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MONOTONE ITERATIVE METHOD FOR SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. In this paper, we apply the monotone iteration method to establish the existence of a positive solution for the fractional differential equation

$$D_{0+}^{\alpha} u(t) + q(t) f(t, u(t)) = 0, \quad 0 < t < 1,$$

together with the boundary conditions (BCs)

$$u(0) = u'(0) = \cdots = u^{n-2}(0) = 0, \quad D_{0+}^{\beta} u(1) = \int\limits_0^1 h(s, u(s))\, dA(s),$$

where $n > 2$, $n - 1 < \alpha \leq n$, $\beta \in [1, \alpha - 1]$, D_{0+}^{α} and D_{0+}^{β} are the standard Riemann–Liouville fractional derivatives of order α and β, respectively, and $f, h : [0,1] \times [0, \infty) \to [0, \infty)$ are continuous functions. The sufficient condition provided in this paper is new, interesting and easy to verify. Our conditions do not require the sublinearity or superlinearity on the nonlinear functions f and h at 0 or ∞. The paper is supplemented with examples illustrating the applicability of our result.

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რეზიუმე. სტატიაში გამოყენებულია მონოტონური იტერაციის მეთოდი, რათა დავადგინოთ დადებითი ამონახსნის არსებობა წილადური დიფერენციალური

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განტოლებისთვის

$$u(0) = u'(0) = \cdots = u^{n-2}(0) = 0, \quad D_{0+}^{\beta} u(1) = \int\limits_0^1 h(s, u(s))\, dA(s)$$

სასაზღვრო პირობებით, სადაც $n > 2$, $n - 1 < \alpha \leq n$, $\beta \in [1, \alpha - 1]$, D_{0+}^{α} და D_{0+}^{β}, შესაბამისად, α და β რიგის სტანდარტული რიმან-ლიუვილის წილადური წარმოებულებია და $f, h : [0,1] \times [0, \infty) \to [0, \infty)$ უწყვეტი ფუნქციებია. ამ სტატიაში წარმოდგენილი საკმარისი პირობა ახალი, საინტერესო და მარტივად შესამოწმებელია. ჩვენი პირობები არ მოითხოვს არაწრფივი f და h ფუნქციების ქვეწრფივობას ან ზეწრფივობას 0-ში ან ∞-ში. სტატიაში აგრეთვე მოყვანილია მაგალითები ჩვენი შედეგის გამოყენების საილუსტრაციოდ.

1 Introduction

The aim of the present paper is to demonstrate the applications of the monotone iteration method for studying the existence of at least one positive solution of the nonlinear fractional differential equation

$$D_{0+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

together with the boundary conditions (BCs)

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta}u(1) = \int_0^1 h(s,u(s))\,dA(s), \tag{1.2}$$

where $n-1 < \alpha \le n$, $n > 2$, $\beta \in [1, \alpha-1]$ is fixed, $q : (0,1) \to [0,\infty)$ is a continuous function, $f, h : (0,1) \times [0,\infty) \to [0,\infty)$ are continuous functions, $\int_0^1 h(s,u(s))\,dA(s)$ is a Riemann–Stieltjes integral with A being nondecreasing and of bounded variation, and D_{0+}^{α}, D_{0+}^{β} are the standard Riemann–Liouville fractional derivatives of order α and β, respectively.

We define the fractional derivative and fractional integral for a function F of order γ, $\gamma \in [0,\infty)$ as follows.

Definition 1.1. The (left-sided) fractional integral of order $\gamma > 0$ of a function $F : (0,\infty) \to \mathbb{R}$ is given by

$$(I_{0+}^{\gamma}F)(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}F(s)\,ds, \quad t > 0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$, where $\Gamma(\gamma)$ is the Euler Gamma function, defined by $\Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1}e^{-t}\,dt$, $\gamma > 0$.

Definition 1.2. The Riemann–Liouville fractional derivative of order $\gamma > 0$ of a function F : $(0,\infty) \to \mathbb{R}$ is given by

$$(D_{0+}^{\gamma}F)(t) = \Big(\frac{d}{dt}\Big)^n (I_{0+}^{n-\gamma}F)(t) = \frac{1}{\Gamma(n-\gamma)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{F(s)}{(t-s)^{\gamma-n+1}}\,ds$$

for $t > 0$, where $n = [\![\gamma]\!] + 1$ ($[\![\gamma]\!]$ is the largest integer, not greater than γ), provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 1.3. By a positive solution of (1.1), (1.2) we mean a function $u \in C[0,1]$ satisfying (1.1), (1.2) with $u(t) > 0$ for all $t \in (0,1]$.

The fixed point theorems have been playing a crucial role in establishing the solutions of fractional differential equations. For instance, one may refer to [4–6, 8, 12, 15–20] on the use of a fixed point index property, Krasnoselskii's, Avery–Peterson's, Schauder's fixed point theorems, the Leray–Schauder alternative, and Guo–Krasnoselskii's fixed point theorem to study the existence of at least one, two or three positive solutions of fractional differential equations of form (1.1) with nonlinear BCs of form (1.2). For a system of fractional differential equations with integral boundary conditions of coupled or uncoupled type, one may refer to [1, 9–11, 13, 14].

In their recent work [16], Padhi et al. have used Schauder's fixed point theorem and the Leray–Schauder's alternative along with the Krasnoselskii's fixed point theorem to study the existence and uniqueness of positive solutions of (1.1), (1.2). Using the Avery–Peterson's fixed point theorem, the authors established the existence of at least three positive solutions of (1.1), (1.2).

In [16], Padhi et al. have shown that the boundary value problem (1.1), (1.2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)q(s)f(s,u(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1} \int_0^1 h(s,u(s))\,dA(s),$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1 \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1. \end{cases}$$

Further, it is proved in [16] that the Green's function $G(t,s)$ satisfies the inequality

$$t^{\alpha-1}G(1,s) = t^{\alpha-1} \max_{0\le t\le 1} G(t,s) \le G(t,s) \le \max_{0\le t\le 1} G(t,s) = G(1,s), \tag{1.3}$$

where

$$G(1,s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-\beta-1}\big[1-(1-s)^{\beta}\big]. \tag{1.4}$$

To establish our results, we assume that the following conditions are satisfied:

(A1) $f, h \in C([0,1]\times[0,\infty),[0,\infty))$;

(A2) $q \in C((0,1),[0,\infty))$, and q does not vanish identically on any subinterval of $(0,1]$;

(A3) for any positive numbers r_1 and r_2 with $r_1 < r_2$, there exist continuous functions p_f and $p_h : (0,1) \to [0,\infty)$ such that

$$f(t,u) \le p_f(t), \quad h(t,u) \le p_h(t) \ \text{ for } \ 0 \le t \le 1, \ \ \frac{r_1}{2^{2(\alpha-1)}} \le u \le r_2,$$

and

$$\int_0^1 G(1,s)q(s)p_f(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \int_0^1 p_h(s)\,dA(s) < \infty,$$

where $G(1,s)$ is given in (1.4).

In this paper, we apply the monotone iterative method to obtain sufficient conditions on the existence of one positive solution and an iterative scheme for approximating the solutions. The following theorem states the main result of this paper.

Theorem 1.1. *Assume that there exist constants r and R with $0 < 2r < R$ such that the following conditions are satisfied:*

(A4)
$$\frac{r}{\int_0^1 G(1,s)q(s)\,ds} \le f(t,u) \le f(t,v) \le \frac{R}{2\int_0^1 G(1,s)q(s)\,ds}$$
$$\text{for } \ \mu^2 r \le u \le v \le R \ \text{ and } \ \frac{1}{2} \le t \le 1$$

and

(A5)
$$h(t,u) \le h(t,v) \le \frac{\Gamma(\alpha)R}{2\Gamma(\alpha-\beta)\int_0^1 dA(s)} \quad \text{for } \ \mu^2 r \le u \le v \le R \ \text{ and } \ \frac{1}{2} \le t \le 1.$$

Then problem (1.1), (1.2) *has at least one positive solution.*

2 Preliminaries

In this section, we provide some basic concepts on the cones in a Banach space and the monotone iteration method.

Definition 2.1. Let X be a real Banach space. A nonempty convex closed set $P \subset X$ is said to be a cone provided that

(i) $ku \in P$ for all $u \in P$ and all $k \geq 0$;

(ii) $u, -u \in P$ implies $u = 0$.

In order to prove Theorem 1.1, we use the following well known monotone iteration method imported from [2, 3, 7] or Theorem 7.A in [21].

Theorem 2.1. *Let X be a real Banach space and K be a cone in X. Assume that there exist constants v_0 and w_0 with $v_0 \leq w_0$ and $[v_0, w_0] \subset X$ such that*

(i) *$T : [v_0, w_0] \to X$ is completely continuous;*

(ii) *T is a monotonic increasing operator on $[v_0, w_0]$;*

(iii) *v_0 is a lower solution of T, that is, $v_0 \leq Tv_0$;*

(iv) *w_0 is an upper solution of T, that is, $Tw_0 \leq w_0$.*

Then T has a fixed point and the iterative sequences $v_{n+1} = Tv_n$ and $w_{n+1} = Tw_n$, $n = 1, 2, 3, \dots$, with

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq w_{n-1} \leq \cdots \leq w_1 \leq w_0$$

converges to v and w, respectively, which are the greatest and smallest fixed points of T in $[v_0, w_0]$.

In this paper, we let $X = C[0, 1]$ to be the Banach space endowed with the norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|.$$

Define a cone K on X as $K = \{u \in C[0, 1] : \ u(t) \geq 0, \ t \in [0, 1]\}$ and an operator $T : K \to X$ as

$$Tu(t) = \int_0^1 G(t, s)q(s)f(s, u(s))\, ds + \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)}\, t^{\alpha-1} \int_0^1 h(s, u(s))\, dA(s). \tag{2.1}$$

Then it is easy to verify that $u(t)$ is a positive solution of problem (1.1), (1.2) if and only if $u(t)$ is a fixed point of the operator T on the cone K.

Let $g(s) = G(1, s)$ with $\int_{1/2}^{1} g(s)\, ds > 0$ and $c(t) = t^{\alpha-1}$. Then (1.3) can be rewritten as

$$c(t)g(s) \leq G(t, s) \leq g(s) \ \text{ for } \ 0 \leq t, s \leq 1. \tag{2.2}$$

Since it is useful to work on a smaller cone than K, we consider a cone K_1 of the type

$$K_1 = \Big\{u \in X : \ u(t) \geq 0 \text{ and } \min_{t \in [a,b]} u(t) \geq c_{a,b}\|u\|\Big\},$$

where $[a, b]$ is some subinterval of $[0, 1]$ and $c_{a,b} > 0$. Condition (2.2) ensures that for $[a, b] \subset [0, 1]$, if $c_{a,b} = \min\{c(t) : \ t \in [a, b]\} > 0$, then T maps K into K_1. Since (2.2) is valid for any $t \in [0, 1]$, we can work on the subinterval $[1/2, 1] \subset$ [0,1] for which the inequality

$$\mu G(1, s) \leq G(t, s) \leq G(1, s)$$

replaces (1.3) or (2.2), where

$$\mu = \frac{1}{2^{\alpha-1}} = \min_{t\in[1/2,1]} c(t) = \min_{t\in[1/2,1]} t^{\alpha-1}.$$

In this case, the operator T, defined in (2.1), maps the cone K into the subcone P, where

$$P = \Big\{u \in C[0,1] : \min_{t\in[1/2,1]} u(t) \geq \mu\|u\|\Big\}. \tag{2.3}$$

Also, $u(t)$ is a positive solution of problem (1.1), (1.2) if and only if $u(t)$ is a fixed point of the operator T on the subcone P.

3 Proof of Theorem 1.1

To prove our theorem, we consider the cone P, defined in (2.3). Let $u \in P$. Then

$$\|Tu\| \leq \int_0^1 G(1,s)q(s)f(s,u(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h(s,u(s))\,dA(s)$$

and

$$\begin{aligned}\min_{t\in[1/2,1]} Tu(t) &\geq \Big(\min_{t\in[1/2,1]} t^{\alpha-1}\Big)\Big[\int_0^1 G(1,s)q(s)f(s,u(s)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h(s,u(s))\,dA(s)\Big]\\ &= \mu\Big[\int_0^1 G(1,s)q(s)f(s,u(s))\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h(s,u(s))\,dA(s)\Big]\\ &\geq \mu\|Tu\|\end{aligned}$$

implies that $T : P \to P$. Also, T is well defined.

Set $v_0 = \mu^2 r$ and $w_0 = R$; then $v_0 < w_0$. We now prove that $T : [v_0, w_0] \to P$ is completely continuous. Let $\{u_n\} \in [v_0, w_0]$ and $u \in [v_0, w_0]$ be such that $\lim\limits_{n\to\infty} u_n = u$. Then $\mu^2 r \leq u_n \leq R$ and $\mu^2 r \leq u \leq R$ for $t \in [0,1]$. Since f is continuous on $[0,1] \times [\mu^2 r, R]$, for $\varepsilon > 0$ there exists $\delta_1 > 0$ with $|u_1 - u_2| < \delta_1$ for $u_1, u_2 \in [\mu^2 r, R]$, and we have

$$|f(t,u_1) - f(t,u_2)| < \frac{\varepsilon}{2\int_0^1 G(1,s)q(s)\,ds}, \quad t \in [0,1].$$

Similarly, from the continuity of h on $[0,1] \times [\mu^2 r, R]$, we get

$$|h(t,u_1) - h(t,u_2)| < \frac{\Gamma(\alpha)\varepsilon}{2\Gamma(\alpha-\beta)\int_0^1 dA(s)}, \quad t \in [0,1],$$

for $\varepsilon > 0$ and $\delta_2 > 0$ with $|u_1 - u_2| < \delta_2$, $u_1, u_2 \in [\mu^2 r, R]$. Set $\delta = \min\{\delta_1, \delta_2\}$; then it follows from $\lim_{n\to\infty} u_n = u$ that there exists a positive number N such that for every $n \geq N$, we have $|u_n(t) - u(t)| < \delta$, $t \in [0,1]$. Then the inequality

$$\begin{aligned}|Tu_n(t) - Tu(t)| \leq& \int_0^1 G(1,s)q(s)\big|f(s,u_n(s)) - f(s,u(s))\big|\,ds\\ &+ \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 \big|h(s,u_n(s)) - h(s,u(s))\big|\,dA(s) < \varepsilon\end{aligned}$$

shows that $T:[v_0,w_0]\to P$ is continuous.

Setting
$$f^*=\max_{t\in[0,1],u\in[\mu^2r,R]} f(t,u)\ \text{ and }\ h^*=\max_{t\in[0,1],\,u\in[\mu^2r,R]} h(t,u),$$
we have
$$|Tu(t)|\le f^*\int_0^1 G(1,s)q(s)\,ds+h^*\,\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 dA(s).$$
Thus, T is uniformly bounded on P.

Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous there. Similarly, the function $t^{\alpha-1}$ is uniformly continuous on $[0,1]$, because it is continuous there. So, for every $\varepsilon>0$, there exists $\delta>0$ such that $|G(t_1,s)-G(t_2,s)|<\varepsilon$ and $|t_1^{\alpha-1}-t_2^{\alpha-1}|<\varepsilon$ for $|t_1-t_2|<\delta$, $(t_1,s),(t_2,s)\in[0,1]\times[0,1]$. Consequently, for any $u\in[\mu^2r,R]:=[v_0,w_0]$ and $t_1,t_2\in[0,1]$ with $|t_1-t_2|<\delta$, we have
$$\begin{aligned}|Tu(t_1)-Tu(t_2)|&\le\int_0^1|G(t_1,s)-G(t_2,s)|q(s)f(s,u(s))\,ds\\&+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,|t_1^{\alpha-1}-t_2^{\alpha-1}|\int_0^1 h(s,u(s))\,dA(s)<\varepsilon\Big[\int_0^1 q(s)p_f(s)\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 p_h(s)\,dA(s)\Big].\end{aligned}$$
Hence the family $\{Tx: x\in[v_0,w_0]\}$ is equicontinuous on $[0,1]$, and so T is relatively compact. By the Arzela–Ascoli theorem, $T:[v_0,w_0]\to P$ is completely continuous.

Let $u,v\in[v_0,w_0]$ be such that $u\le v$. Then $v_0\le u\le v\le w_0$. By (A4) and (A5), we have
$$\begin{aligned}Tu(t)&=\int_0^1 G(t,s)q(s)f(s,u(s))\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,u(s))\,dA(s)\\&\le\int_0^1 G(t,s)q(s)f(s,v(s))\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,v(s))\,dA(s)\\&=Tv(t).\end{aligned}$$
Thus, T is monotonic increasing in $[v_0,w_0]$.

Now we prove that $v_0=\mu^2r$ is a lower solution of T, that is, $v_0\le Tv_0$. Indeed, for $v_0\in P$, we have $Tv_0\in P$ and so
$$\begin{aligned}Tv_0(t)&\ge\mu\|Tv_0(t)\|\ge\mu\min_{t\in[1/2,1]}Tv_0(t)\\&=\mu\Big(\min_{t\in[1/2,1]}\int_0^1 G(t,s)q(s)f(s,v_0(s))\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,v_0(s))\,dA(s)\Big)\\&\ge\mu\int_0^1\Big(\min_{t\in[1/2,1]}G(t,s)\Big)q(s)f(s,v_0(s))\,ds\ge\mu^2\int_0^1 G(1,s)q(s)f(s,u(s))\,ds\ge\mu^2r=v_0(t).\end{aligned}$$

Finally, we show that $w_0=R$ is an upper solution of T, that is, $Tw_0\le w_0$. Clearly,
$$Tw_0(t)\le\int_0^1 G(1,s)q(s)f(s,w_0(s))\,ds+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\,t^{\alpha-1}\int_0^1 h(s,w_0(s))\,dA(s)\le R=w_0(t),$$
so $w_0=R$ is an upper solution of T.

If we construct the sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ as

$$v_n = Tv_{n-1}, \quad w_n = Tw_{n-1}, \ \ n = 1, 2, \dots,$$

then it follows that

$$v_0 \le v_1 \le v_2 \le \dots \le v_n \le \dots \le w_n \le w_{n-1} \le \dots \le w_1 \le w_0,$$

and $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ converge, respectively, to v and w, which are the greatest and smallest fixed points of T in $[v_0, w_0]$. Since $v \le w$, Theorem 2.1 guarantees that w is the positive solution of problem (1.1), (1.2). This completes the proof of the theorem. □

Remark. One may observe from the assumptions (A4) and (A5) that we do not require any superlinearity or sublinearity on f and h either at 0 or ∞. The only assumption we require on f and g is that they must be monotonically nondecreasing in the subinterval $[1/2, 1]$, which shows that the functions f and h may be decreasing or nonincreasing and also may be identically zero or zero at some points in $[0, 1/2)$. This fact is evident from Examples 4.1 and 4.2.

4 An illustration

In this section, we provide two examples illustrating Theorem 1.1.

Example 4.1. Consider the fractional differential equation

$$D_{0+}^{5/2}u(t) + \Gamma\Big(\frac{5}{2}\Big)\big[1 - (1-t)^{3/2}\big]^{-1} f(t, u(t)) = 0, \ \ 0 < t < 1, \tag{4.1}$$

with the multipoint BCs

$$u(0) = u'(0) = 0, \quad D_{0+}^{3/2}u(1) = \int_0^1 h(s, u(s))\, dA(s), \tag{4.2}$$

where

$$A(t) = \begin{cases} t & \text{if } \ t \in \Big[0, \dfrac{4}{9}\Big) \cup \Big[\dfrac{5}{9}, \dfrac{8}{9}\Big), \\ \dfrac{4}{9} & \text{if } \ t \in \Big[\dfrac{4}{9}, \dfrac{5}{9}\Big), \\ \dfrac{8}{9} & \text{if } \ t \in \Big[\dfrac{8}{9}, 1\Big], \end{cases} \tag{4.3}$$

$$f(t, u) = \begin{cases} \dfrac{1}{2}\,(35 + e^{-\frac{1}{u-32}}) & \text{if } \ u > 32, \\ \dfrac{35}{2} & \text{if } \ u \le 32, \end{cases}$$

and

$$h(t, u) = \begin{cases} 28 + e^{-\frac{1}{u-2}} & \text{if } \ u > 2, \\ 28 & \text{if } \ u \le 2. \end{cases}$$

Here $\alpha = \frac{5}{2}$, $\beta = \frac{3}{2}$ and $q(t) = \Gamma(\frac{5}{2})[1 - (1-t)^{3/2}]^{-1}$. Clearly,

$$G(1, t) = \frac{1}{\Gamma(\frac{5}{2})}\,\big[1 - (1-t)^{3/2}\big], \ \ 0 < t \le 1,$$

implies that $q(t)G(1, t) \equiv 1$, hence

$$\int_0^1 G(1, t)q(t)\, dt \equiv 1.$$

Also,

$$\mu = \frac{1}{2^{\alpha-1}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}\,.$$

For $u \le v$, we have $e^{-\frac{1}{u-32}} \le e^{-\frac{1}{v-32}}$, which implies that $f(t,u) \le f(t,v)$ for $u \le v$. In a similar way, we can prove that $h(t,u) \le h(t,v)$ for $u \le v$.

Set $r = 16$ and $R = 40$; then

$$f(t,u) \ge \frac{35}{2} = 17.5 > 16 = r$$

and

$$f(t,u) \le \frac{1}{2}\,(35 + e^{-\frac{1}{u-32}}) \le \frac{1}{2}\,(35 + e^{-\frac{1}{40-32}}) \le \frac{1}{2}\,(35 + e^{-\frac{1}{8}}) \le 18 < 20 = \frac{R}{2}$$

imply that

$$r \le f(t,u) \le f(t,v) \le \frac{R}{2} \ \text{ for } \ \frac{r}{8} \le u \le v \le R \ \text{ and } \ \frac{1}{2} \le t \le 1,$$

that is, condition (A4) is satisfied. Similarly, $h(t,u) \le 29 < \frac{135}{8}\sqrt{\pi}$ implies that condition (A5) is satisfied. Thus, by Theorem 1.1, problem (4.1), (4.2) has at least two positive solutions.

Example 4.2. Consider the fractional differential equation (4.1) together with the BCs (4.2) and $A(t)$ in (4.3) with $f(t,u(t)) = \frac{1}{2} + t\sin\frac{u}{3}$ and $h(t,u) = t + \frac{1}{2} + 0.88\sin u$. Set $r = \frac{1}{2}$ and $R = 3$. Since $\sin u$ is an increasing function for $\frac{1}{16} \le u \le 1$, then $f(t,u)$ and $h(t,u)$ satisfy the properties $f(t,u) \le f(t,v)$ and $h(t,u) \le h(t,v)$ for $u \le v$, $\frac{1}{2} \le t \le 1$ and $\frac{1}{16} = \mu^2 r \le u \le v \le R = 3$. Further, since $\sin u > 0$ for $\frac{1}{16} \le u \le 3$, we have

$$r \le \frac{1}{2} \le \frac{1}{2} + t\sin\frac{u}{3} = f(t,u) \le \frac{1}{2} + \sin 1 \le \frac{3}{2} = \frac{R}{2}$$

and

$$\begin{aligned} h(t,u) &\le 1 + \frac{1}{2} + 0.88\sin u \\ &\le 1 + \frac{1}{2} + (0.88)(0.8415) \\ &\le 2.24049 \\ &\le 2.243216 \\ &= \frac{27\sqrt{\pi}}{64}\,R, \end{aligned}$$

that is, conditions (A4) and (A5) are satisfied. Hence, by Theorem 1.1, problem (4.1), (4.2), with the considered $f(t,u(t))$ and $h(t,u)$, has at least two positive solutions.

5 Discussion and conclusions

The fixed point theorems are playing a vital role in studying, analysing the systems of fractional differential equations and also in establishing positive solutions. These fixed point theorems are also helpful in examining the existence/non-existence conditions for various coexistence equilibria in many dynamical systems with applications to natural, biological and epidemiological sciences. Many of the existing fixed point theorems require the superlinearity and sublinearity conditions.

In [16], Padhi et al. applied Schauder's fixed point theorem (see [16, Theorems 4.2 and 4.4]) to prove the existence of a positive solution of (1.1), (1.2), where the function f is assumed to be either superlinear or sublinear at 0 or ∞. In another attempt, Theorem 4.5 in [16] requires the existence of two reals r_1 and r_2 with $0 < r_1 < r_2$ such that either one of the following conditions is required to prove the existence of a positive solution of (1.1), (1.2):

(A6)

$$
\begin{aligned}
r_1 \le \int_0^1 G(1,s)q(s)f_1(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s,r_1)\,dA(s) < \infty,\\
\int_0^1 G(1,s)q(s)f_2(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s,r_2)\,dA(s) \le r_2,
\end{aligned}
$$

(A7)

$$
\begin{aligned}
\int_0^1 G(1,s)q(s)f_2(s,r_1)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_2(s,r_1)\,dA(s) < \infty,\\
r_2 \le \int_0^1 G(1,s)q(s)f_1(s,r_2)\,ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\int_0^1 h_1(s,r_2)\,dA(s) < \infty,
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(t,r) &= \min\left\{f(t,u): \ t^{\alpha-1}r \le u \le r\right\}, \ 0<t<1,\\
f_2(t,r) &= \max\left\{f(t,u): \ t^{\alpha-1}r \le u \le r\right\}, \ 0<t<1,\\
h_1(t,r) &= \min\left\{h(t,u): \ t^{\alpha-1}r \le u \le r\right\}, \ 0<t<1,\\
h_2(t,r) &= \max\left\{h(t,u): \ t^{\alpha-1}r \le u \le r\right\}, \ 0<t<1.
\end{aligned}
$$

The present work proposes the fixed point theorem with the use of the monotone iterative method for establishing the existence of one positive solution and also the method for approximating the solution. In this process, the obtained sufficient conditions require no superlinearity and/or sublinearity on the functions under consideration at 0 or ∞. Thus, Theorem 1.1 cannot be comparable with Theorems 4.2 and 4.4 in [16]. Instead, the conditions in Theorem 1.1 require the only monotonic increase of the functions in the subinterval $[1/2,1]$ and they may decrease or nonincrease or identically be zero in the other half of the interval $[0,1/2)$. This shows that assumptions (A4) and (A5) are not comparable with (A6) and (A7). We strongly feel that Theorem 1.1 simplifies the calculations in establishing the existence of positive solutions of the boundary value fractional differential equations.

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