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EXPONENTIAL STABILITY FOR COUPLED FLEXIBLE STRUCTURE SYSTEM WITH DISTRIBUTED DELAY

Abstract. In this paper, we consider a coupled flexible structure system with distributed delay in two equations. We first give the well-posedness of the system by using a semigroup method. Then, by using the perturbed energy method and constructing some Lyapunov functionals, we obtain the exponential decay result.

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1 Introduction

In this article, we study the well-posedness and exponential stability for coupled flexible structure system with distributed delay in two equations

$$\begin{cases} m_1(x)u_{tt} - (p_1(x)u_x + 2\delta_1(x)u_{xt})_x + \mu_0 u_t + \int\limits_{\tau_1}^{\tau_2} \mu_1(s)u_t(x, t-s)\, ds = 0, \\ m_2(x)v_{tt} - (p_2(x)v_x + 2\delta_2(x)v_{xt})_x + \mu_0' v_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s)v_t(x, t-s)\, ds = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, with the following initial and boundary conditions:

$$\begin{gathered} u(\,\cdot\,,0) = u_0(x), \;\; u_t(\,\cdot\,,0) = u_1(x), \;\; \forall\, x \in [0,L], \\ u(0,t) = u(L,t) = 0, \;\; \forall\, t \geq 0, \\ v(\,\cdot\,,0) = v_0(x), \;\; v_t(\,\cdot\,,0) = v_1(x), \;\; \forall\, x \in [0,L], \\ v(0,t) = v(L,t) = 0, \;\; \forall\, t \geq 0, \\ u_t(x,-t) = f_0(x,t), \;\; 0 < t \leq \tau_2, \\ v_t(x,-t) = g_0(x,t), \;\; 0 < t \leq \tau_2, \end{gathered} \tag{1.2}$$

where $u(x,t), v(x,t)$ are the displacements of a particle at position $x \in (0,L)$ and time $t > 0$. u_0, v_0 are initial data, and f_0, g_0 are the history function. The parameters $m_i(x)$, $\delta_i(x)$ and $p_i(x)$ (for $i = 1,2$) are responsible for the non-uniform structure of the body, where $m_i(x)$ denotes mass per unit length of the structure, $\delta_i(x)$ is a coefficient of internal material damping and $p_i(x)$ is a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m_i(x), \delta_i(x)$ and $p_i(x)$ in [1] such that

$$m_i, \delta_i, p_i \in W^{1,\infty}(0,L), \;\; m_i(x), \delta_i(x), p_i(x) > 0, \;\; \forall\, x \in [0,L] \;\text{ for }\; i = 1,2.$$

The coefficients μ_0, μ_0' are positive constants, and $\mu_1, \mu_2 : [\tau_1; \tau_2] \to \mathbb{R}$ are the bounded functions, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for the problem on the under the assumption

$$\begin{cases} \mu_0 > \int\limits_{\tau_1}^{\tau_2} |\mu_1(s)|\, ds, \\ \mu_0' > \int\limits_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds. \end{cases} \tag{1.3}$$

During the last few decades, the theory of stabilisation of flexible structural system has been a topic of interest in view of vibration control of various structural elements. In [6], Gorain established the uniform exponential stability of the problem

$$m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x = f(x) \;\text{ on }\; (0,L) \times \mathbb{R}^+,$$

which describes the vibrations of an inhomogeneous flexible structure with an exterior disturbing force f. Indeed, it is physically relevant to take into account thermal effects in flexible structures: in 2014, M. Siddhartha et al. [9] showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{cases} m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x + \kappa\theta_x = f, \\ \theta_t - \theta_{xx} + \kappa u_{tx} = 0. \end{cases}$$

It is known that the dynamic systems with delay terms have become a major research subject in the differential equation since the 1970s of the last century (see, e.g., [2–4, 7, 8, 11–15, 18]). It may not only destabilize a system which is asymptotically stable in the absence of delay, but may also lead to the well-posedness (see [5, 17] and the references therein). Therefore, the stability issue of systems with delay is of great theoretical and practical importance. In [8], the authors consider a non-uniform flexible structure system with time delay under Cattaneo's law of heat condition

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \eta\theta_x + \mu u_t(x, t-\tau_0) = 0, & x \in (0, L), \ \ t > 0, \\ \theta_t + \kappa q_x + \eta u_{tx} = 0, & x \in (0, L), \ \ t > 0, \\ \tau q_t + \beta q + \kappa\theta_x = 0, & x \in (0, L), \ \ t > 0, \end{cases} \tag{1.4}$$

with the boundary condition

$$u(0, t) = u(L, t) = 0, \ \ \theta(0, t) = \theta(L, t) = 0, \ \ t \geq 0, \tag{1.5}$$

and the initial condition

$$u(x, 0) = u_0(x), \ \ u_t(x, 0) = u_1(x), \ \ \theta(x, 0) = \theta_0(x), \ \ q(x, 0) = q_0(x), \ \ x \in [0, L]. \tag{1.6}$$

They proved that system (1.4)–(1.6) is well-posed, and the system is an exponential decay under a small condition on time delay. M. S. Alves et al. (see [1]) considered system (1.4)–(1.6) without delay term, and obtained an exponential stability result for one set of boundary conditions and at least a polynomial for another set of boundary conditions.

In [14], Nicaise and Pignotti considered the wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \triangle u + \mu_1 u_t + a(x) \int\limits_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)\, ds = 0$$

in $\Omega \times (0, \infty)$, with initial and mixed Dirichlet–Neumann boundary conditions and a as a function, chosen in an appropriate space. They established exponential stability of the solution under the assumption

$$\|a\|_\infty \int\limits_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1.$$

The authors also obtained the same result when the distributed delay acted on a part of the boundary.

Motivated by the above results, in the present work we consider system (1.1), (1.2), prove the well-posedness and establish exponential stability results.

We now briefly sketch the outline of the paper. In Section 2, we state and prove the well-posedness of system (1.1), (1.2) by using the semigroup method. In Section 3, we establish an exponential stability by using the perturbed energy method and construct some Lyapunov functionals.

2 The well-posedness

In this section, we give a brief idea about the existence and uniqueness of solutions for (1.1), (1.2) using the semigroup theory [16]. As in [14], we introduce the new variables

$$\begin{aligned} z_1(x, \rho, t, s) &= u_t(x, t-\rho s), \ \ x \in (0, L), \ \ \rho \in (0, 1), \ \ s \in (\tau_1, \tau_2), \ \ t > 0, \\ z_2(x, \rho, t, s) &= v_t(x, t-\rho s), \ \ x \in (0, L), \ \ \rho \in (0, 1), \ \ s \in (\tau_1, \tau_2), \ \ t > 0. \end{aligned}$$

Then we have

$$s z_{it}(x, \rho, t, s) + z_{i\rho}(x, \rho, t, s) = 0 \ \text{ in } \ (0, L) \times (0, 1) \times (0, \infty) \times (\tau_1, \tau_2) \ \text{ for } \ i = 1, 2.$$

Therefore, problem (1.1) takes the form

$$\begin{cases} m_1(x)u_{tt} - (p_1(x)u_x + 2\delta_1(x)u_{xt})_x + \mu_0 u_t + \int\limits_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\, ds = 0, \\ s z_{1t}(x,\rho,t,s) + z_{1\rho}(x,\rho,t,s) = 0, \\ m_2(x)v_{tt} - (p_2(x)v_x + 2\delta_2(x)v_{xt})_x + \mu_0' v_t + \int\limits_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\, ds = 0, \\ s z_{2t}(x,\rho,t,s) + z_{2\rho}(x,\rho,t,s) = 0, \end{cases} \tag{2.1}$$

with the following initial and boundary conditions:

$$\begin{cases} u(\,\cdot\,,0) = u_0(x), \quad u_t(\,\cdot\,,0) = u_1(x), \quad \forall\, x \in [0,L], \\ u(0,t) = u(L,t) = 0, \quad \forall\, t \geq 0, \\ v(\,\cdot\,,0) = v_0(x), \quad v_t(\,\cdot\,,0) = v_1(x), \quad \forall\, x \in [0,L], \\ v(0,t) = v(L,t) = 0, \quad \forall\, t \geq 0, \\ z_1(x,0,t,s) = u_t(x,t) \ \text{ on } \ (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z_2(x,0,t,s) = v_t(x,t) \ \text{ on } \ (0,L)\times(0,\infty)\times(\tau_1,\tau_2), \\ z_1(x,\rho,0,s) = f_0(x,\rho s) \ \text{ on } \ (0,L)\times(0,1)\times(\tau_1,\tau_2), \\ z_2(x,\rho,0,s) = g_0(x,\rho s) \ \text{ on } \ (0,L)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.2}$$

Introducing the vector function $U = (u, \varphi, z_1, v, \psi, z_2)^T$, where $\varphi = u_t$ and $\psi = v_t$, system (2.1), (2.2) can be written as

$$\begin{cases} U'(t) + \mathcal{A}U(t) = 0, \quad t > 0, \\ U(0) = U_0 = (u_0, u_1, f_0, v_0, v_1, g_0)^T, \end{cases} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -\varphi \\ -\dfrac{1}{m_1(x)}\,(p_1(x)u_x + 2\delta_1(x)\varphi_x)_x + \dfrac{\mu_0}{m_1(x)}\,\varphi + \dfrac{1}{m_1(x)} \int\limits_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\, ds \\ s^{-1} z_{1\rho} \\ -\psi \\ -\dfrac{1}{m_2(x)}\,(p_2(x)v_x + 2\delta_2(x)\psi_x)_x + \dfrac{\mu_0'}{m_2(x)}\,\psi + \dfrac{1}{m_2(x)} \int\limits_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\, ds \\ s^{-1} z_{2\rho} \end{pmatrix}.$$

Next, we define the energy space as

$$\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)) \\ \times H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),$$

equipped with the inner product

$$\langle U, \widetilde{U} \rangle_{\mathcal{H}} = \int\limits_0^L p_1(x) u_x \widetilde{u}_x\, dx + \int\limits_0^L m_1(x) \varphi \widetilde{\varphi}\, dx + \int\limits_0^L \int\limits_0^1 \int\limits_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1(x,\rho,s) \widetilde{z}_1(x,\rho,s)\, ds\, d\rho\, dx \\ + \int\limits_0^L p_2(x) v_x \widetilde{v}_x\, dx + \int\limits_0^L m_2(x) \psi \widetilde{\psi}\, dx + \int\limits_0^L \int\limits_0^1 \int\limits_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2(x,\rho,s) \widetilde{z}_2(x,\rho,s)\, ds\, d\rho\, dx.$$

Then the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H} \mid \ u, v \in H^2(0,L) \cap H_0^1(0,L), \ \ \varphi, \psi \in H_0^1(0,L), \\ z_1, z_{1\rho}, z_2, z_{2\rho} \in L^2((0,L) \times (0,1) \times (\tau_1, \tau_2)), \\ z_1(x,0,s) = \varphi(x), \ \ z_2(x,0,s) = \psi(x) \end{array} \right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

The well-posedness of problem (2.3) is ensured by

Theorem 2.1. *Assume that $U_0 \in \mathcal{H}$ and (1.3) holds, then problem (2.3) has a unique solution $U \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}^+; D(\mathcal{A})) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

Proof. The result follows from the Lumer–Phillips theorem provided we prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $U = (u, \varphi, z_1, v, \psi, z_2)^T \in D(\mathcal{A})$, by using the inner product and integrating by parts, we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = 2 \int_0^L \delta_1(x) \varphi_x^2 \, dx + \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) \, ds \, dx + \mu_0 \int_0^L \varphi^2 \, dx \\ + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1(x,\rho,s) z_{1\rho}(x,\rho,s) \, ds \, d\rho \, dx + 2 \int_0^L \delta_2(x) \psi_x^2 \, dx \\ + \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s) \, ds \, dx + \mu_0' \int_0^L \psi^2 \, dx + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2(x,\rho,s) z_{2\rho}(x,\rho,s) \, ds \, d\rho \, dx. \end{aligned}$$

Integrating by parts in ρ, we have

$$\begin{aligned} \int_0^L \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_i(s)| z_i(x,\rho,s) z_{i\rho}(x,\rho,s) d\rho \, ds \, dx \\ = \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_i(s)| \left[z_i^2(x,1,s) - z_i^2(x,0,s) \right] \, ds \, dx \ \text{ for } \ i = 1,2. \end{aligned}$$

Using the fact that $z_1(x,0,s,t) = \varphi$ and $z_2(x,0,s,t) = \psi$, we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = 2 \int_0^L \delta_1(x) \varphi_x^2 \, dx + \int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) \, ds \, dx + \left(\mu_0 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(s)| \, ds \right) \int_0^L \varphi^2 \, dx \\ + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s) \, ds \, dx + 2 \int_0^L \delta_2(x) \psi_x^2 \, dx + \int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s) \, ds \, dx \\ + \left(\mu_0' - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, ds \right) \int_0^L \psi^2 \, dx + \frac{1}{2} \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s) \, ds \, dx. \end{aligned} \tag{2.4}$$

Now, using Young's inequality, we can estimate

$$\int_0^L \varphi \int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s) \, ds \, dx \geq -\frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_1(s)| \, ds \int_0^1 \varphi^2 \, dx - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s) \, ds \, dx \tag{2.5}$$

and

$$\int_0^L \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\,ds\,dx \geq -\frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \int_0^1 \psi^2\,dx - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s)\,ds\,dx. \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4), and using (1.3), we obtain

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \geq 2\int_0^L \delta_1(x)\varphi_x^2\,dx + \left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds\right)\int_0^L \varphi^2\,dx$$
$$+ 2\int_0^L \delta_2(x)\psi_x^2\,dx + \left(\mu_0' - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\right)\int_0^L \psi^2\,dx \geq 0.$$

Hence, $\mathcal{A}$ is monotone. Next, we prove that the operator $I+\mathcal{A}$ is surjective, i.e., for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, there exists $U = (u, \varphi, z_1, v, \psi, z_2)^T \in D(\mathcal{A})$ satisfying

$$(I+\mathcal{A})U = F, \tag{2.7}$$

which is equivalent to

$$\begin{cases} u - \varphi = f_1, \\ (m_1(x)+\mu_0)\varphi - (p_1(x)u_x + 2\delta_1(x)\varphi_x)_x + \displaystyle\int_{\tau_1}^{\tau_2} \mu_1(s) z_1(x,1,t,s)\,ds = m_1(x) f_2, \\ s z_1 + z_{1\rho} = s f_3, \\ v - \psi = f_4, \\ (m_2(x)+\mu_0')\psi - (p_2(x)v_x + 2\delta_2(x)\psi_x)_x + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(s) z_2(x,1,t,s)\,ds = m_2(x) f_5, \\ s z_2 + z_{2\rho} = s f_6. \end{cases} \tag{2.8}$$

Suppose that we have found u and v. Then equations $(2.8)_1$ and $(2.8)_4$ yield

$$\begin{cases} \varphi = u - f_1, \\ \psi = v - f_4. \end{cases} \tag{2.9}$$

It is clear that $\varphi \in H_0^1(0,L)$ and $\psi \in H_0^1(0,L)$. Equations $(2.8)_3$ and $(2.8)_6$ with (2.9), recalling $z_1(x,0,t,s) = \varphi$, $z_2(x,0,t,s) = \psi$, yield

$$z_1(x,\rho,s) = u(x)e^{-\rho s} - f_1(x)e^{-\rho s} + s e^{-\rho s}\int_0^\rho f_3(x,\tau,s)e^{\tau s}\,d\tau \tag{2.10}$$

and

$$z_2(x,\rho,s) = v(x)e^{-\rho s} - f_4(x)e^{-\rho s} + s e^{-\rho s}\int_0^\rho f_6(x,\tau,s)e^{\tau s}\,d\tau. \tag{2.11}$$

Clearly, $z_1, z_{1\rho}, z_2, z_{2\rho} \in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$.

Inserting $(2.9)_1$ and (2.10) into $(2.8)_2$, and inserting $(2.9)_2$ and (2.11) into $(2.8)_5$, we get

$$\begin{cases} \eta_1 u - (p_1(x)u_x + 2\delta_1(x)\varphi_x)_x = g_1, \\ \eta_2 v - (p_2(x)v_x + 2\delta_2(x)\psi_x)_x = g_2, \\ u_x - \varphi_x = g_3, \\ v_x - \psi_x = g_4, \end{cases} \tag{2.12}$$

where

$$\eta_1 = m_1(x) + \mu_0 + \int_{\tau_1}^{\tau_2} \mu_1(s)e^{-s}\,ds, \quad \eta_2 = m_2(x) + \mu_0' + \int_{\tau_1}^{\tau_2} \mu_2(s)e^{-s}\,ds,$$

$$g_1 = \eta_1 f_1 + m_1(x) f_2 - \int_{\tau_1}^{\tau_2} s\mu_1(s)e^{-s} \int_0^1 f_3(x,\tau,s)e^{\tau s}\,d\tau\,ds,$$

$$g_2 = \eta_2 f_4 + m_2(x) f_5 - \int_{\tau_1}^{\tau_2} s\mu_2(s)e^{-s} \int_0^1 f_6(x,\tau,s)e^{\tau s}\,d\tau\,ds,$$

$$g_3 = f_{1x}, \quad g_4 = f_{4x}.$$

The variational formulation corresponding to equation (2.12) takes the form

$$B((u,v)^T, (\widetilde{u},\widetilde{v})^T) = G(\widetilde{u},\widetilde{v})^T, \tag{2.13}$$

where

$$B : \left[H_0^1(0,L) \times H_0^1(0,L)\right]^2 \longrightarrow \mathbb{R}$$

is the bilinear form given by

$$B\big((u,v)^T, (\widetilde{u},\widetilde{v})^T\big) = \eta_1 \int_0^L u\widetilde{u}\,dx + \int_0^L (p_1(x) + 2\delta_1(x))u_x\widetilde{u}_x\,dx + \eta_2 \int_0^L v\widetilde{v}\,dx + \int_0^L (p_2(x) + 2\delta_2(x))v_x\widetilde{v}_x\,dx,$$

and

$$G : \left[H_0^1(0,L) \times H_0^1(0,L)\right] \longrightarrow \mathbb{R}$$

is the linear form defined by

$$G(\widetilde{u},\widetilde{v})^T = \int_0^L g_1\widetilde{u}\,dx + \int_0^L g_2\widetilde{v}\,dx + \int_0^L 2\delta_1(x)g_3\widetilde{u}_x\,dx + \int_0^L 2\delta_2(x)g_4\widetilde{v}_x\,dx.$$

Now, we introduce the Hilbert space $V = H_0^1(0,L) \times H_0^1(0,L)$ equipped with the norm

$$\|(u,v)\|_V^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|v\|_2^2 + \|v_x\|_2^2.$$

It is clear that $B(\cdot,\cdot)$ and $G(\cdot)$ are bounded. Furthermore, we can find that there exists a positive constant α such that

$$B((u,v)^T, (u,v)^T) = \eta_1 \int_0^L u^2\,dx + \int_0^L (p_1(x) + 2\delta_1(x))u_x^2\,dx + \eta_2 \int_0^L v^2\,dx + \int_0^L (p_2(x) + 2\delta_2(x))v_x^2\,dx \geq \alpha\|(u,v)\|_V^2,$$

which implies that $B(\cdot,\cdot)$ is coercive.

Consequently, applying the Lax–Milgram lemma, we obtain that (2.13) has a unique solution $(u,v)^T \in V$.

Then, by substituting u, v into (2.9), we get

$$\varphi, \psi \in H_0^1(0, L).$$

Next, it remains to show that

$$u, v \in H^2(0, L) \cap H_0^1(0, L).$$

Furthermore, if $\widetilde{v} \equiv 0 \in H_0^1(0, L)$, then (2.13) reduces to

$$-\int_0^L \left[(p_1(x) + 2\delta_1(x))u_x\right]_x \widetilde{u}\, dx = \int_0^L g_1 \widetilde{u}\, dx - \int_0^L 2(\delta_1(x)g_3)_x \widetilde{u}\, dx - \eta_1 \int_0^L u\widetilde{u}\, dx$$

for all $\widetilde{u}$ in $H_0^1(0, L)$, which implies

$$\left[(p_1(x) + 2\delta_1(x))u_x\right]_x = \eta_1 u - g_1 + 2(\delta_1(x)g_3)_x \in L^2(0, L).$$

Thus, by the L^2 theory for the linear elliptic equations, we obtain

$$u \in H^2(0, L) \cap H_0^1(0, L).$$

In a similar way, we have

$$v \in H^2(0, L) \cap H_0^1(0, L).$$

Finally, the application of the classical regularity theory for the linear elliptic equations guarantees the existence of unique solution $U \in D(\mathcal{A})$ which satisfies (2.7). Therefore, the operator $\mathcal{A}$ is maximal.

Hence, the result of Theorem 2.1 follows. □

3 Exponential stability

In this section, we prove the exponential decay for problem (2.1), (2.2). This will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$\begin{aligned} &E(t) = E_1(t) + E_2(t), \\ &E_1(t) = \frac{1}{2}\int_0^L \left[m_1(x)u_t^2 + p_1(x)u_x^2\right] dx + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1^2(x, \rho, z, t)\, ds\, d\rho\, dx, \\ &E_2(t) = \frac{1}{2}\int_0^L \left[m_2(x)u_t^2 + p_2(x)u_x^2\right] dx + \frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2^2(x, \rho, z, t)\, ds\, d\rho\, dx. \end{aligned} \tag{3.1}$$

We have the following exponentially stable result.

Theorem 3.1. *Let* $(u, u_t, z_1, v, v_t, z_2)$ *be a solution of* (2.1), (2.2) *and assume that* (1.3) *holds. Then there exists positive constants* λ_0, λ_1 *such that the energy* $E(t)$ *associated with problem* (2.1), (2.2) *satisfies*

$$E(t) \le \lambda_0 e^{-\lambda_1 t}, \quad t \ge 0. \tag{3.2}$$

To prove this result, we will state and prove some useful lemmas in advance.

Lemma 3.2 (**Poincaré-type Scheeffer's inequality**, [10])**.** *Let* $h \in H_0^1(0, L)$*. Then*

$$\int_0^L |h|^2\, dx \le \frac{L^2}{\pi^2}\int_0^L |h_x|^2\, dx. \tag{3.3}$$

Lemma 3.3 (**Mean value theorem**, [1]). *Let (u, u_t, v, v_t) be a solution to system (1.1), (1.2) with an initial datum in $D(\mathcal{A})$. Then, for any $t > 0$, there exists a sequence of real numbers (depending on t), denoted by $\zeta_i, \xi_i \in [0, L] (i = 1, \dots, 6)$, such that*

$$\int_0^L p_1(x)u_x^2\,dx = p_1(\zeta_1)\int_0^L u_x^2\,dx, \quad \int_0^L m_1(x)u_t^2\,dx = m_1(\zeta_2)\int_0^L u_t^2\,dx,$$

$$\int_0^L m_1(x)u^2\,dx = m_1(\zeta_3)\int_0^L u^2\,dx, \quad \int_0^L \delta_1(x)u^2\,dx = \delta_1(\zeta_4)\int_0^L u^2\,dx,$$

$$\int_0^L \delta_1(x)u_x^2\,dx = \delta_1(\zeta_5)\int_0^L u_x^2\,dx, \quad \int_0^L \delta_1(x)u_{xt}^2\,dx = \delta_1(\zeta_6)\int_0^L u_{xt}^2\,dx,$$

$$\int_0^L p_2(x)v_x^2\,dx = p_2(\xi_1)\int_0^L v_x^2\,dx, \quad \int_0^L m_2(x)v_t^2\,dx = m_2(\xi_2)\int_0^L v_t^2\,dx,$$

$$\int_0^L m_2(x)v^2\,dx = m_2(\xi_3)\int_0^L v^2\,dx, \quad \int_0^L \delta_2(x)v^2\,dx = \delta_2(\xi_4)\int_0^L v^2\,dx,$$

$$\int_0^L \delta_2(x)v_x^2\,dx = \delta_2(\xi_5)\int_0^L v_x^2\,dx, \quad \int_0^L \delta_2(x)v_{xt}^2\,dx = \delta_2(\xi_6)\int_0^L v_{xt}^2\,dx.$$

Lemma 3.4. *Let $(u, u_t, z_1, v, v_t, z_2)$ be a solution of (2.1), (2.2). Then the energy functional satisfies*

$$E'(t) = E_1'(t) + E_2'(t) \le 0, \quad \forall\, t \ge 0,$$

$$E_1'(t) \le -2\int_0^L \delta_1(x)u_{xt}^2\,dx + \left(\int_{\tau_1}^{\tau_2} |\mu_1(s)|\,ds - \mu_0\right)\int_0^L u_t^2\,dx \le 0,$$

$$E_2'(t) \le -2\int_0^L \delta_2(x)v_{xt}^2\,dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds - \mu_0'\right)\int_0^L v_t^2\,dx \le 0.$$

Proof. Multiplying $(2.1)_1$ and $(2.1)_3$ by u_t and v_t, respectively, and integrating over $(0, L)$, using integration by parts and the boundary conditions in (2.2), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\int_0^L \left[m_1(x)u_t^2 + p_1(x)u_x^2\right]dx \\ &= -2\int_0^L \delta_1(x)u_{xt}^2\,dx - \mu_0\int_0^L u_t^2\,dx - \int_0^L u_t\int_{\tau_1}^{\tau_2} \mu_1(s)z_1(x,1,t,s)\,ds\,dx, \end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\int_0^L \left[m_2(x)v_t^2 + p_2(x)v_x^2\right]dx \\ &= -2\int_0^L \delta_2(x)v_{xt}^2\,dx - \mu_0'\int_0^L v_t^2\,dx - \int_0^L v_t\int_{\tau_1}^{\tau_2} \mu_2(s)z_2(x,1,t,s)\,ds\,dx. \end{aligned} \tag{3.5}$$

On the other hand, multiplying $(2.1)_2$ and $(2.1)_4$ by $|\mu_1(s)|z_1$ and $|\mu_2(s)|z_2$, respectively, and integrating over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, and recalling $z_1(x, 0, t, s) = u_t$ and $z_2(x, 0, t, s) = v_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)|z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx$$
$$= -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2}|\mu_1(s)|\,ds\,dx, \tag{3.6}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx$$
$$= -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_0^L v_t^2\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\,dx. \tag{3.7}$$

A combination of (3.4) and (3.6) gives

$$E_1'(t) = -2\int_0^L \delta_1(x)u_{xt}^2\,dx - \mu_0\int_0^L u_t^2\,dx - \int_0^L u_t\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,t,s)\,ds\,dx$$
$$-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(s)|\,ds\int_0^L u_t^2\,dx. \tag{3.8}$$

Also, (3.5) and (3.7) give

$$E_2'(t) = -2\int_0^L \delta_2(x)v_{xt}^2\,dx - \mu_0'\int_0^L v_t^2\,dx - \int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z_2(x,1,t,s)\,ds\,dx$$
$$-\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^L v_t^2\,dx. \tag{3.9}$$

Now, using Young's inequality, we obtain

$$-\int_0^L u_t\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,t,s)\,ds\,dx \le \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(s)|\,ds\int_0^1 u_t^2\,dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s)\,ds\,dx, \tag{3.10}$$

$$-\int_0^L v_t\int_{\tau_1}^{\tau_2}\mu_2(s)z_2(x,1,t,s)\,ds\,dx \le \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_0^1 v_t^2\,dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s)\,ds\,dx. \tag{3.11}$$

Substituting (3.10) into (3.8), (3.11) into (3.9), and using (1.3), we obtain (3.4), which completes the proof. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.5. *Let* $(u, u_t, z_1, v, v_t, z_2)$ *be a solution of* (2.1), (2.2). *Then the functions*

$$I_1(t) := \int_0^L \delta_1(x)u_x^2\,dx + \int_0^L m_1(x)u_t u\,dx,$$

$$F_1(t) := \int_0^L \delta_2(x)v_x^2\,dx + \int_0^L m_2(x)v_t v\,dx$$

satisfy, for all $\varepsilon_1, \varepsilon_2 > 0$ *and* $\varepsilon_1', \varepsilon_2' > 0$, *the estimates*

$$I_1'(t) \leq -\Big(p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\,\varepsilon_1 - \frac{L^2\varepsilon_2}{\pi^2}\Big)\int_0^L u_x^2\,dx + \Big(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\Big)\int_0^L u_t^2\,dx + \frac{\mu_0}{4\varepsilon_2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx, \tag{3.12}$$

$$F_1'(t) \leq -\Big(p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\,\varepsilon_1' - \frac{L^2\varepsilon_2'}{\pi^2}\Big)\int_0^L v_x^2\,dx + \Big(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\Big)\int_0^L v_t^2\,dx + \frac{\mu_0'}{4\varepsilon_2'}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_2^2(x,1,s,t)\,ds\,dx. \tag{3.13}$$

Proof. By differentiating $I_1(t)$ with respect to t, using $(2.1)_1$ and integrating by parts, we obtain

$$I_1'(t) = -\int_0^L p_1(x)u_x^2\,dx - \mu_0\int_0^L u_t u\,dx - \int_0^L u\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,s,t)\,ds\,dx + \int_0^L m_1(x)u_t^2\,dx.$$

By using Young's inequality, Lemma 3.2 and $(1.3)_1$, for $\varepsilon_1, \varepsilon_2 > 0$ we get

$$-\mu_0\int_0^L u_t u\,dx \leq \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1\int_0^L u_x^2\,dx + \frac{1}{4\varepsilon_1}\int_0^L u_t^2\,dx, \tag{3.14}$$

$$-\int_0^L u\int_{\tau_1}^{\tau_2}\mu_1(s)z_1(x,1,s,t)\,ds\,dx \leq \frac{L^2\varepsilon_2}{\pi^2}\int_0^L u_x^2\,dx + \frac{\mu_0}{4\varepsilon_2}\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx. \tag{3.15}$$

Consequently, using Lemma 3.3, (3.14) and (3.15), we establish (3.12).
Similarly, we prove (3.13). □

Lemma 3.6. *Let* $(u, u_t, z_1, v, v_t, z_2)$ *be a solution of* $(2.1), (2.2)$. *Then the functions*

$$I_2(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_1(s)|z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx,$$

$$F_2(t) := \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx,$$

satisfy, for some positive constants n_1 *and* n_2, *the estimates*

$$I_2'(t) \leq -n_1\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)|z_1^2(x,\rho,s,t)\,ds\,d\rho\,dx - n_1\int_0^L\int_{\tau_1}^{\tau_2}|\mu_1(s)|z_1^2(x,1,s,t)\,ds\,dx + \mu_0\int_0^L u_t^2\,dx, \tag{3.16}$$

$$F_2'(t) \leq -n_2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z_2^2(x,\rho,s,t)\,ds\,d\rho\,dx$$

$$- n_2 \int_0^L \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s,t)\, ds\, dx + \mu_0' \int_0^L v_t^2\, dx. \tag{3.17}$$

Proof. By differentiating $I_2(t)$ with respect to t and using equation $(2.1)_2$, we obtain

$$\begin{aligned} I_2'(t) &= -2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_1(s)| z_1(x,\rho,s,t) z_{1\rho}(x,\rho,s,t)\, ds\, d\rho\, dx \\ &= -\frac{d}{d\rho} \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx \\ &= - \int_0^L \int_{\tau_1}^{\tau_2} |\mu_1(s)| [e^{-s} z_1^2(x,1,s,t) - z_1^2(x,0,s,t)]\, ds\, dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx. \end{aligned}$$

Using the fact that $z_1(x,0,s,t) = u_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} I_2'(t) \leq & - \int_0^L \int_{\tau_1}^{\tau_2} e^{-s} |\mu_1(s)| z_1^2(x,1,s,t)\, ds\, dx \\ & + \int_{\tau_1}^{\tau_2} |\mu_1(s)|\, ds \int_0^L u_t^2\, dx - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx. \end{aligned}$$

Since $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$.

Finally, setting $n_1 = e^{-\tau_2}$ and recalling $(1.3)_1$, we obtain (3.16).

Similarly, we prove (3.17). □

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

Lemma 3.7. *Let $N, N_1, N_2 > 0$ and a functional be defined by*

$$L(t) := NE(t) + I_1(t) + N_1 I_2(t) + F_1(t) + N_2 F_2(t). \tag{3.18}$$

For two positive constants c_1 and c_2, we have

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \quad \forall\, t \geq 0. \tag{3.19}$$

Proof. Let

$$\mathcal{L}(t) := I_1(t) + N_1 I_2(t) + F_1(t) + N_2 F_2(t).$$

Then

$$\begin{aligned} |\mathcal{L}(t)| \leq & \int_0^L \delta_1(x) u_x^2\, dx + \frac{1}{2} \int_0^L m_1(x) u_t^2\, dx + \frac{1}{2} \int_0^L m_1(x) u^2\, dx \\ & + N_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx + \int_0^L \delta_2(x) v_x^2\, dx + \frac{1}{2} \int_0^L m_2(x) v_t^2\, dx \\ & + \frac{1}{2} \int_0^L m_2(x) v^2\, dx + N_2 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z_2^2(x,\rho,s,t)\, ds\, d\rho\, dx \leq c' E_1(t) + c'' E_2(t) \leq c_0 E(t), \end{aligned}$$

where $c_0 = \max\{c', c''\}$, with

$$c' = 1 + \frac{L^2 m_1(\zeta_3)}{\pi^2 p_1(\zeta_1)} + \frac{2\delta_1(\zeta_5)}{p_1(\zeta_1)} + 2N_1, \quad c'' = 1 + \frac{L^2 m_2(\xi_3)}{\pi^2 p_2(\xi_1)} + \frac{2\delta_2(\xi_5)}{p_2(\xi_1)} + 2N_2.$$

Consequently, $|L(t) - NE(t)| \le c_0 E(t)$, which yields

$$(N - c_0)E(t) \le L(t) \le (N + c_0)E(t).$$

Choosing N large enough, we obtain estimate (3.19). □

Now, we prove the main result of this section.

Proof of Theorem 3.1. Differentiating (3.18) and recalling (3.4), (3.12), (3.13), (3.16) and (3.17), we obtain

$$\begin{aligned}
L'(t) \le{}& \Bigg[\Bigg(\int_{\tau_1}^{\tau_2} |\mu_1(s)|\, ds - \mu_0\Bigg)N + \Big(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\Big) + N_1\mu_0\Bigg] \int_0^L u_t^2\, dx \\
&- \Big[p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\,\varepsilon_1 - \frac{L^2}{\pi^2}\,\varepsilon_2\Big] \int_0^L u_x^2\, dx - 2N\int_0^L \delta_1(x) u_{xt}^2\, dx \\
&- n_1N_1 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx - \Big[n_1N_1 - \frac{\mu_0}{4\varepsilon_2}\Big] \int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s,t)\, ds\, dx \\
&+ \Bigg[\Bigg(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \mu_0'\Bigg)N + \Big(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\Big) + N_2\mu_0'\Bigg] \int_0^L v_t^2\, dx \\
&- \Big[p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\varepsilon_1' - \frac{L^2}{\pi^2}\varepsilon_2'\Big] \int_0^L v_x^2\, dx - 2N\int_0^L \delta_2(x) v_{xt}^2\, dx \\
&- n_2N_2 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2^2(x,\rho,s,t)\, ds\, d\rho\, dx - \Big[n_2N_2 - \frac{\mu_0'}{4\varepsilon_2'}\Big] \int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s,t)\, ds\, dx.
\end{aligned}$$

Using Lemma 3.2 and Lemma 3.3, we get

$$\begin{aligned}
L'(t) \le{}& -\Big[\gamma_1 N - \frac{L^2}{\pi^2}\Big(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\Big) - \frac{L^2\mu_0}{}N_1\Big] \int_0^L u_{tx}^2\, dx - \Big[p_1(\zeta_1) - \frac{L^2\mu_0^2}{\pi^2}\,\varepsilon_1 - \frac{L^2}{\pi^2}\varepsilon_2\Big] \int_0^L u_x^2\, dx \\
&- n_1N_1 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_1(s)| z_1^2(x,\rho,s,t)\, ds\, d\rho\, dx - \Big[n_1N_1 - \frac{\mu_0}{4\varepsilon_2}\Big] \int_0^L\int_{\tau_1}^{\tau_2} |\mu_1(s)| z_1^2(x,1,s,t)\, ds\, dx \\
&- \Big[\gamma_2 N - \frac{L^2}{\pi^2}\Big(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\Big) - \frac{L^2\mu_0'}{\pi^2}\,N_2\Big] \int_0^L v_{tx}^2\, dx - \Big[p_2(\xi_1) - \frac{L^2\mu_0'^2}{\pi^2}\,\varepsilon_1' - \frac{L^2}{\pi^2}\,\varepsilon_2'\Big] \int_0^L v_x^2\, dx \\
&- n_2N_2 \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z_2^2(x,\rho,s,t)\, ds\, d\rho\, dx - \Big[n_2N_2 - \frac{\mu_0'}{4\varepsilon_2'}\Big] \int_0^L\int_{\tau_1}^{\tau_2} |\mu_2(s)| z_2^2(x,1,s,t)\, ds\, dx, \quad (3.20)
\end{aligned}$$

where

$$\gamma_1 = 2\delta_1(\zeta_6) - \frac{L^2}{\pi^2}\Bigg(\int_{\tau_1}^{\tau_2} |\mu_1(s)|\, ds - \mu_0\Bigg) > 0,$$

$$\gamma_2 = 2\delta_2(\xi_6) - \frac{L^2}{\pi^2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds - \mu_0'\right) > 0.$$

At this point, we need to choose our constants very carefully.

First, we choose $\varepsilon_2 < \frac{\pi^2}{2L^2} p_1(\zeta_1)$ and $\varepsilon_2' < \frac{\pi^2}{2L^2} p_2(\xi_1)$ so that

$$p_1(\zeta_1) - \frac{L^2}{\pi^2}\varepsilon_2 > \frac{p_1(\zeta_1)}{2}, \quad p_2(\xi_1) - \frac{L^2}{\pi^2}\varepsilon_2' > \frac{p_2(\xi_1)}{2}.$$

Next, we choose N_1 and N_2 large enough so that

$$n_1 N_1 - \frac{\mu_0}{4\varepsilon_2} > 0, \quad n_2 N_2 - \frac{\mu_0'}{4\varepsilon_2'} > 0.$$

Then, we choose ε_1 and ε_1' small enough satisfying

$$\frac{p_1(\zeta_1)}{2} - \frac{L^2\mu_0^2}{\pi^2}\varepsilon_1 > 0, \quad \frac{p_2(\xi_1)}{2} - \frac{L^2\mu_0'^2}{\pi^2}\varepsilon_1' > 0.$$

Finally, we choose N large enough so that

$$\gamma_1 N - \frac{L^2}{\pi^2}\left(m_1(\zeta_2) + \frac{1}{4\varepsilon_1}\right) - \frac{L^2\mu_0}{\pi^2} N_1 > 0,$$
$$\gamma_2 N - \frac{L^2}{\pi^2}\left(m_2(\xi_2) + \frac{1}{4\varepsilon_1'}\right) - \frac{L^2\mu_0'}{\pi^2} N_2 > 0.$$

By (3.1), we deduce that there exists a positive constant c_3 such that (3.20) becomes

$$L'(t) \le -c_3 E(t), \ \ \forall\, t \ge 0. \tag{3.21}$$

The combination of (3.19) and (3.21) gives

$$L'(t) \le -\lambda_1 L(t), \ \ \forall\, t \ge 0, \tag{3.22}$$

where $\lambda_1 = \frac{c_3}{c_2}$. Then a simple integration of (3.22) over $(0,t)$ yields

$$c_1 E(t) \le L(t) \le L(0) e^{-\lambda_1 t}, \ \ \forall\, t \ge 0. \tag{3.23}$$

Finally, combining (3.19) and (3.23), we obtain (3.2) with $\lambda_0 = \frac{c_2 E(0)}{c_1}$, which completes the proof. □

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