lambda-4} < 2^{-4\lambda}(b-a)^{4\lambda-2}\Big(\int_a^{b}[p_1(t)]_-^\lambda\,dt\Big)^2.$$

Consequently,

$$\int_a^{b}[p_1(t)]_-^\lambda\,dt > \frac{4(b-a)}{\pi^2}\Big(\frac{\pi}{b-a}\Big)^{2\lambda},$$

which contradicts inequality (1.10). The contradiction obtained provers the lemma. □

Lemma 2.6. *Let $p_1 : [a,b] \to \mathbb{R}_-$ and $p_2 : [a,b] \to \mathbb{R}$ be integrable functions satisfying inequalities* (1.3) *and* (1.9). *Let, moreover, there exist $t_0 \in \,]a,b[$ such that the function p_1 is non-increasing and non-decreasing in the intervals $]a,t_0[$ and $]t_0,b[$, respectively, and inequalities* (1.11) *are satisfied. Then*

$$(p_1, p_2) \in \mathcal{N}\mathbf{eum}([a,b]).$$

Proof. Let $p \in L([a,b])$ be an arbitrary function satisfying inequality (1.4), and let u be an arbitrary solution of problem (1.5), (1.6).

Inequalities (1.4) and (1.9) result in inequalities (1.14). Hence by Lemma 2.4 follows the existence at least one zero of the function u in $]a,b[$. Consequently, there exists $t_1 \in \,]a,b[$ such that

$$u'(a) = 0, \quad u(t_1) = 0, \tag{2.21}$$

$$u(t_1) = 0, \quad u'(b) = 0. \tag{2.22}$$

If along with (1.11) we take into account the monotonicity of the function p_1 in the intervals $]a, t_0[$ and $]t_0, b[$, then it becomes clear that either

$$a < t_1 \le t_0, \quad \int\limits_a^{t_1} \sqrt{|p_1(t)|}\, dt < \frac{\pi}{2}, \tag{2.23}$$

or

$$t_0 \le t_1 < b, \quad \int\limits_{t_1}^{b} \sqrt{|p_1(t)|}\, dt < \frac{\pi}{2}. \tag{2.24}$$

However, if condition (2.23) (condition (2.24)) holds, then by Lemma 2.3 problem (1.5), (2.21) (problem (1.5), (2.22)) has only the trivial solution. Thus we have proved that $u(t) \equiv 0$. Hence, in view of the arbitrariness of a solution u of problem (1.5), (1.6) and a function p, we have $(p_1, p_2) \in \mathcal{N}\mathbf{eum}([a, b])$. □

2.4. Lemma on the solvability of problem (1.1), (1.2). Along with problem (1.1), (1.2) we consider the auxiliary problem

$$u'' = (1-\lambda)p(t)u + \lambda f(t, u), \tag{2.25}$$

$$u'(a) = \lambda c_1, \quad u'(b) = \lambda c_2, \tag{2.26}$$

where $p \in L([a, b])$, and λ is a parameter.

According to Corollary 2 from [9], the following lemma is valid.

Lemma 2.7. *Let problem* (1.5), (1.6) *have only the trivial solution and let there exist a positive constant r such that for any $\lambda \in]0, 1[$ an arbitrary solution u of problem* (2.25), (2.26) *admits the estimate*

$$|u(t)| + |u'(t)| < r \quad \text{for } a \le t \le b. \tag{2.27}$$

Then problem (1.1), (1.2) *has at least one solution.*

3 Proof of the main results

Proof of Theorem 1.1. By Lemma 2.1, there exists a positive constant r_0 such that every solution u of the differential inequality

$$p_1(t)|u(t)| - q(t, |u(t)|) \le u''(t)\operatorname{sgn}(u(t)) \le p_2(t)|u(t)| + q(t, |u(t)|) \tag{3.1}$$

admits the estimate

$$\|u\| \le r_0\bigg(|u'(a)| + |u'(b)| + \int\limits_a^b q(s, \|u\|)\, ds\bigg), \tag{3.2}$$

where

$$\|u\| = \max\{|u(t)| : \ a \le t \le b\}.$$

On the other hand, according to equality (1.7), there exists a number r_1 such that

$$r_0\bigg(|c_1| + |c_2| + \int\limits_a^b q(s, x)\, ds\bigg) < x \quad \text{for } x \ge r_1. \tag{3.3}$$

Put

$$r_2 = \bigg(\frac{1}{r_0} + \int\limits_a^b (|p_1(s)| + |p_2(s)|)\, ds\bigg) r_1, \quad r = r_1 + r_2.$$

Let $p \in L([a,b])$ be an arbitrary function satisfying inequality (1.4), $\lambda \in]0,1[$, and u be an arbitrary solution of problem (2.25), (2.26). By Lemma 2.7 and condition (2.2), it suffices to state that u admits estimate (2.27).

By virtue of inequality (1.8), the function u is a solution of problem (3.1), (2.26). Thus it admits the estimate

$$\|u\| \le r_0\bigg(|c_1| + |c_2| + \int\limits_a^b q(s, \|u\|)\, ds\bigg).$$

Hence in view of (3.3) we have

$$\|u\| \le r_1.$$

If along with this inequality we take into account conditions (2.26) and (3.3), we find

$$|u'(t)| \le |u'(a)| + \int\limits_a^b |u''(s)|\, ds \le |c_1| + \int\limits_a^b q(s, r_1)\, ds + \int\limits_a^b (|p_1(s)| + |p_2(s)|)\, |u(s)|\, ds$$

$$\le r_1/r_0 + r_1 \int\limits_a^b (|p_1(s)| + |p_2(s)|)\, ds = r_2 \quad \text{for } a \le t \le b.$$

Therefore estimate (2.27) is valid. □

Proof of Theorem 1.2. Inequality (1.12) yields inequality (1.8), where $q(t,|x|) \equiv |f(t,0)|$. Consequently, all the conditions of Theorem 1.1 are fulfilled which guarantees the solvability of problem (1.1), (1.2).

Let u_1 and u_2 be arbitrary solutions of the above mentioned problem. Put

$$u(t) = u_1(t) - u_2(t).$$

In view of condition (1.12), the function u is a solution of the differential inequality

$$p_1(t)|u(t)| \le u''(t)\operatorname{sgn}(u(t)) \le p_2(t)|u(t)|,$$

satisfying the boundary conditions (1.6). Hence by Lemma 2.1 it follows that $u(t) \equiv 0$. Consequently, problem (1.1), (1.2) has one and only one solution. □

By Lemma 2.5, Theorems 1.1 and 1.2 yield Corollaries 1.1 and 1.3, respectively. By Lemma 2.6, Theorems 1.1 and 1.2 yield Corollaries 1.2 and 1.4, respectively.

In the case, where $f(t,x) \equiv p(t)x + q(t)$, Corollary 1.3 results in Corollary 1.5, and Corollary 1.4 results in Corollary 1.6.

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(Received 12.03.2018)

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