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GLOBAL EXISTENCE AND CONTROLLABILITY FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY IN FRÉCHET SPACES

Abstract. The sufficient conditions are given ensuring the existence and the controllability of mild solutions for a semi-linear fractional differential equation with state-dependent delay in Fréchet space. We use in the study a generalization of Darboux's fixed point theorem combined with measures of non-compactness.

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რეზიუმე. მოცემულია საკმარისი პირობები, რომლებიც უზრუნველყოფს სუსტი ამონახსნების არსებობას და მართვას ფრეშეს სივრცეში ნახევრადწრფივი ფრაქციული დიფერენციალური განტოლებისთვის შინაგან მდგომარეობაზე დამოკიდებული დაგვიანებით. კვლევაში გამოყენებულია დარბუს უძრავი წერტილის თეორემის განზოგადების კომბინაცია არაკომპაქტურობის ზომებთან.

1 Introduction

This paper deals with the existence and controllability of mild solutions for a semi-linear fractional differential equation with state-dependent delay in Fréchet spaces. In Section 3, we examine semilinear fractional differential equations with state-dependent delay given by

$$^{c}D^{\alpha}y(t) = Ay(t) + f(t, y(t-\rho(y(t)))), \text{ a.e. } t \in J = [0, +\infty), \ 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \phi(t), \ t \in [-r, 0], \tag{1.2}$$

and, in Section 4, we investigate the controllability of semi-linear fractional differential equation with state-dependent delay

$$^{c}D^{\alpha}y(t) = Ay(t) + f(t, y(t-\rho(y(t)))) + Bu(t), \text{ a.e. } t \in J = [0, +\infty), \ 0 < \alpha < 1, \tag{1.3}$$

$$y(t) = \phi(t), \ t \in [-r, 0], \tag{1.4}$$

where $^{c}D^{\alpha}$ is the standard Caputo fractional derivative, $f : J \times E \to E$ is a given function, $A : D(A) \subset E \to E$ is an almost sectorial operator, that is, $A \in \Theta_{\omega}^{\gamma}(E)$ $(-1 < \gamma < 0,\ 0 < \omega < \frac{\Pi}{2})$, $\Theta_{\omega}^{\gamma}(E)$ is a space of almost sectorial operator to be specified later, the control function u is given in $L^2(J, U)$, a Banach space of admissible control functions, B is a bounded linear operator from U into E, $\phi : [-r, 0] \to E$ is a given continuous function and $(E, \|\cdot\|)$ is a Banach space, ρ is a positive bounded continuous function on $C([-r, 0], E)$, r is the maximal delay defined by

$$r = \sup_{y \in C} |\rho(y)| < \infty.$$

Recently, fractional calculus takes a great interest, in cause, in part to both the intensive development of the theory of fractional calculus itself and the applications of such constructions to different sciences such as physics, mechanics, chemistry, engineering, etc. (for details, see the monographs [17, 21, 23] and the references therein). Newly, several works have been published on the existence and uniqueness of mild solutions for various types of fractional differential equations using different approaches and techniques such as fixed point theorems, probability density functions, lower and upper solutions method, coincidence degree theory, etc. (see, e.g., [2, 3, 12, 15, 28]).

Moreover, the existence of solutions on the half-line of the integer order differential equations has been investigated in [1, 5, 6, 8, 16, 22]. Quite recently, in [25], Su considered the existence of solutions to the boundary value problems of fractional differential equations on unbounded domains by using the Darboux fixed point theorem. The attractiveness of fractional evolution equations with almost sectorial operators has been proved by Zhou [29].

The problem of controllability for linear and nonlinear systems shown by ODEs in a finite-dimensional space has been extensively examined. Certain authors have enlarged the controllability concept to the infinite-dimensional systems in Banach space with unbounded operators (for more details see [11, 20]). N. Carmichael and M. D. Quinn [24] proved that the controllability problem can be translated into a fixed point problem. Interesting controllability results of various classes of fractional differential equations defined on a bounded and unbounded intervals are given in many papers (see e.g., [4, 7, 10, 19]).

Our investigations are considered in the Fréchet spaces by using a generalization of the classical Darboux fixed point theorem with the concept of a family of measures of noncompactness.

The paper is organized as follows. In Section 2, we recall briefly some basic definitions and preliminary facts that will be used throughout the paper. In Section 3, we discuss the existence of mild solutions for problem (1.1), (1.2). In Section 4, we testify the controllability of mild solutions for problem (1.3), (1.4). The investigation on semilinear fractional differential equations with almost sectorial operators have not been shown yet in the Fréchet spaces, so the present results make a valuable contribution to this study.

2 Preliminaries

Let $J = [0, b]$, $b > 0$, be a compact interval in $\mathbb{R}$, $C(J, E)$ be the Banach space of all continuous functions from J to E with the norm

$$\|y\|_\infty = \sup_{t \in J} \|y(t)\|.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable.

Let $L^1(J, E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b \|y(t)\|\, dt.$$

Definition 2.1. A function $f : J \times E \to E$ is said to be Carathéodory if

(i) for each $t \in J$ the function $f(t, \cdot) : E \to E$ is continuous;

(ii) for each $y \in E$ the function $f(\cdot, y) : J \to E$ is measurable.

Definition 2.2 ([17])**.** The fractional primitive of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to E$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

Definition 2.3 ([17])**.** The Riemann–Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : \mathbb{R}^+ \to E$ is given by

$$D^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad t > t_0, \quad n-1 < \alpha < n.$$

Definition 2.4 ([17])**.** The Caputo fractional derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : \mathbb{R}^+ \to E$ is given by

$${}^cD^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds.$$

We denote by $D(A)$ the domain of A, by $\sigma(A)$ its spectrum, while $\rho(A) = \mathbf{C} \setminus \sigma(A)$ is the resolvent set of A, and denote by $R(z, A) = (zI - A)^{-1}$, $z \in \rho(A)$, the family of bounded linear operators which are the resolvents of A.

Definition 2.5. Let $-1 < \gamma < 0$ and $0 < \omega < \frac{\Pi}{2}$. By $\Theta_\omega^\gamma(E)$ we denote the family of all linear closed operators $A : D(A) \subset E \to E$ which satisfy the following conditions:

(a) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \le \omega\} \cup \{0\}$;

(b) for every $\omega < \mu < \Pi$, there exists a constant C_μ such that

$$\|R(z; A)\| \le C_\mu |z|^\gamma \quad \text{for all} \quad z \in \mathbb{C} \setminus S_\mu.$$

A linear operator A is said to be an almost sectorial operator on E if $A \in \Theta_\omega^\gamma(E)$.

Let A be an operator in the class $\Theta_\omega^\gamma(E)$ and $-1<\gamma<0$, $0<\omega<\frac{\Pi}{2}$. Define the operator families $\{\mathcal{S}_\alpha(t)\}_{t\in S^0_{\frac{\Pi}{2}-\omega}}$, $\{\mathcal{P}_\alpha(t)\}_{t\in S^0_{\frac{\Pi}{2}-\omega}}$ by

$$\mathcal{S}_\alpha(t)=E_\alpha(-zt^\alpha)(A)=\frac{1}{2\Pi i}\int\limits_{\Gamma_\theta}E_\alpha(-zt^\alpha)R(z,A)\,dz,$$

$$\mathcal{P}_\alpha(t)=e_\alpha(-zt^\alpha)(A)=\frac{1}{2\Pi i}\int\limits_{\Gamma_\theta}e_\alpha(-zt^\alpha)R(z,A)\,dz,$$

where the integral contour $\Gamma_\theta=\{\mathbb{R}_+e^{i\theta}\}\cup\{\mathbb{R}_+e^{-i\theta}\}$ is oriented counter-clockwise and $\omega<\theta<\mu<\frac{\Pi}{2}-|\arg t|$. Now, we present the following important results about the operators $\mathcal{S}_\alpha$ and $\mathcal{P}_\alpha$.

Theorem 2.6 ([27])**.** *For each fixed $t\in S^0_{\frac{\Pi}{2}-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are the bounded linear operators on E. Moreover, there exist the constants $C_s=C(\alpha,\gamma)>0$, $C_p=C(\alpha,\gamma)>0$ such that for all $t>0$,*

$$\|\mathcal{S}_\alpha(t)\|\le C_st^{-\alpha(1+\gamma)},\quad \|\mathcal{P}_\alpha(t)\|\le C_pt^{-\alpha(1+\gamma)}.$$

Also,

$$\mathcal{S}_\alpha(t)x=\int\limits_0^\infty\Psi_\alpha(s)T(st^\alpha)x\,ds,\ \ t\in S^0_{\frac{\Pi}{2}-\omega},\ \ x\in E,$$

and

$$\mathcal{P}_\alpha(t)x=\int\limits_0^\infty\alpha s\Psi_\alpha(s)T(st^\alpha)x\,ds,\ \ t\in S^0_{\frac{\Pi}{2}-\omega},\ \ x\in E,$$

where $T(\,\cdot\,)$ is a semigroup associated with A.

Theorem 2.7 ([27])**.** *For $t>0$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology.*

Consider the problem

$$^cD^\alpha y(t)-Ay(t)=f(t),\ \ t\in(0,b],\tag{2.1}$$

$$y(0)=y_0,\tag{2.2}$$

where $^cD^\alpha$, $0<\alpha<1$, is the Caputo fractional derivative, $f\in L^1(J,E)$ and $y_0\in E$.

Definition 2.8 ([27])**.** A function $y\in C([0,b],E)$ is called a mild solution of Problem (2.1), (2.2) if

$$y(t)=\mathcal{S}_\alpha(t)y_0+\int\limits_0^t(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)\,ds,\ \ t\in[0,b].$$

Let $C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions ν from $\mathbb{R}_+$ into E, equipped with the family semi-norms

$$\|\nu\|_n=\sup_{t\in[0,n]}\|\nu(t)\|,\ \ n\in\mathbb{N},$$

and the distance

$$d(u,v)=\sum_{n=1}^\infty 2^{-n}\,\frac{\|u-v\|_n}{1+\|u-v\|_n}\,,\ \ u,v\in C(\mathbb{R}_+).$$

(For more details about measures of noncompactness see [13, 14].)

Definition 2.9. Let $\mathcal{M}_X$ be the family of all nonempty and bounded subsets of a Fréchet space X. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$, where $\mu_n:\mathcal{M}_X\to[0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if for all $B,B_1,B_2\in\mathcal{M}_X$ it satisfies the following conditions:

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is, $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact;

(b) $\mu_n(B_1) < \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$;

(c) $\mu(ConvB) = \mu(B)$ for $n \in \mathbb{N}$;

(d) if $\{B\}$ is a sequence of closed sets from $\mathcal{M}_X$ such that $B_{i+1} \subset B_i$, $i = 1, \dots,$ and if $\lim_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty = \bigcap_{i=1}^{\infty} B_i$ is nonempty.

Definition 2.10. A nonempty subset $B \subset X$ is said to be bounded if for $n \in \mathbb{N}$, there exists $M_n > 0$ such that

$$\|y\|_n \le M_n, \ \text{ for each } \ y \in B.$$

Lemma 2.11 ([9])**.** *If Y is a bounded subset of the Banach space X, then for each $\varepsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu(Y) \le 2\mu\big(\{y_k\}_{k=1}^{\infty}\big) + \varepsilon.$$

Lemma 2.12 ([18])**.** *If $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ is uniformly integrable, then $\mu(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}$ and*

$$\mu\Bigg(\bigg\{\int_0^t u_k(s)\,ds\bigg\}_{k=1}^{\infty}\Bigg) \le 2\int_0^t \mu\big(\{u_k(s)\}_{k=1}^{\infty}\big)\,ds$$

for each $t \in [0, n]$.

Definition 2.13. Let Ω be a nonempty subset of a Fréchet space X, and let $A : \Omega \to X$ be a continuous operator which transforms bounded subsets onto the bounded ones. One says that A satisfies the Darboux condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $(\mu_n)_{n\in\mathbb{N}}$ if

$$\mu_n(A(B)) \le k_n \mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$. If $k_n < 1$, $n \in \mathbb{N}$, then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel, we will make use of the following generalization of the classical Darboux fixed point theorem for the Fréchet spaces.

Theorem 2.14 ([13,14])**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Fréchet space F and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

3 The main result

Influenced by [27] with $\phi(0) \in D(A^\beta)$, $\beta > 1 + \gamma$, we define a mild solution of problem (1.1), (1.2) by the following

Definition 3.1. We say that a continuous function $y : \mathbb{R} \to E$ is a mild solution of problem (1.1), (1.2) if $y(t) = \phi(t)$ for all $t \in [-r, 0]$ and y satisfies the integral equation

$$y(t) = \mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s - \rho(y(s)))\big)\,ds \ \text{ for each } \ t \in J.$$

Let us include the hypotheses.

(H1) The function $f : J \times E \to E$ is Carathéodory.

(H2) There exist a function $p \in L^1_{loc}(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : J \to [0, +\infty)$ such that
$$\|f(t,u)\| \leq p(t)\psi(\|u\|) \ \text{ for a.e. } \ t \in J \ \text{ and each } \ u \in E.$$

(H3) There exists a function $l \in L^1_{loc}(J, \mathbb{R}^+)$ such that for any bounded set $B \subset E$, and for each $t \in J$, we have
$$\alpha((f,B)) \leq l(t)\alpha(B).$$

(H4) There exists $r_n > 0$ such that
$$C_s n^{-\alpha(1+\gamma)}|\phi(0)| + C_p \psi(r_n) \sup_{t\in[0,n]} \left\{ \int_0^t (t-s)^{-(1+\alpha\gamma)} p(s)\, ds \right\} \leq r_n.$$

For $n \in \mathbb{N}$, we define on $C([-r,+\infty), E)$ the family of measures of noncompactness by
$$\mu_n(V) = \omega_0^n(V) + \sup_{t\in[0,n]} e^{-Lt}\mu(V(t)),$$
where $V(t) = \{v(t) \in E : v \in V)\}, t \in [0,n]$, and $L > 0$ is a constant chosen so that
$$l_n = 4C_p \sup_{t\in[0,n]} \int_0^t e^{-L(t-s)}(t-s)^{-(1+\alpha\gamma)} l(s)\, ds < 1.$$

Remark 3.2. Notice that if the set V is equicontinuous, then $\omega_0^n(V) = 0$.

Theorem 3.3. *Assume* (H1)–(H4) *are satisfied. Then problem* (1.1), (1.2) *admits at least one mild solution.*

Proof. Consider the operator $N : C([-r,+\infty), E) \to C([-r,+\infty), E)$ given by
$$(Ny)(t) = \begin{cases} \phi(t) & \text{if } \ t \in [-r, 0]; \\ \mathcal{S}_\alpha(t)\phi(0) + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s-\rho(y(s)))\big)\, ds & \text{if } \ t \in J. \end{cases}$$

We shall check that the operator N satisfies all conditions of Theorem 2.14. The proof is given in several steps.

Let
$$B_{r_n} = \big\{u \in C([-r,+\infty), E) : \ \|u\|_n \leq r_n\big\},$$
where r_n is the constant given by (H4). It is obvious that the subset B_{r_n} is closed, bounded and convex.

Step 1. $N(B_{r_n}) \subset B_{r_n}$.

For any $n \in \mathbb{N}$ and for each $y \in B_{r_n}$ and $t \in [0,n]$, we have
$$\begin{aligned} \|(Ny)(t)\| &\leq \|\mathcal{S}_\alpha(t)\|\,|\phi(0)| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\, \big\|f\big(s, y(s-\rho(y(s)))\big)\big\|\, ds \\ &\leq C_s t^{-\alpha(1+\gamma)}|\phi(0)| + \int_0^t (t-s)^{-(1+\alpha\gamma)} C_p p(s)\psi(\|y(s)\|)\, ds \\ &\leq C_s n^{-\alpha(1+\gamma)}|\phi(0)| + C_p\psi(r_n) \sup_{t\in[0,n]} \left\{ \int_0^t (t-s)^{-(1+\alpha\gamma)} p(s)\, ds \right\} \\ &\leq r_n. \end{aligned}$$

Thus

$$\|N(y)\|_n \leq r_n.$$

Step 2. N is continuous on B_{r_n}.

Let y_n be a sequence such that $y_n \longrightarrow y$ in B_{r_n}. Then for each $t \in [0, n]$, we have

$$\begin{aligned}
&\big\|(Ny_n)(t) - (Ny)(t)\big\| \\
&\quad\leq \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\| \Big\|f\big(s, y_n(s-\rho(y_n(s)))\big) - f\big(s, y(s-\rho(y(s)))\big)\Big\| \, ds \\
&\qquad\leq C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} \Big\|f\big(s, y_n(s-\rho(y_n(s)))\big) - f\big(s, y(s-\rho(y(s)))\big)\Big\| \, ds.
\end{aligned}$$

Since f is a Carathéodory function for $t \in [0, n]$, from the continuity of ρ, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_n \longrightarrow 0 \text{ as } n \to \infty.$$

Step 3. $N(B_{r_n})$ is bounded which is clear.

Step 4. For each bounded equicontinuous subset V of B_{r_n}, $\mu_n(N(V)) \leq k_n \mu_n(V)$.

From Lemmas 2.11 and 2.12, for any $V \subset B_{r_n}$ and any $\epsilon > 0$, there exists a sequence $\{y_k\}_{k=0}^{\infty} \subset V$ such that for all $t \in [0, n]$,

$$\begin{aligned}
\mu((NV)(t)) &= \mu\Bigg(\Bigg\{\mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s-\rho(y(s)))\big)\, ds, v \in V\Bigg\}\Bigg) \\
&\leq 2\mu\Bigg(\Bigg\{\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(t, y_k(s-\rho(y_k(s)))\big)\, ds\Bigg\}_{k=1}^{\infty}\Bigg) + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} \mu\Big(\Big\{f\big(t, y_k(s-\rho(y_k(s)))\big)\Big\}_{k=1}^{\infty}\Big)\, ds + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} l(s) \mu\big(\{(y_k(s))\}_{k=1}^{\infty}\big)\, ds + \epsilon \\
&\leq 4C_p \int_0^t e^{Ls}(t-s)^{-(1+\alpha\gamma)} e^{-Ls} l(s) \mu\big(\{(y_k(s))\}_{k=1}^{\infty}\big)\, ds + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\mu(N(V)) \leq 4C_p \int_0^t e^{-L(t-s)}(t-s)^{-(1+\alpha\gamma)} l(s)\mu_n(V)\, ds.$$

Thus

$$\mu_n(N(V)) \leq l_n \mu_n(V).$$

As a conclusion, N has at least one fixed point in B_{r_n}. □

4 Controllability of semilinear fractional differential equations with state-dependent delay

In this section, we prove a controllability result for system (1.3), (1.4).

Definition 4.1. System (1.3), (1.4) is said to be controllable if for any continuous function $\phi \in [-r, 0]$, any $y_1 \in E$ and for each $n \in \mathbb{N}$ there exists a control $u \in L^2([0, n], E)$ such that the mild solution $y(\cdot)$ of (1.3), (1.4) satisfies $y(n) = y_1$.

Let us introduce the following hypotheses:

(H4′) There exists $r'_n > 0$ such that

$$C_s n^{-\alpha(1+\gamma)}|\phi(0)|\Big[1+\frac{n^{-\alpha\gamma}}{-\alpha\gamma}\Big] + |y_1| C_p M_1 M_2 \frac{n^{-\alpha\gamma}}{-\alpha\gamma} + C_p \psi(r'_n) \int_0^n (t-s)^{-(1+\alpha\gamma)} p(s)\, ds \cdot \Big[1 + \frac{n^{-\alpha\gamma}}{-\alpha\gamma} C_p M_1 M_2\Big] \le {r'}_n.$$

(H5) For each $n > 0$, the linear operator $W : L^2([0, n], U) \to E$ is defined by

$$Wu = \int_0^n (t-s)^{\alpha-1} P_\alpha(n-s)(Bu(s))\, ds,$$

and

(i) the operator W has a pseudo-invertible operator W^{-1} which takes values in $L^2([0, n], U)/\operatorname{Ker} W$ and there exist positive constants M_1, M_2 such that

$$\|B\| \le M_1 \ \text{ and } \ \|W^{-1}\| \le M_2,$$

(ii) there exist $\eta_W(t) \in L^\infty(J, \mathbb{R}^+)$, $C_B \ge 0$, for any bounded sets $V_1 \subset E$, $V_2 \subset U$,

$$\mu((W^{-1}V_1)(t)) \le \eta_W(t)\mu(V_1(t)), \quad \mu((BV_2)) \le C_B \mu_U(V_2).$$

Theorem 4.2. *Suppose that hypotheses* (H1)–(H3) *and* (H4′)–(H5) *hold. Further, assume that the inequality*

$$l_n\Big(1 + 2C_p C_B \|\eta_W\|_{L^\infty} \frac{n^{-\alpha\gamma}}{\alpha\gamma}\Big) < 1$$

holds, then problem (1.3), (1.4) *is controllable.*

Proof. We define in $C((-\infty, r], E)$ the family of measures of noncompactness by

$$\mu_n(V) = \omega_0^n(V) + \sup_{t\in[0,n]} e^{-Lt}\mu(V(t)),$$

where $V(t) = \{v(t) \in E : \ v \in V\}$.

Consider the operator $N_1 : C((-\infty, r], E) \to C((-\infty, r], E)$ defined by

$$(N_1 y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0]; \\ \mathcal{S}_\alpha(t)\phi(0) + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f\big(s, y(s-\rho(y(s)))\big)\, ds \\ \quad + \displaystyle\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) B u_y(s)\, ds & \text{if } t \in J. \end{cases}$$

Using assumption (H5), for an arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = W^{-1}\Big[y_1 - \mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f\big(s, y(s-\rho(y(s)))\big)\,ds\Big](t).$$

Noting that

$$\|u_y(t)\| \le \|W^{-1}\|\,\Big[|y_1| + \|\mathcal{S}_\alpha(t)\phi(0)\| + \int_0^n (n-\tau)^{\alpha-1}\mathcal{P}_\alpha(n-\tau)f\big(\tau, y(\tau-\rho(y(\tau)))\big)\,d\tau\Big],$$

by (H2) we get

$$\|u_y(t)\| \le M_2\Big[|y_1| + C_s t^{-\alpha(1+\gamma)}|\phi(0)| + \int_0^n C_p(n-\tau)^{-(1+\alpha\gamma)}p(\tau)\|y(\tau)\|\,d\tau\Big]. \tag{4.1}$$

Next, for any $n \in \mathbb{N}$,

$$B_{r'_n} = B(0, r'_n) = \big\{w \in C([-r,\infty), E) :\ \|w\|_n \le r'_n\big\},$$

where $r'_n > 0$ is the constant defined in (H4′). Obviously, the subset $B_{r'_n}$ is closed, bounded and convex.

Step 1. $N_1(B_{r_n}) \subset B_{r_n}$.
For any $n \in \mathbb{N}$, and each $y \in B_{r'_n}$, by (4.1) we have

$$\begin{aligned}
\|(N_1y)(t)\| &\le \|\mathcal{S}_\alpha(t)\|\,|\phi(0)| + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\,\big\|f\big(s, y(s-\rho(y(s)))\big)\big\|\,ds\\
&\quad + \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\|Bu_y(s)\|\,ds\\
&\le C_s n^{-\alpha(1+\gamma)}|\phi(0)| + C_p\psi(r'_n)\int_0^t (t-s)^{-(1+\alpha\gamma)}p(s)\,ds\\
&\quad + C_pM_1M_2\int_0^t (t-s)^{-(1+\alpha\gamma)}\Big[|y_1| + C_s n^{-\alpha(1+\gamma)}|\phi(0)|\\
&\qquad + C_p\psi(r'_n)\int_0^n (n-\tau)^{-(1+\alpha\gamma)}p(\tau)\,d\tau\Big]\,ds\\
&\le C_s n^{-\alpha(1+\gamma)}|\phi(0)|\,\Big[1 + \frac{n^{-\alpha\gamma}}{-\alpha\gamma}\Big] + |y_1|C_pM_1M_2\,\frac{n^{-\alpha\gamma}}{-\alpha\gamma}\\
&\quad + C_p\psi(r'_n)\int_0^n (t-s)^{-(1+\alpha\gamma)}p(s)\,ds\cdot\Big[1 + \frac{n^{-\alpha\gamma}}{-\alpha\gamma}\,C_pM_1M_2\Big]\\
&\le r'_n.
\end{aligned}$$

Step 2. N_1 is continuous on $B_{r'_n}$.

Let y_n be a sequence such that $y_n \longrightarrow y$ in $B_{r'_n}$. Then for each $t \in [0,n]$, and by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\|(N_1 y_n)(t) - (N_1 y)(t)\| \\
&\quad \leq \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \left\| f\big(s, y_n(s-\rho(y_n(s)))\big) - f\big(s, y(s-\rho(y(s)))\big) \right\| ds \\
&\qquad + \int_0^t (t-s)^{\alpha-1} \|\mathcal{P}_\alpha(t-s)\| \left\| Bu_{y_n}(s) - Bu_y(s) \right\| ds \longrightarrow 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Thus N_1 is continuous.

Step 3. Since $N_1(B_{r_n}) \subset B_{r'_n}$ and $B_{r'_n}$ is bounded, we find that $N_1(B_{r'_n})$ is bounded.

Step 4. For each bounded subset V of $B_{r'_n}$, $\mu_n(N_1(V)) \leq k_n \mu_n(V)$. □

From Lemmas 2.11 and 2.12, for any $V \subset B_{r'_n}$ and any $\epsilon > 0$, there exists a sequence $\{y_k\}_{k=0}^\infty \subset V$ such that for all $t \in [0,n]$, we have

$$\begin{aligned}
\mu((N_1 V)(t)) &= \mu\bigg(\bigg\{ \mathcal{S}_\alpha(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) \Big[f(s, y(s-\rho(y(s)))) + Bu_y(s) \Big] ds, v \in V \bigg\}\bigg) \\
&\leq 2\mu\bigg(\bigg\{ \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) \Big[f\big(s, y_k(s-\rho(y_k(s)))\big) + Bu_{y_k}(s) \Big] ds \bigg\}_{k=1}^\infty \bigg) + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} \mu\Big(\Big\{ f\big(s, y_k(s-\rho(y_k(s)))\big) + Bu_{y_k}(s) \Big\}_{k=1}^\infty \Big) + \epsilon \\
&\leq 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} l(s) \mu\big(\{y_k(s)\}_{k=1}^\infty\big) + \epsilon \\
&\qquad + 4C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} C_B \mu\big(\{u_{y_k}(s)\}_{k=1}^\infty\big)\, ds.
\end{aligned}$$

Now, let us calculate $\mu(\{u_{y_k}(s))\}_{k=1}^\infty)$.

By (H5) we have

$$\begin{aligned}
\mu\big(\{u_{y_k}(t)\}_{k=1}^\infty\big) &\leq 2\eta_W(t) C_p \int_0^t (t-s)^{-(1+\alpha\gamma)} l(s) \mu\big(\{(y_k(s))\}_{k=1}^\infty\big)\, ds \\
&\leq \frac{1}{2}\, \eta_W(t) C_p 4 \int_0^t (t-s)^{-(1+\alpha\gamma)} e^{Ls} e^{-Ls} l(s) \mu\big(v\{(y_k(s))v\}_{k=1}^\infty v\big)\, ds.
\end{aligned}$$

Then

$$\mu_n(u(V)) \leq \frac{1}{2}\, l_n \eta_W(t) \mu_n(V). \tag{4.2}$$

Since $\epsilon > 0$ is arbitrary, by (4.2) we obtain

$$\mu(N_1(V)) \leq l_n \mu_n(V) + 2 l_n C_p C_B \,\frac{t^{-\alpha\gamma}}{\alpha\gamma}\, \|\eta_W\|_{L^\infty} \mu_n(V).$$

Thus

$$\mu_n(N_1(V)) \leq l_n \Big(1 + 2C_p C_B \|\eta_W\|_{L^\infty} \,\frac{n^{-\alpha\gamma}}{\alpha\gamma} \Big) \mu_n(V).$$

As a conclusion, we have achieved that N_1 has at least one fixed point in $B_{r'_n}$.

5 An example

We consider the fractional differential equation with state-dependent delay of the form

$$\begin{cases} {}_0^c\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) + Q(t)|u(t-\tau(u(t,x)),x)|, & x\in[0,\pi],\ \ t\in[0,\infty),\\ u(t,x) = u_0(t,x), & x\in[0,\pi],\ \ -\tau_{\max}\le t\le 0,\\ u(t,0)=u(t,\pi)=0, & t\in[0,\infty), \end{cases} \tag{5.1}$$

where $u_0 \in C^2([-\tau_{\max},0]\times[0,\pi],\mathbb{R})$ Q is a continuous function from $[0,+\infty)$ to $\mathbb{R}$, the delay function τ is the bounded positive continuous function in $\mathbb{R}^n$, and $\tau_{\max}$ is the maximal delay which is defined by

$$\tau_{\max} = \sup_{x\in\mathbb{R}} \tau(x).$$

Consider the space of Hölder continuous functions $E = C^l([0,\pi],\mathbb{R})$ $(0<l<1)$, and let ${}_0^c\partial^\alpha$ be the regularized Caputo fractional partial derivative of order $0<\alpha<1$ with respect to t defined by

$$({}_0^c\partial^\alpha u)(t,x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha}u(t,x)\,ds - t^{-\alpha}u(0,x)\right).$$

Next, we introduce the operator

$$A = -\partial_x^2, \quad D(A) = \big\{u\in C^{2+l}([0,\pi]) :\ u(t,0)=u(t,\pi)=0\big\}$$

in the space $C^l([0,\pi],\mathbb{R})$. It follows from [26] that ν exists, $\epsilon>0$ such that $A+\nu \in \Theta^{\frac{l}{2}-1}_{\frac{\pi}{2}-\epsilon}(X)$. Set

$$\begin{aligned} y(t)(x) &= u(t,x),\ \ t\in(-\infty,0],\ \ x\in[0,\pi],\\ \phi(t)(x) &= u_0(t,x),\ \ t\in[-\tau_{\max},0],\ \ x\in[0,\pi],\\ f(t,\varphi)(x) &= Q(t)\big|u(t-\tau(u(t,x)),x)\big|,\ \ \varphi\in E,\quad t\in[0,+\infty),\ \ -\infty<\theta\le 0,\ \ x\in[0,\pi]. \end{aligned}$$

Then system (5.1) can be written in the abstract form as (1.1), (1.2). As a consequence of Theorem 2.14, system (5.1) has a mild solution.

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